Instability of the phonon spectrum of a superfluid Bose system near the termination point of the homogeneous phase

V. A. Podol'skiĭ

A. I. Herzen State Pedagogical Institute

(Submitted 14 May 1982)
Zh. Eksp. Teor. Fiz. **83**, 2102–2109 (December 1982)

We study the instability of the phonon spectrum of a "compressed" Bose system with a condensate. We show that the phonon instability is caused by a preceding roton instability that occurs when the roton minimum is lowered with increasing pressure.

PACS numbers: 67.90. + z

1. INTRODUCTION

It is well known that the quantum liquids ^3He and ^4He change to a crystalline state when the pressure is increased. It was indicated in Ref. 1 that a periodic structure differing from the usual crystalline state can exist if the interaction between the particles is long range and possesses sufficiently large Fourier components of the potential $\nu_p < 0$ (attraction) in some interval of momentum transfer $p \sim p_0$. The possibility for the formation of such crystals, called coherent ones, was studied in Ref. 1 using the model of a "compressed" system (characteristic momentum transfer p_0 small compared to the reciprocal of the distance between the particles $n^{1/3}$, energy of interaction small compared to the kinetic energy, but the average potential energy per unit volume of the order of or larger than the kinetic energy). The transition of the system into an inhomogeneous state usually corresponds to the occurrence of an instability of the homogeneous state relative to arbitrarily small density perturbations which violate the homogeneity.

The compressed model was used in Ref. 2 to analyze the behavior of a quantum liquid near the point of the absolute instability of the homogeneous phase. For a Bose system in the Bogolyubov approximation the instability is connected with the vanishing of the roton minimum when the density increases after which the Green function pole enters the nonphysical region. The energy of the system then ceases to correspond to a local minimum and the homogeneous state turns out to be unstable with respect to short-wavelength density perturbations. However, this does not agree with the experimental situation, as the roton minimum in ^4He decreases only insignificantly when the pressure is increased up to crystallization. It was noted in Ref. 2 that an exact picture of the occurrence of the instability differs from the Bogolyubov picture in two respects: 1) the short-wavelength instability starts at a nonzero value of the roton minimum $\Delta_c \neq 0$; 2) the roton instability precedes the phonon one connected with it: as $n \to n_c$ (i.e., as $\Delta \to \Delta_c$) the roton region makes a negative divergent contribution to the elasticity so that in some point $n = n' < n_c$ (in the immediate vicinity of n_c) the sound speed

$$u = (n/m)\,(d\mu/dn)^{1/2}$$

vanishes. Thus, in the true termination point of the homogeneous phase the phonon pole frequencies become imaginary after passing through zero. This result was obtained in Ref. 2 indirectly as a consequence of thermodynamic relations. In the present paper it is confirmed by a direct study of the phonon spectrum in the framework of the same model.

2. THE BOGOLYUBOV-ZUBAREV HAMILTONIAN AND THE GREEN FUNCTION

To study the phonon section of the spectrum it is convenient to use a modification of the Bogolyubov-Zugarev technique developed in Refs. 3 to 6. Its salient feature is that the self-energy parts to all orders of perturbation theory are proportional in the long-wavelength limit to the external momentum. Thus, in contrast to the Belyaev technique one is guaranteed the correct behavior of the spectrum for small momenta, regardless of which subsequence of diagrams is summed.

The Hamiltonian of the system is written in the form

$$\hat{H}_{BZ} = \hat{H}_0 + \hat{H}_1,$$

where

$$\hat{H}_0 = \mathscr{E}_0^B + \sum_{\mathbf{k}\neq 0} \varepsilon_B(\mathbf{k})\, b_{\mathbf{k}}^+ b_{\mathbf{k}}, \tag{1}$$

$$\begin{aligned}\hat{H}_1 = \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \delta_{\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3,0} [\gamma_a(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\, b_{\mathbf{k}_1} b_{\mathbf{k}_2} b_{\mathbf{k}_3} \\ + \gamma_b(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\, b_{\mathbf{k}_1}^+ b_{\mathbf{k}_2}^+ b_{\mathbf{k}_3}^+ \\ + \gamma_c(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\, b_{\mathbf{k}_1} b_{-\mathbf{k}_2} b_{-\mathbf{k}_3} \\ + \gamma_d(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\, b_{-\mathbf{k}_3}^{+} b_{-\mathbf{k}_2}^{\pm} b_{\mathbf{k}_1}],\end{aligned} \tag{2}$$

