Neutron diffusion and heat transfer in the crusts of neutron stars

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The coefficients that determine the neutron heat transfer and diffusion in the crusts of neutron stars are calculated on the basis of a solution of the Boltzmann equation with allowance for degeneracy. Exact expressions are obtained for the cases of strong and weak degeneracy together with interpolation formulas valid for all values of the degeneracy parameter.

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1. INTRODUCTION

The processes that take place in the solid crusts of neutron stars are very important for their observational manifestations. The crusts can crack, this providing an explanation for the observed glitches in pulsar periods.[1] When a neutron star is formed, a nonequilibrium layer can arise in its crust, this layer consisting of superheavy ($A\approx 300$–400) nuclei which form a crystal lattice and free neutrons moving between them.[2,3] The matter density in the region of the nonequilibrium layer is in the range $\sim 10^{10} < \rho < \sim 10^{12}$ g/cm^3.

The neutrons in the nonequilibrium layer form an almost ideal degenerate Fermi gas. Because of the strong gravity force, the neutrons diffuse into the star. To investigate the diffusion, it is necessary to know the neutron transfer coefficients in the crystal lattice formed by the nuclei. The present paper is devoted to calculation of the neutron diffusion and thermal conductivity coefficients under the given conditions on the basis of solution of the Boltzmann equation.

Investigation of the diffusion process on the basis of a rough estimate for the diffusion coefficient[4] showed that diffusion can result in stretching of the crust and, possibly, the formation of cracks and fractures. Starquakes in the presence of an equilibrium layer can result in nonequilibrium matter being carried to the surface of a neutron star and to a rapid nuclear explosion. Such explosions are invoked to explain the observed cosmic bursts of γ radiation.[5–7] The use of the exact value of the diffusion coefficient calculated in the present paper will make it possible to draw more definite conclusions concerning the processes that take place in the crusts of neutron stars. The expressions we obtain can be used to describe transport properties in other objects containing free neutrons if their dimensions appreciably exceed the neutron mean free path.

To find the diffusion coefficients, we solve the Boltzmann equation with allowance for degeneracy using the methods developed in Refs. 8–11, in which the transport coefficients of a simple gas are calculated. In the present paper, we consider a neutron gas in the crystal lattice of heavy nuclei. We take into account the interaction of the neutrons with the nondegenerate nuclei and also with one another. We obtain exact expressions for the cases of weak and strong degeneracy (as in Refs. 10 and 11, respectively) and interpolation formulas for the diffusion and thermal conductivity coefficients for arbitrary value of the degeneracy parameter.

2. BOLTZMANN EQUATION AND TRANSFER EQUATIONS

To calculate the transport coefficients in the present paper, we use a Boltzmann equation with allowance for degeneracy[8–11] but only two-body collisions. This equation is valid for the neutrons if the neutron gas is nearly ideal. At the same time, the neutron interaction energy must be much less than the kinetic energy of the random motion. For nuclear forces, these energies become approximately equal at a nuclear density of $\rho\approx 2\times 10^{14}$ g/cm^3. Because of the short range of the nuclear forces, a neutron gas rapidly becomes ideal when the density decreases at any temperature, this being due to the Fermi energy of the degenerate neutrons. The densities in the nonequilibrium layers are between two and four orders of magnitude less than the nuclear density, and therefore the Boltzmann equation applies well.

The nuclear component of the matter in the nonequilibrium layer is evidently in the crystal state, and therefore the isotropic part of the distribution function f_{N0} may differ from the Maxwellian distribution. However, if the mass m_N of a nucleus is much greater than the neutron mass m_n, then to terms $\sim m_n/m_N$ the details of the distribution function f_{N0} of the nuclei are unimportant, and the calculation can be made for effectively arbitrary f_{N0}. A magnetic field could lead to anisotropy of the nuclear distribution function, which would affect the transport properties. In an anisotropic medium, diffusion and heat transfer are determined by second-rank tensors rather than scalar coefficients. Because of the large mass of the nuclei, and also the high density and temperature of the matter in the nonequilibrium layers, the anisotropy in the distribution of the nuclei is slight up to magnetic fields of 10^{16} G if the density ρ exceeds 10^{10} g/cm^3 and the temperature is $T\gtrsim 10^7$ °K. Then the Larmor frequency of the nuclei is much less than the lattice vibration frequency, and the Larmor frequency quantum energy is much less than the mean thermal energy of the nuclei.

