Unambiguity of matching of the fields of a nonlinear ionizing electromagnetic wave at a discontinuity near a plasma resonance point

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The selection of solutions (discontinuous near the plasma-resonance point) of the locally nonlinear electrodynamic problem of the action of a strong ionizing high-frequency electromagnetic wave is considered. To this end, the locally nonlinear problem is treated from the viewpoint of a more complete but singularly perturbed problem, that of finding the field, the temperature, and the density of the electrons. To select the solutions of the locally nonlinear problem it is proposed to use a criterion from the theory of singularly perturbed equations, namely choose from among all its solutions only the one which is simultaneously a solution of the initial problem. For solutions with jumps whose structure is determined by the equation of heat conduction (through the point of the plasma resonance due to superheating of the plasma, disregarding diffusion), this criterion calls for satisfaction of a certain integral condition that leads to continuity of the heat flux through the jump. This integral equation is expounded in detail for a jump through the point of the plasma resonance that appears in the field of a strong high-frequency ionizing electromagnetic wave.

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1. The locally nonlinear electrodynamic problem (the dielectric constant of the medium is a local function of the field amplitude, $\varepsilon = \varepsilon(|E|^2)$), which describes the behavior of a high-frequency ($\omega \gg \nu$) strong electromagnetic wave with a field component parallel to the electron density in the region containing a plasma-resonance point, is of great practical importance. It is encountered, for example, in the theory of autonomous gas discharges at ultrahigh frequencies, in which the plasma is produced from a neutral gas by the field. Within the framework of this problem, the field of the wave and the plasma parameters determined by the field are discontinuous in the vicinity of the plasma-resonance point.[1,2] Therefore the solution is made up of two continuous solutions defined respectively in regions with positive and negative dielectric constants ε; these solutions are then matched on the boundaries of the discontinuity.[3,4] This procedure, however, does not yield an unambiguous solution in the entire region, since the position of the jump in the volume or, equivalently, the jump of the dielectric constant, is unknown. These jumps depend on the amplitude, phase velocity, or incidence angle of the wave and are themselves the unknowns in the problem. Formally, the situation is that we are short one matching condition on the discontinuity.

Such a situation is typical on the whole of the locally nonlinear electrodynamic problem; it is encountered both for discontinuous solutions with jumps due to plasma superheating[5] and in the case of continuous solutions.[6,5]

Under sufficiently general assumptions, the complete problem of the behavior of an ionizing electromagnetic wave in a plasma is described by the wave equation for the field E and by the balance equations for the temperature T and for the electron density n. Each of these quantities is characterized by a separate inhomogeneity scale: the field by the amplitude inhomogeneity scale L_E, the temperature by the heat-conduction length L_T, and the density by the diffusion length L_n. These scales, depending on the conditions of the problem, can be differently related.

If however, $L_E \gg L_n$ and $L_E \gg L_T$, the higher-order derivatives in the dimensionless temperature and density balance equations (the characteristic scale is chosen to be the largest one, L_E) acquire small parameters ($\Theta = T/T_g$):

$$\mu_T^2 \frac{d}{d\zeta}\left[\varkappa \frac{d\Theta}{d\zeta}\right] = \frac{\delta(\Theta)}{\delta_0}(\Theta-1)n\nu - n\nu\frac{|E|^2}{E_p^2},$$

$$\mu_n^2 \frac{d}{d\zeta}\left[D\left(\frac{dn}{d\zeta}+\frac{n}{2\Theta}\frac{d\Theta}{d\zeta}\right)\right] = n(\rho n - \nu_i(\Theta)), \qquad (1.1)$$

$$\mu_T \equiv L_T/L_E \ll 1, \qquad \mu_n \equiv L_n/L_E \ll 1$$

(κ, D, and ρ are the heat-conduction, ambipolar-diffusion, and recombination coefficients; ν and ν_i are the collision and ionization frequencies; E_p is the "plasma" field, δ is the parameter of energy transfer in collisions; T_g is the gas temperature), and the problem becomes singularly perturbed.[7]

If now, using the smallness of μ_n and μ_T, we neglect in (1.1) the terms with the derivatives, the temperature and density of the electrons, and with them also the dielectric constant $\varepsilon = \varepsilon(|E|^2)$, of the plasma, become local functions of the amplitude and for Maxwell's equations in the zeroth approximation in μ_n and μ_T we obtain a locally nonlinear problem that raises a number of difficulties.