$$\mathscr{E}_0^B = \frac{N(N-1)}{2\Omega}\nu_0 - \frac{1}{4}\sum_{\mathbf{k}\neq 0} \varepsilon_B(\mathbf{k}) \left(1 - \frac{1}{\lambda_{\mathbf{k}}}\right)^2 \lambda_{\mathbf{k}}, \tag{3}$$

$$\lambda_{\mathbf{k}} = \left(1 + \frac{4m}{k^2}\frac{N}{\Omega}\nu_{\mathbf{k}}\right)^{-1/2}, \quad \varepsilon_B(\mathbf{k}) = \frac{k^2}{2m\lambda_{\mathbf{k}}}, \tag{4}$$

$$\gamma_a(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \frac{1}{24mN^{1/2}} \frac{1}{(\lambda_{\mathbf{k}_1}\lambda_{\mathbf{k}_2}\lambda_{\mathbf{k}_3})^{1/2}} \times \sum_{P(123)} (\mathbf{k}_1\mathbf{k}_2)(\lambda_{\mathbf{k}_1}+1)(\lambda_{\mathbf{k}_2}+1)\lambda_{\mathbf{k}_3},$$

 0038-5646/82/121219-05$04.00

$$\gamma_b(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=\frac{1}{24mN^{1/2}}\frac{1}{(\lambda_{\mathbf{k}_1}\lambda_{\mathbf{k}_2}\lambda_{\mathbf{k}_3})^{1/2}}\times\sum_{P(123)}(\mathbf{k}_1\mathbf{k}_2)(\lambda_{\mathbf{k}_1}-1)(\lambda_{\mathbf{k}_2}-1)\lambda_{\mathbf{k}_3},\tag{5}$$

$$\gamma_c(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=\frac{1}{8mN^{1/2}}\frac{1}{(\lambda_{\mathbf{k}_1}\lambda_{\mathbf{k}_2}\lambda_{\mathbf{k}_3})^{1/2}}\{(\mathbf{k}_2\mathbf{k}_3)(\lambda_{\mathbf{k}_2}+1)(\lambda_{\mathbf{k}_3}+1)\lambda_{\mathbf{k}_1}+[(\mathbf{k}_1\mathbf{k}_2)(\lambda_{\mathbf{k}_2}+1)\lambda_{\mathbf{k}_3}+(\mathbf{k}_1\mathbf{k}_3)(\lambda_{\mathbf{k}_3}+1)\lambda_{\mathbf{k}_2}](\lambda_{\mathbf{k}_1}-1)\},$$

$$\gamma_d(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)=\frac{1}{8mN^{1/2}}\frac{1}{(\lambda_{\mathbf{k}_1}\lambda_{\mathbf{k}_2}\lambda_{\mathbf{k}_3})^{1/2}}\{(\mathbf{k}_2\mathbf{k}_3)(\lambda_{\mathbf{k}_2}-1)(\lambda_{\mathbf{k}_3}-1)\lambda_{\mathbf{k}_1}+[(\mathbf{k}_1\mathbf{k}_2)(\lambda_{\mathbf{k}_2}-1)\lambda_{\mathbf{k}_3}+(\mathbf{k}_1\mathbf{k}_3)(\lambda_{\mathbf{k}_3}-1)\lambda_{\mathbf{k}_2}](\lambda_{\mathbf{k}_1}+1)\},$$

here $\nu_{\mathbf{k}}$ is the Fourier transform of the interaction potential; $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^{+}$ the particle annihilation and creation operators satisfying the usual commutation relations for bosons; $P(123)$ denotes a cyclical permutation of the indexes; $\gamma_c(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$,$\gamma_d(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)$ are symmetrical in $(\mathbf{k}_2,\mathbf{k}_3)$. In contrast to the Belyaev technique we take here for $\hat{H}_0$ the quadratic part of the Bogolyubov-Zubarev Hamiltonian $\hat{H}_{\mathrm{BZ}}$ while the remaining terms are considered as a perturbation with $\hat{H}_1$ the non-Hermitean part of the Hamiltonian $\hat{H}_{\mathrm{BZ}}$:

The unperturbed Green function is given by the expression

$$G^0(p)=(\varepsilon-\varepsilon_B(\mathbf{p})+i\delta)^{-1},\tag{6}$$

and for the exact Green function $G(p)$ we can write down a matrix Dyson equation[4] which is shown graphically in Fig. 1.

The normal and anomalous Green functions are given by the formulae

$$G(p)=(\varepsilon+\varepsilon_B(\mathbf{p})+\Sigma_{22}(p))/M(p),\tag{7}$$
$$\hat{G}(p)=-\Sigma_{12}(p)/M(p),\qquad \check{G}(p)=-\Sigma_{21}(p)/M(p),$$
$$M(p)=[\varepsilon-{}^1/_2(\Sigma_{11}-\Sigma_{22})]^2-[\varepsilon_B(p)+{}^1/_2(\Sigma_{11}+\Sigma_{22})]^2+\Sigma_{12}\Sigma_{21}.$$

In contrast to the expressions obtained by Belyaev[7] the Bogolyubov spectrum occurs in (7) instead of the free particle spectrum while the $\Sigma_{12}(p)\neq\Sigma_{21}(p)$ [and therefore $\hat{G}(p)\neq\check{G}(p)$] because the Hamiltonian is non-Hermitean. The energy spectrum of the system is determined by the poles of the Green functions from the equation

$$M(p)=0.\tag{8}$$

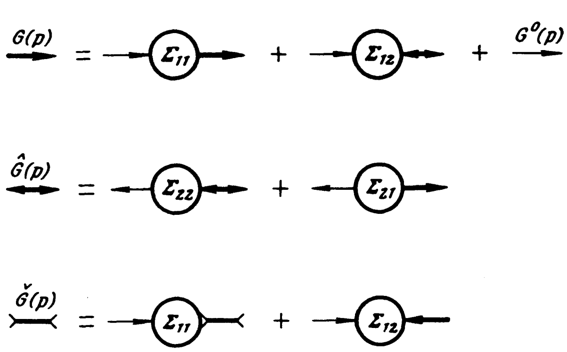


FIG. 1.

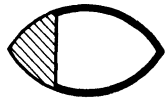
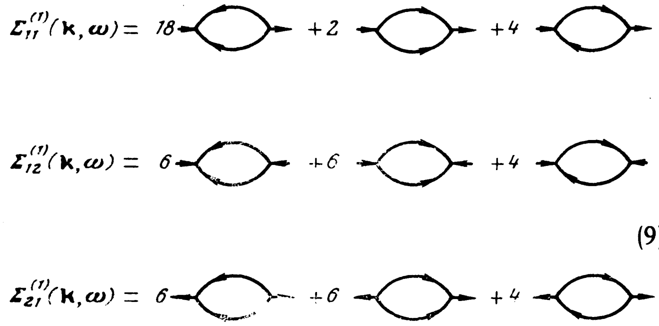


FIG. 2.

3. EQUATION FOR THE RENORMALIZED VERTICES

The very general form of the graphs for Σ_{ik} is a loop with a renormalized three-particle vertex and the exact Green functions (Fig. 2). In the first approximation the expressions for Σ_{ik} were found in Refs. 3, 4:



(9)

$\Sigma_{22}^{(1)}(\mathbf{k},\omega)$ is obtained from $\Sigma_{11}^{(1)}(\mathbf{k},\omega)$ by replacing all internal lines with the opposite ones.

The first-order diagrams for Σ_{ik} can be made more complicated in three ways to find the higher-order diagrams: a) by "dressing" the Green functions; b) by renormalizing the point vertices; c) by "dressing" the functions and renormalizing the vertices (see, respectively Figs. 3a,b,c). The renormalization of the vertices leads to the occurrence of graphs of a purely "ladder" type, of a "web" type, and mixed graphs (respectively, Figs. 4a,b,c).

We consider the compressed Bose system satisfying the conditions

$$\alpha\equiv m|\nu_{\mathbf{p}_0}|p_0\ll 1,\quad p_0\ll n^{1/3},\quad m|\nu_{\mathbf{p}_0}|np_0^{-2}\lesssim 1.\tag{10}$$

Studying the effect of the roton instability on the phonon section of the spectrum we shall assume that the external 4-momenta in Σ_{ik} are small ($|\mathbf{k}|\sim 0$, $\omega\sim 0$) while all internal ones are close to the "roton" values [$\varepsilon=\varepsilon_B(\mathbf{p}_0)$, $|\mathbf{p}|=p_0$ where p_0 is a characteristic momentum corresponding to the roton section of the spectrum].