The first difficulty is the appearance of discontinuities in the solution. This difficulty is due to the fact that the locally linear approximation is of large scale (L_E) and is not suitable for the behavior of the fields of n and T in the regions where they change strongly (over scales L_n and L_T). These regions are called boundary layers and can appear both on the boundaries and in the interior.[7] The discontinuity regions in the vicinity of the plasma resonance, or discontinuities due to superheating, are examples of boundary layers inside a volume.

The second difficulty is that the solutions of the locally nonlinear electrodynamic problem are multiply-valued. The reason is that by discarding the derivatives in (1.1) we replace the initial problem by another simpler but not equivalent one. Consequently, both the character and the sets of solutions of the initial singularly perturbed problem (1.1) and the simplified locally nonlinear problem can be entirely different. Therefore a special principle of selecting the solutions of locally nonlinear problems (called matched in the theory) has been formulated in the theory of singularly perturbed equations. Its gist is to choose from all the solutions $u(z)$ of the locally nonlinear problem only the one which is simulataneously also the solution $u(z;\mu)$ of the initial problem as the small parameter μ tends to zero and is written in the form[7]

$$\lim_{\mu\to 0} u(z;\mu)=\bar{u}(z). \tag{1.2}$$

This is the analog of the usual well known test and selection of solutions for algebraic equations.

It is this selection principle which is proposed here for the construction of a single-valued solution of the locally nonlinear electrodynamic problem in various situations.

The general formulation of the criterion (1.2) is simple and is the same for all cases; its detailed meaning and consequences, however, depend on the type of equation, on the character of the nonlinearity, and on the boundary conditions, and must be expounded separately in each concrete case. In the upshot, the criterion (1.2) spells out the conditions of the problem for the initial equations (1.1), and the solution of the problem can be constructed with the aid of the locally nonlinear approximation. Under other conditions, when (1.2) is not satisfied, the solutions of the locally nonlinear problem do not include solutions of the initial problem (1.1). The solution of the initial singularly perturbed problem must then be constructed directly.

2. The condition (1.2) touches mainly on the structure of the boundary layers, where the solutions of the problem (1.1) and of the locally nonlinear problem can differ greatly. The characteristics of the "large scale" locally nonlinear problem are then contained in this condition as parameters on which its satisfaction depends.

The structure of the jump in the field of an ionizing electromagnetic wave is determined in the general case by the balance equations for n and T. The ratio of the lengths L_n and L_T governs both the thickness of the discontinuity and the possibility of its existence. At $L_T \gg L_n$, the case typical of low gas pressure when the consition $\omega \gg \nu$ is satisfied and considered below, the electron density is a local function of the temperature:

$$n=\nu_i(\Theta)/\rho,\qquad \nu_i(\Theta)=\nu_i^0\exp[-U/T_g\Theta],\quad (U/T_g\Theta\gg 1) \tag{2.1}$$

(U is the ionization potential), and the structure of the jump in the vicinity of the plasma-resonance point is determined completely by the heat-condition equation

$$\frac{d}{d\tau}\left[\Theta^{1/2}n\frac{d\Theta}{d\tau}\right]=n(\Theta)\Theta^{1/2}F(\Theta;E_\parallel;D),\quad \tau=\frac{z}{L_T}, \tag{2.2}$$

$$F(\Theta;E_\parallel;D)=\frac{\delta(\Theta)}{\delta_0}(\Theta-1)-E_\parallel^2-\frac{D^2}{[1-n(\Theta)/n_s]^2+\nu_0^2\Theta/\omega^2}, \tag{2.3}$$

$$\varkappa(\Theta)=\varkappa_0\Theta^{1/2}n(\Theta),\quad \nu=\nu_0\Theta^{1/2},\quad \delta_0=2m/M. \tag{2.4}$$

Here $E_\parallel$ and D are respectively the moduli, normalized to E_p, of the electric-field component tangential to the surface of the jump and of the normal component of the electric-induction vector. They are assumed constant in the region of the jump, since they vary over a scale $L_E \gg l$, where $l \sim L_T$ is the characteristic thickness of the jump.