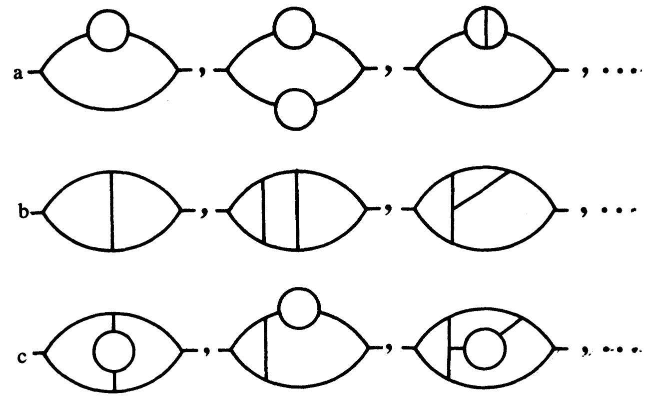


FIG. 3.

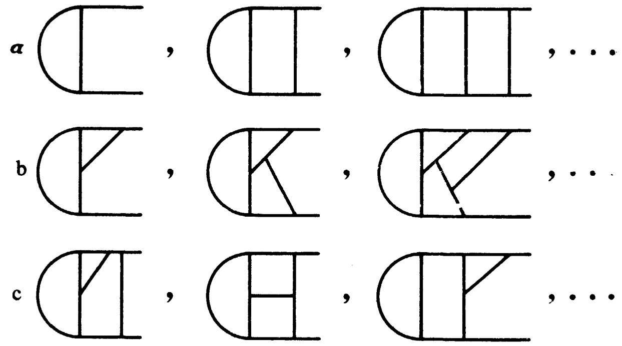


FIG. 4.

The first approximation for the self-energy part is a sum of terms of the form

$$i\int \frac{d^4p}{(2\pi)^4}\gamma_1(\mathbf{k},-\mathbf{p},\mathbf{p}-\mathbf{k})\gamma_2(\mathbf{k},-\mathbf{p},\mathbf{p}-\mathbf{k})G^0(p)G^0(k-p) \quad (11)$$

(γ_1 and γ_2 are vertices from the set $\gamma_a, \gamma_b, \gamma_c, \gamma_d$). The main contribution to the integral over $\mathbf{p}$ comes from the region $|\mathbf{p}| \sim p_0$, as $|\mathbf{k}| \sim 0$ and both intermediate momenta $\mathbf{p}$ and $\mathbf{p}-\mathbf{k}$ are then close to p_0 in order of magnitude. As $|\mathbf{k}| \to 0$ we use for γ approximate expressions retaining in them the lowest powers of k:

$$\gamma_a(\mathbf{k},-\mathbf{p},\mathbf{p}-\mathbf{k}) \approx \gamma_a(\mathbf{k},\mathbf{p}_0) = \gamma_b(\mathbf{k},\mathbf{p}_0) = \frac{1}{24mn^{1/2}}\left(\frac{k}{c_0}\right)^{1/2}\frac{\varepsilon^0(\mathbf{p}_0)}{\varepsilon_B(\mathbf{p}_0)}p_0^2,$$

$$\gamma_c(\mathbf{k},-\mathbf{p},\mathbf{p}-\mathbf{k}) \approx \gamma_c(\mathbf{k},\mathbf{p}_0) = \gamma_d(\mathbf{k},\mathbf{p}_0) = \frac{1}{8mn^{1/2}}\left(\frac{k}{c_0}\right)^{1/2}\frac{\varepsilon^0(\mathbf{p}_0)}{\varepsilon_B(\mathbf{p}_0)}p_0^2, \quad (12)$$

$$n \equiv N/\Omega, \quad c_0 = 2(mn\nu_0)^{1/2}, \quad \varepsilon^0(\mathbf{p}_0) = p_0^2/2m.$$

Substituting (6) and (12) into (11) and expanding $\varepsilon_B(\mathbf{p})$ near the roton minimum:

$$(\varepsilon_B(\mathbf{p}))^2 \approx \Delta_B^2 + {}^1/_2 c_B(p-p_0)^2, \quad c_B \approx p_0^2/m^2, \quad \Delta_B = \varepsilon_B(\mathbf{p}_0),$$

we get after integration over the frequency an approximate estimate for the loop:

$$-\frac{1}{m^2n}\left(\frac{\varepsilon^0(\mathbf{p}_0)}{\Delta_B}\right)^2\frac{k}{c_0}\frac{p_0^4}{\Delta_B}\int\limits_{|\mathbf{p}|\sim p_0} d^3\mathbf{p} \sim -\frac{p_0^5}{mn}\left(\frac{\varepsilon^0(\mathbf{p}_0)}{\Delta_B}\right)^2\frac{k}{c_0}. \quad (13)$$

We used the fact that $\varepsilon_B(\mathbf{p}) \approx \Delta_B$ in the region $|\mathbf{p}| \sim p_0$ and assumed that n was sufficiently close to $n_B^c = \varepsilon^0(\mathbf{p}_0)/2|\nu_{p_0}|$. In that case

$$\int\limits_{|\mathbf{p}|\sim p_0} d^3\mathbf{p} \sim p_0 m\Delta_B, \quad ((p-p_0)^2 \leqslant \Delta_B^2/c_B).$$

Separating in (13) the small dimensionless parameter α

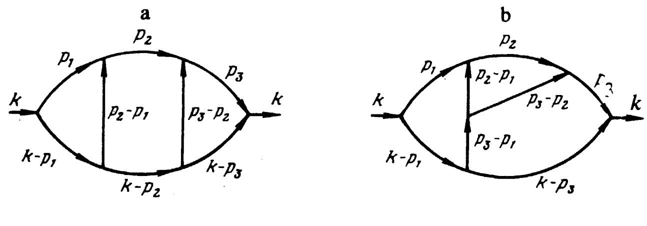


FIG. 5.

$$\frac{p_0^5}{mn} \sim \varepsilon^0(\mathbf{p}_0)\frac{p_0^3}{n} = \varepsilon^0(\mathbf{p}_0)\frac{p_0^3}{n_B^c}\frac{n_B^c}{n} \sim \varepsilon^0(\mathbf{p}_0)\,(p_0|\nu_{p_0}|m),$$

we get for $\Sigma_{ik}^{(1)}(\mathbf{k})$ the final estimate

$$\Sigma_{ik}^{(1)}(\mathbf{k}) \sim -\alpha(k/c_0)(\varepsilon^0(\mathbf{p}_0)/\Delta_B)^2\varepsilon^0(\mathbf{p}_0). \quad (14)$$

We note that $\Sigma_{ik} \propto k$ which guarantees a phonon type of spectrum.

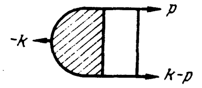
The integral over the intermediate momenta is proportional to the product of the small parameter α and the factor $(\varepsilon^0(\mathbf{p}_0)/\Delta_B)^2$ which diverges as $n \to n_B^c$. Higher-order diagrams can be split into two classes. We put diagrams with a ladder-type vertex in the first one. In our approximation each integration over $d^3\mathbf{p}$ in them leads to the occurrence of an additional factor $\alpha(\varepsilon^0)\mathbf{p}_0)/\Delta_B)^3$, whereas in the diagrams of the second class with web-type vertices such an integration gives a factor $\alpha(\varepsilon^0(\mathbf{p}_0)/\Delta_B)^2$ which has a lower power of divergence and the contribution from those diagrams is thus less than the one from the diagrams from the first class. We consider, for instance, two third-order diagrams (Fig. 5a,b). For the diagram of Fig. 5a the main contribution comes from the region

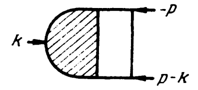
$$|\mathbf{p}_1| \sim p_0, \quad |\mathbf{p}_2| \sim p_0, \quad |\mathbf{p}_3| \sim p_0, \quad |\mathbf{p}_2-\mathbf{p}_1| \sim p_0, \quad |\mathbf{p}_3-\mathbf{p}_2| \sim p_0,$$

while for the diagram of Fig. 5b there is added to these conditions yet another one: $|\mathbf{p}_3 - \mathbf{p}_1| \sim p_0$ which leads to a corresponding diminution of the phase volume.

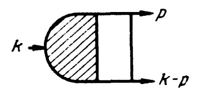
Diagrams with loop insertions in the internal lines give the same contribution as the ladder ones. This means that we can take for the internal Green funcitons in them, in general, "exact roton" functions which leads everywhere to replacing Δ_B by the exact value Δ. However, $\Delta_c \approx \Delta_B^c \neq 0$ at the point where the instability appears,[2] and the unperturbed Green functions which are essentially Bogolyubov functions become a sufficiently good approximation to the "exact" ones. We shall, however, show below that the instability of the phonon section of the spectrum is connected precisely with the renormalization of the three-particle vertex γ. We can thus write down for $\Sigma_{ik}(\mathbf{k},\omega)$ equations which differ from Eqs. (9) for $\Sigma_{ik}^{(1)}(\mathbf{k},\omega)$ only in that in each of their diagrams one of the point vertices γ is replaced by the vertex Γ taken in the ladder approximation.