The salient feature of the nonlinearity of Eq. (2.2) is that its right-hand side vanishes in the vicinity of the plasma-resonance point ($F=0$) at three values of the temperature: Θ_+, Θ_*, and Θ_-. A similar triple-valuedness is observed also at a constant electron density in cases when the parameter δ of energy transfer in collisions, or the collision frequency, decreases with increasing temperature (the region of the Ramsauer effect and of collision with ions).[8]

Mathematically rigorous solutions of boundary-value problems with jumps in the volume, for a singularly perturbed second-order equation with this type of nonlinearity, is given in Refs. 9 and 7. It turns out that the condition (1.2) will be satisfied if (in terms of the considered problem):

a) the temperatures Θ_+ and Θ_- are singular saddle points of Eq. (2.2), separated by the "center" Θ_*;

b) the fields $E_\parallel$ and D are connected by the condition

$$\int_{\Theta_*}^{\Theta_+} n(\Theta)\varkappa(\Theta)\nu(\Theta)F(\Theta;E_\parallel,D)\,d\Theta = \int_{\Theta_*}^{\Theta_-} n(\Theta)\varkappa(\Theta)\nu(\Theta)F(\Theta;E_\parallel,D)\,d\Theta. \tag{2.5}$$

Equation (2.2) has in this case in the vicinity of the jump a solution of the "continuous step" type, describing a transition between two states with temperatures Θ_+ and Θ_-. We note that the possible types of solutions of the nonlinear heat-conduction equation, including those of the "step" type, were considered in Ref. 8. A condition of the type (2.5) for a inhomogeneous-plasma jump connected with the "thermal" nonlinearity due to the crowding out of the plasma from the region of the strong longitudinal ($E_\parallel=0$) field because of electron heating, was suggested in Ref. 10 in an analysis of the analogy with phase transitions.

The physical meaning of the condition (2.5) can be seen from Eq. (2.2). Multiplying this equation by $\kappa d\Theta/d\tau$ and integrating, we arrive under condition (2.5) to a conservation law for the heat flux through the jump:

$$\varkappa\frac{d\Theta}{dz}\bigg|_{\Theta_+}=\varkappa\frac{d\Theta}{dz}\bigg|_{\Theta_-}. \tag{2.6}$$

According to (2.3), the function F has the meaning of the difference between the amounts of heat released by the field and absorbed in the volume by collision; therefore under conditions (2.5) the excess heat released by the field in the region of a jump with a temperature from Θ_* to Θ_- should be absorbed in collisions in the other part of the jump with temperature from Θ_+ to Θ_*. The conservation law (2.6) holds here, too. Thus, the electron-density and temperature jumps behave in analogy with the surface that separates two media. In contrast to the latter, however, the jump has a finite thickness and therefore condition (2.6) can be satisfied only for a definite heat-release law and for a definite temperature distribution in it.

Condition (2.5) solves the problem of single-valued matching of the fields on the jump through the plasma-resonance point only in principle, but cannot be used directly in the scheme of the locally nonlinear problem, since it is integral in character. In addition, the dependences of Θ_+, Θ_*, and Θ_- on $E_\|$ and D are unknown; nor is the character of the connection between $E_\|$ and D known. The exposition that follows is devoted to details of this character for the case of a high-frequency ionizing wave.

3. The singular points of Eq. (2.3), which determine the plasma parameters in the local-coupling approximation, are the roots of the equation

$$\frac{\delta(\Theta)}{\delta_0}(\Theta-1)-E_\|^2=\frac{D^2}{[1-n(\Theta)/n_c]^2+\nu_0^2\Theta/\omega^2}, \qquad (3.1)$$

where $n(\Theta)$ is defined by (2.1) and $n_c = m\omega^2/4\pi e^2$ is the critical density. The right-hand side of (3.1) has the meaning of the square of the amplitude of the electric-field component, $E_\perp^2 = D^2/|\varepsilon|^2$, directed along the inhomogeneity, and as a function of temperature it has the form of a narrow

$$\frac{\Delta\Theta}{\Theta_c}\sim 2\frac{\nu_c}{\omega}\frac{T_g\Theta_c}{U}\ll 1$$

(Θ_c is the temperature of the plasma-resonance point) and resonant line (see Fig. 1, curve 1), whose maximum is shifted insignificantly, by

$$\frac{\Delta n}{n_c}\sim\frac{\nu_c^2}{2\omega^2}\frac{T_g\Theta_c}{U}\ll 1,$$

into the transparency region ($n < n_c$) of the plasma and its height is $D^2\omega^2/\nu_c^2$, where $\nu_c = \nu_0\Theta_c^{1/2}$ is the collision frequency at the plasma-resonance point.