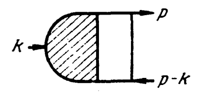
We must add to the equations for the self-energy parts a set of equations for the vertices

$$\Gamma_a(-k, p, k-p) = \gamma_a(-\mathbf{k}, \mathbf{p}, \mathbf{k}-\mathbf{p}) +$$ 

$$\Gamma_b(k, -p, p-k) = \gamma_b(\mathbf{k}, -\mathbf{p}, \mathbf{p}-\mathbf{k}) +$$ 

(15)

$$\Gamma_c(k, p, k-p) = \gamma_c(\mathbf{k}, \mathbf{p}, \mathbf{k}-\mathbf{p}) +$$ 

$$\Gamma_d(p, k, p-k) = \gamma_d(\mathbf{p}, \mathbf{k}, \mathbf{p}-\mathbf{k}) +$$ 

We must here in the internal vertices γ arrange all possible directions of the arrows. The set of equations for Γ is closed and has a unique solution, since its determinant, which is a function of Δ_B, does not vanish. We show below that the phonon instability starts before the roton one, and it is just with the appearance of the roton instability that the vanishing of the determinant is connected.

4. CONNECTION BETWEEN THE PHONON AND THE ROTON INSTABILITIES

When studying the phonon instability we look for the zero solution for the spectrum ($\omega = 0$) and we therefore write Eq. (8) in the form

$$\varepsilon_B^2(\mathbf{k}) + \varepsilon_B(\mathbf{k})[\Sigma_{11}(\mathbf{k}) + \Sigma_{22}(\mathbf{k})] + [\Sigma_{11}(\mathbf{k})\Sigma_{22}(\mathbf{k}) - \Sigma_{12}(\mathbf{k})\Sigma_{21}(\mathbf{k})] = 0. \quad (16)$$

Here $\Sigma_{ik}(\mathbf{k})$ is the sum of terms of the form

$$i\int \frac{d^4p}{(2\pi)^4} \gamma_\alpha(\mathbf{k}, \mathbf{p}) \Gamma_\beta(\mathbf{k}, \mathbf{p}, \varepsilon) G^0(p) G^0(k-p) \quad (17)$$

(α, β enumerate the indexes a, b, c, d).

The integral (17) consists of a term originating from regions far from the point $\varepsilon = \varepsilon_B(\mathbf{p}_0), |\mathbf{p}| = p_0$ and an integral over the vicinity of that point which makes the main contribution, since $\varepsilon_B(\mathbf{p}_0)$ [the pole of the function $G^0(p)$] vanishes as $n \to n_B^c$. If the external momentum k is small we can take this vicinity sufficiently small and only the contribution from the poles of the Green functions will contribute appreciably to the integral. After integration, (17) is reduced to the form

$$-\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\gamma_\alpha(\mathbf{k}, \mathbf{p}) \Gamma_\beta(\mathbf{k}, \mathbf{p}, \varepsilon_B(\mathbf{p}))}{2\varepsilon_B(\mathbf{p})}$$

$$\approx -\gamma_\alpha(\mathbf{k}, \mathbf{p}_0) \Gamma_\beta(\mathbf{k}, \mathbf{p}_0, \Delta_B)\, m p_0 R, \quad (18)$$

$$R = (1 + {}^3/_{8\pi})/8\pi^2 \approx 0.03,$$

and for the $\Sigma_{ik}(\mathbf{k})$ we get the expressions

$$\Sigma_{11}(\mathbf{k}) = -2mp_0R[9\gamma_a(\mathbf{k}, \mathbf{p}_0)\Gamma_b(\mathbf{k}, \mathbf{p}_0, \Delta_B) + \gamma_d(\mathbf{k}, \mathbf{p}_0)\Gamma_c(\mathbf{k}, \mathbf{p}_0, \Delta_B)],$$

$$\Sigma_{12}(\mathbf{k}) = -6mp_0R[\gamma_b(\mathbf{k}, \mathbf{p}_0)\Gamma_c(\mathbf{k}, \mathbf{p}_0, \Delta_B) + \gamma_c(\mathbf{k}, \mathbf{p}_0)\Gamma_b(\mathbf{k}, \mathbf{p}_0, \Delta_B)],$$

$$\Sigma_{21}(\mathbf{k}) = -6mp_0R[\gamma_d(\mathbf{k}, \mathbf{p}_0)\Gamma_a(\mathbf{k}, \mathbf{p}_0, \Delta_B) + \gamma_a(\mathbf{k}, \mathbf{p}_0)\Gamma_d(\mathbf{k}, \mathbf{p}_0, \Delta_B)], \quad (19)$$

$$\Sigma_{22}(\mathbf{k}) = -2mp_0R[9\gamma_b(\mathbf{k}, \mathbf{p}_0)\Gamma_a(\mathbf{k}, \mathbf{p}_0, \Delta_B) + \gamma_c(\mathbf{k}, \mathbf{p}_0)\Gamma_d(\mathbf{k}, \mathbf{p}_0, \Delta_B)].$$

The vertices γ are taken in the approximation (12).

To find the vertices Γ we need to solve the set of Eqs. (15). After integrating over the frequency (in the pole approximation) we get for Γ_a the equation

$$\Gamma_a(\mathbf{k}, \mathbf{p}_0, \Delta_B) = \gamma_a(\mathbf{k}, \mathbf{p}_0) + \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left\{ \frac{\varepsilon_B(\mathbf{p}) + \varepsilon_B(\mathbf{p}-\mathbf{p}_0)}{\varepsilon_B(\mathbf{p})[(\varepsilon_B(\mathbf{p}) + \varepsilon_B(\mathbf{p}-\mathbf{p}_0))^2 - \Delta_B^2]} [\gamma_c^2(\mathbf{p}, \mathbf{p}_0)\Gamma_a(\mathbf{k}, \mathbf{p}_0, \Delta_B) + \gamma_c(\mathbf{p}, \mathbf{p}_0)\gamma_a(\mathbf{p}, \mathbf{p}_0)\Gamma_d(\mathbf{k}, \mathbf{p}, \Delta_B)] + \frac{\gamma_c(\mathbf{p}, \mathbf{p}_0)\gamma_a(\mathbf{p}, \mathbf{p}_0)\Gamma_c(\mathbf{k}, \mathbf{p}, \Delta_B)}{[\varepsilon_B(\mathbf{p}) + \varepsilon_B(\mathbf{p}-\mathbf{p}_0) - \Delta_B]^2} \right\}. \quad (20)$$

The equations for the other Γ vertices are similar. The main contribution to the integral over $\mathbf{p}$ comes from the region $|\mathbf{p}| \sim p_0$, $|\mathbf{p} - \mathbf{p}_0| \sim p_0$ in which we can use for $\gamma(\mathbf{p}, \mathbf{p}_0)$ the approximate expressions

$$\gamma_a(\mathbf{p}, \mathbf{p}_0) = \gamma_b(\mathbf{p}, \mathbf{p}_0) \approx -\frac{1}{24mn^{1/2}} \left(\frac{\varepsilon^0(\mathbf{p}_0)}{\Delta_B} \right)^{1/2} (p_0^2 + p^2 + \mathbf{p}\mathbf{p}_0),$$

$$\gamma_c(\mathbf{p}, \mathbf{p}_0) = \gamma_d(\mathbf{p}, \mathbf{p}_0) \approx -\frac{1}{8mn^{1/2}} \left(\frac{\varepsilon^0(\mathbf{p}_0)}{\Delta_B} \right)^{1/2} (p_0^2 + p^2 + \mathbf{p}\mathbf{p}_0).$$

As a result (15) reduces to a set of linear algebraic equations

$$A\Gamma = \gamma, \quad (21)$$

where $\tilde{\Gamma}$ is a column with elements $(\Gamma_a, \Gamma_b, \Gamma_c, \Gamma_d)$, $\tilde{\gamma}$ a column with elements $(\gamma_a, \gamma_b, \gamma_c, \gamma_d)$, and A a matrix with elements

$$a_{11} = a_{22} = a_{33} = 1 - (c_1 + c_2)\xi, \quad a_{13} = a_{24} = a_{34} = -{}^1/_3 c_3 \xi,$$

$$a_{14} = a_{23} = a_{32} = -{}^1/_3 (c_1 + c_2)\xi, \quad a_{42} = a_{43} = -(c_1 + {}^1/_3 c_2)\xi,$$

$$a_{44} = 1 - {}^1/_3 c_3 \xi, \quad a_{12} = a_{21} = a_{31} = a_{41} = 0.$$

Here

$$\xi = \frac{\alpha}{64}\frac{n_B^c}{n}\left(\frac{\varepsilon^0(\mathbf{p}_0)}{\Delta_B}\right)^3 \sim \frac{\gamma^2(\mathbf{p}_0)m^2}{p_0} \sim \frac{p_0^3}{n}\left(\frac{\varepsilon^0(\mathbf{p}_0)}{\Delta_B}\right)^3.$$