At $\delta(\Theta) = \delta_0$, the case that will be considered first, the plot of the left-hand side of (3.1) is a straight line (curve 2 of Fig. 1) that shifts to the right (towards larger Θ) parallel to itself with increasing $E_\|$.

Independently of the value of $D^2\omega^2/\nu_c^2$, at $E_\|^2 \geqslant \Theta_c - 1$ there exists only one solution of (3.1), corresponding to a superficial state of the plasma. At $E_\|^2 < \Theta_c - 1$ the structure

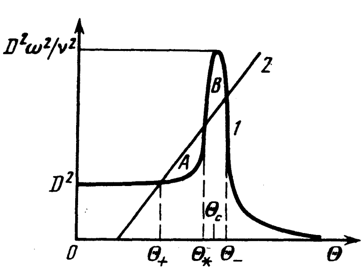

FIG. 1. Illustrating the graphic solution of equation (3.1): 1—$E_\perp^2$; 2—$(\Theta-1-E_\|^2)$

of the solution of (3.1) becomes already dependent on the ratio of $E_\|$ and $D\omega/\nu_c$. If the straight line 2 on Fig. 1 passes above the maximum of curve 1,

$$\Theta_c-1-E_\|^2>D^2\omega^2/\nu_c^2, \qquad (3.2)$$

or to the right of the left corner of the resonance line,

$$\Theta_c-1-E_\|^2<D^2+\frac{\nu_c}{\omega}\frac{T_g\Theta_c}{U},$$

Eq. (3.1) also has one solution corresponding to the subcritical or supercritical value of the electron density.

In the intermediate field range, curves 1 and 2 intersect at three points, to which correspond two subcritical states of the plasma, namely (Θ_+, n_+) and (Θ_*, n_*), and one supercritical state (Θ_-, n_-). The first to point out that a plasma in a longitudinal ($E_\| = 0$) field can have three values in the vicinity of the plasma-resonance point were Gurevich and Pitaevskiĭ.[1]

If we neglect the small difference between the half-width of the resonance line at $\Theta > \Theta_c$ and $\Theta < \Theta_c$, due to the temperature dependence of the collision frequency, then the roots Θ_* and Θ_- will by symmetric about Θ_c.

The curve intersection points that determine the roots Θ_* and Θ_- move with changing $E_\|$ along almost vertical lines, so that the values of Θ_* and Θ_- should not be very sensitive to the slope of the straight line 2. Recognizing furthermore, that the left-hand side of (3.1) changes little over the width of the resonance line, we obtain

$$n_*=n_c(1-\gamma),\quad n_-=n_c(1+\gamma),$$
$$\gamma=\left[\frac{D^2}{\Theta_c-1-E_\|^2}-\frac{\nu_c^2}{\omega^2}\right]^{1/2}. \qquad (3.3)$$

In this case $\gamma < 1$ and curves 1 and 2 cross below the maximum (the inequality (3.2) does not hold); for γ to be positive, on the other hand, we must have

$$D^2>\frac{\nu_c^2}{\omega^2}(\Theta_c-1-E_\|^2). \qquad (3.4)$$

The concentration n_* and n_- correspond to the temperatures

$$\frac{\Theta_*}{\Theta_c}=\left[1-\frac{T_g\Theta_c}{U}\ln(1-\gamma)\right]^{-1},$$
$$\frac{\Theta_-}{\Theta_c}=\left[1-\frac{T_g\Theta_c}{U}\ln(1+\gamma)\right]^{-1}. \qquad (3.5)$$

The value of the temperature Θ_+ is determined by the intersection of the line 2 with the left-hand gently sloping branch of curve 1 in a region where the electron density is substantially lower than critical (by virtue of the exponential dependence), and the collision frequency can be neglected:

$$\Theta_+\approx 1+E_\|^2+D^2+2D^2n_+/n_c, \qquad (3.6)$$

where $n_+ = n(\Theta_+)$ and $n_+ \ll n_c$. In Eqs. (3.3)–(3.6) the case of coalescence of the roots Θ_*, Θ_- or Θ_*, Θ_+ is excluded from consideration, for in this case of the areas, A or B, in Fig. 1 becomes equal to zero, a fact not allowed by the condition (2.5).