The coefficients c_1, c_2, c_3 are dimensionless integrals:

$$c_1 = \frac{1}{8\pi^2}\frac{(1+\cos^2\theta)^2}{\cos\theta}\int_0^{2\pi} d\varphi \int_0^1 \frac{dx}{(3+x)[1+x(1+\sin\varphi)]} \approx 0.05,$$

$$c_2 = \frac{1}{8\pi^2}\frac{(1+\cos^2\theta)^2}{\cos\theta}\int_0^{2\pi} d\varphi \int_0^1 \frac{dx}{(1+x)[1+x(1+\sin\varphi)]} \approx 0.13,$$

$$c_3 = \frac{1}{4\pi}\frac{(1+\cos^2\theta)^2}{\cos\theta}\int_0^1 \frac{dx}{(1+x)^2} \approx 0.13,$$

in which $\theta \approx \pi/3$ is the angle between the vectors $\mathbf{p}$ and $\mathbf{p}_0$.

As we indicated earlier, the determinant of the set (21) is the function Δ_B (the Δ_B dependence enters via ξ). It is equal to

$$\Lambda = (1-0.12\xi)(0.02\xi^2 - 0.37\xi + 1). \qquad (22)$$

The smallest value of ξ for which (22) vanishes and the system (21) has no solution equals 3.25. It corresponds to critical values of the Bogolyubov roton minimum and of the density:

$$\Delta_B^c \approx 0.17\alpha^{1/3}\varepsilon^0(\mathbf{p}_0), \qquad n_c \approx n_B^c(1-0.03\alpha^{2/3})$$

and is connected with the appearance of the roton instability. Indeed, for this value of the density all terms in the perturbation theory series for the vertex Γ, in which the expansion is taken with respect to the "small" parameter $\alpha(\varepsilon^0(\mathbf{p}_0)/\Delta_B)^3$, become of the same order of magnitude and the series diverges. The values of Δ_B^c and n_c obtained here are, apart from numerical coefficients, the same as those found in Ref. 2.

Solving the set (21) we evaluate $\Sigma_{ik}(\mathbf{k})$ and we put Eq. (16) in the form

$$1 - \frac{0.22(\Delta_B/\varepsilon^0(\mathbf{p}_0))\xi(1-0.31\xi+0.03\xi^2)}{(1-0.18\xi)\Lambda} = 0. \qquad (23)$$

Its smallest root turns out to be equal to

$$\xi = 3.25(1-\Delta_B/\varepsilon^0(\mathbf{p}_0)), \qquad (24)$$

and the values of the density and the Bogolyubov roton minimum corresponding to it are

$$n_c' = n_B^c[1-0.03\alpha^{2/3}(1+0.14\alpha^{1/3})], \qquad (25)$$

$$\Delta_B^{c\prime} = 0.17\alpha^{2/3}(1+0.07\alpha^{1/3})\varepsilon^0(\mathbf{p}_0). \qquad (26)$$

The vanishing of the sound velocity thus occurs earlier than the solution of the set (21) disappears. In other words, the phonon instability occurs before the roton one although it is caused just by being close to the point of the roton instability. We note that the vanishing of the sound velocity does not yet lead to the appearance of divergences in Σ_{ik} and Γ and, thus, the point where the long-wavelength instability appears can, in fact, be reached which is in complete agreement with remarks made in Ref. 2.

The author expresses his deep gratitude to Yu. A. Nepomnyashchiĭ for suggesting the problem and for discussions of preliminary results and to V. N. Popov for many useful discussions and helpful remarks.

[1]D. A. Kirzhnits and Yu. A. Nepomnyashchiĭ, Zh. Eksp. Teor. Fiz. **59**, 2203 (1970) [Sov. Phys. JETP **32**, 1191 (1971)].
[2]Yu. A. Nepomnyashchiĭ and V. A. Podol'skiĭ, Zh. Eksp. Teor. Fiz. **69**, 555 (1975) [Sov. Phys. JETP **42**, 283 (1975)].
[3]A. K. Rajagopal and G. S. Grest, Phys. Rev. A **10**, 1395 (1974).
[4]A. K. Rajagopal and G. S. Grest, Phys. Rev. A **10**, 1837 (1974).
[5]J. P. Straley, Phys. Rev. A **6**, 498 (1972).
[6]P. Berdahl and D. K. Lee, Phys. Rev. A **7**, 1376 (1973).
[7]S. T. Belyaev, Zh. Eksp. Teor. Fiz. **34**, 417 (1958) [Sov. Phys. JETP **7**, 289 (1958)].

Translated by D. ter Haar.