We now write down the condition (2.5) in greater detail:

$$\int_{\Theta_*}^{\Theta_+}\Theta n^2(\Theta)\left\{\frac{D^2}{(1-n/n_c)^2+\nu_0^2\Theta/\omega^2}-(\Theta-1-E_\parallel^2)\right\}d\Theta$$

$$=\int_{\Theta_*}^{\Theta_-}\Theta n^2(\Theta)\left\{\frac{D^2}{(1-n/n_c)^2+\nu_0^2\Theta/\omega^2}-(\Theta-1-E_\parallel^2)\right\}d\Theta. \qquad (3.7)$$

The integrands here are quite unwieldy, but they contain the strongly varying function $n(\Theta)$. Changing over to integration with respect to the concentration and replacing the slowly varying functions by their averages over the considered interval, we arrive at the following transcendental equation in γ from (3.3):

$$\Theta_*^3\left\{\left(\gamma^2+\frac{\nu_*^2}{\omega^2}\right)\times\left[\frac{\operatorname{arctg}[(1-n/n_c)(\omega/\nu_*)]-\operatorname{arctg}[(1-n_+/n_c)(\omega/\nu_*)]}{\nu_*/\omega}+\frac{1}{2}\ln\frac{\Theta_+-1-E_\parallel^2}{\Theta_*-1-E_\parallel^2}\right]+\frac{\Theta_*-1-E_\parallel^2}{\Theta_c-1-E_\parallel^2}\,\frac{n_*^2-n_+^2}{n_c^2}\right\}$$

$$=\Theta_c^3\left\{\left(\gamma^2+\frac{\nu_c^2}{\omega^2}\right)\cdot\times\left[\frac{\operatorname{arctg}[(1-n_*/n_c)(\omega/\nu_c)]+\operatorname{arctg}[(n_-/n_c-1)(\omega/\nu_c)]}{\nu_c/\omega}+\frac{1}{2}\ln\frac{\Theta_--1-E_\parallel^2}{\Theta_*-1-E_\parallel^2}\right]+\frac{n_*^2-n_-^2}{n^2}\right\}, \qquad (3.8)$$

where $\nu_* = \nu_0\Theta_*^{1/2}$ and $\nu_c = \nu_0\Theta_c^{1/2}$. This equation is also quite complicated, but it contains the small parameter ν/ω, so that its analytic solution can ultimately be obtained.

To determine the dependence of γ on ν/ω we use the following qualitative considerations. The left-hand side of (3.8) is connected with integration over the area A (Fig. 1) and will remain finite in the limit as $\nu/\omega\to0$. The right-hand side of (3.8) is connected with integration over the area B, which is equal to

$$S_B\sim\left[D^2\frac{\omega^2}{\nu_c^2}-(\Theta_c-1-E_\parallel^2)\right]\nu_c/\omega.$$

Equation (3.8) can therefore have a solution only if in the limit as $\nu/\omega\to0$ the area S_B of the figure remains constant, i.e.,

$$\lim_{\nu/\omega\to0}S_B\approx D^2\frac{\omega}{\nu_c}\approx S_0.$$

This is possible if $D^2\sim\nu_c/\omega$ and consequently

$$\gamma=\left[\frac{D^2}{\Theta_c-1-E_\parallel^2}-\frac{\nu_c^2}{\omega^2}\right]^{1/2}\sim\left(\frac{\nu_c}{\omega}\right)^{1/2}.$$

Taking these considerations into account, we seek a solution of (3.8) in the form

$$\gamma=\gamma_0\left(\frac{\nu_c}{\omega}\right)^{1/2}+\left(\frac{\nu_c}{\omega}\right)\gamma_1+\dots, \qquad (3.9)$$

where γ_i are quantities as yet unknown. After substitution in (3.8) and simplification we obtain in the zeroth approximation in $(\nu_c/\omega)^{1/2}$

$$\gamma_0^2\pi=1-n_+^2/n_c^2,$$

whence

$$\gamma_0\approx\left(\frac{1}{\pi}\right)^{1/2}\left(1-\frac{n_+^2}{2n_c^2}\right)\approx\frac{1}{\pi^{1/2}}=0.564, \qquad (3.10)$$

since $n_+^2\ll n_c^2$. With the aid of (3.3) and (3.5) we arrive at the following expressions for the plasma parameters in the supercritical state:

$$|\varepsilon_-|=n_-/n_c-1\approx0.564(\nu_c/\omega)^{1/2}, \qquad (3.11)$$

$$\Theta_-=\Theta_c\left[1+\frac{T_g\Theta_c}{U}\left(\frac{\nu_c}{\omega}\right)^{1/2}\gamma_0\right]. \qquad (3.12)$$

From (3.3) we obtain at the same time also the connection between the field components $E_\parallel$ and D (in dimensional units):

$$\frac{D^2}{E_p^2}=\frac{\nu_c}{\omega}\gamma_0^2\left(\Theta_c-1-\frac{E_\parallel^2}{E_p^2}\right). \qquad (3.13)$$

This condition, or its equivalent (3.11), is in fact the additional condition that leads, together with the continuity conditions for the tangential field components on the jump, to unambiguity of the solution of the locally nonlinear problem. The use of condition (3.11) is in this case more convenient, since usually the conditions of field coupling are formulated in terms of the dielectric constant.

It can be seen from (3.11)–(3.13) that the supercritical plasma parameters and the connection between the field components $E_\parallel$ and D are determined by the frequency of the collisions at the plasma-resonance point. The reason is that a change in plasma temperature from Θ_+ to Θ_- in the vicinity of the plasma-resonance point is due to plasma heating by the "resonant" heat, whose magnitude is regulated by the collision frequency. As already noted, the condition (2.5) requires in this case that the excess of the resonant heat released in the hot region (Θ_*, Θ_-) of the jump be absorbed by collisions with the neutrals in cold (Θ_+, Θ_*) region. In this case the heat flux through the jump will be conserved. If the condition (2.5) is not satisfied, the resonant heat will either be insufficient, or will be so plentiful that large plasma volumes will be heated by it. Under these conditions the locally nonlinear approximation can no longer be used to construct the solution in a region in which the dielectric constant reverses sign. At least one of the regions ($\varepsilon>0$ or $\varepsilon<0$) will in this case be subject to a strong influence of the thermal conductivity, and the initial singularly perturbed problem should be directly solvable in it.

The positive boundary of the jump of the dielectric constant, ε_+, is determined from the conditions for the matching of the tangential field components on the jump. In this case the conditions must be formulated so as to exclude from consideration that branch of the solution which corresponds to the root Θ_*, since the "center" cannot be reached from the "saddle" Θ_+. Together with the condition $\varepsilon_+\leqslant\varepsilon_0$ (ε_0 is the dielectric constant of the unperturbed plasma) this limits the range of variation of the parameter ν_c/ω and in the upshot limits the gas pressure. For example, for a surface wave ν_c/ω can vary in the range

$$0.261 \geqslant \nu_c/\omega \geqslant 0.03,$$

which corresponds, for a discharge in argon at a frequency $\omega = 6\times 10^{10}\ \mathrm{sec}^{-1}$, to gas pressures

$$5.22 \geqslant p(\mathrm{Torr}) \geqslant 0.6.$$

To each given value of ν_c/ω or p there corresponds a single solution of the locally nonlinear electrodynamic problem. The preceding results were obtained under the assumption that the energy-transfer parameter in (3.1) is $\delta = 2m/M$. If δ depends on temperature, the line 2 in Fig. 1 is no longer straight. Recognizing, however, that the supercritical plasma parameters allowed by the condition (2.5) depend little on the positions of the intersection points of curves 1 and 2, it can be assumed that the conditions (3.11) and (3.13) remain in force in this case, too. The "plasma" field, however, must be renormalized by substituting $\delta(\Theta_c)$ for δ_0.

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