The neutron-neutron interaction must be taken into account because the neutron component can make an appreciable contribution, up to $\sim 50\%$, to the matter density in the nonequilibrium layers. For an estimate it can be assumed, as for elastic spheres, that the neutron-nucleus interaction cross section σ_{Nn} is $\sim A^{2/3}$ times greater than the neu-

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tron-neutron cross section σ_{nn}. At neutron density ρ_n, the number density n_N of the nuclei is $n_N = (\rho - \rho_n)/Am_p$, and then allowance for the neutron-neutron interactions becomes necessary when $\sigma_{nn} n_n / \sigma_{nN} n_N > 1$, which corresponds to the inequality $A^{1/3}\rho_n/(\rho - \rho_n) > 1$. For $A = 300$–400, it is already necessary to take into account the neutron-neutron interactions when $\rho_n/\rho \gtrsim 0.1$.

In what follows, we shall restrict ourselves to a Boltzmann collision integral with the cross section of elastic collisions. Under the high-density conditions in the nonequilibrium layers, the superheavy nuclei are stable. They do not break up as a result of interaction with thermal or even fast neutrons, and beta decay is forbidden because of the high energy of the Fermi electrons.[2,3] Under these conditions, the only inelastic interaction of a thermal neutron with a nucleus is a process of "charge exchange" type, consisting of exchange between a free and a bound neutron. Because the binding energy of the last neutron in the nuclei in the nonequilibrium layers is near zero, the inelasticity of this process, i.e., the fraction of kinetic energy transformed into thermal energy, is small, and the charge exchange process can be regarded formally as electric and its cross section taken into account as a term in the total elastic scattering cross section σ_{nN}.

The transfer equations for the neutron number and energy in the two-component mixture of nuclei and neutrons, and also for the total momentum can be obtained in the usual manner[8] from the Boltzmann equation for Fermi particles.[8–11]

$$\frac{dn_n}{dt} + n_n \frac{\partial c_{0i}}{\partial r_i} + \frac{\partial}{\partial r_i}(n_n\langle v_i\rangle) = 0, \tag{1}$$

$$\frac{3}{2} n_n \frac{d}{dt}\left(\frac{P_n}{\rho_n}\right) - \frac{3}{2}\frac{P_n}{\rho_n}\frac{\partial}{\partial r_i}(n_n\langle v_i\rangle) + \frac{\partial q_i^{(n)}}{\partial r_i} + \Pi_{ik}^{(n)}\frac{\partial c_{0i}}{\partial r_k} + \rho_n\langle v_i\rangle\left(\frac{dc_{0i}}{dt} - f_i\right) = 0, \tag{2}$$

$$\rho\frac{dc_{0i}}{dt} + \frac{\partial \Pi_{ik}}{\partial r_k} - \rho f_i = 0. \tag{3}$$

Here

$$\Pi_{ik}^{(n)} = n_n m_n \langle v_i v_k\rangle, \quad \Pi_{ik} = \Pi_{ik}^{(n)} + \Pi_{ik}^{(N)},$$
$$q_i = \frac{1}{2} n_n m_n \langle v^2 v_i\rangle, \quad P_n = \frac{1}{3} n_n m_n \langle v^2\rangle, \tag{4}$$
$$\frac{d}{dt} = \frac{\partial}{\partial t} + c_{0i}\frac{\partial}{\partial r_i},$$

f_i is the acceleration produced by the external forces, c_{0i} is the mass-average velocity, v_i is the thermal velocity of the neutrons, and the quantities in the angular brackets are calculated by averaging over the neutron distribution function f. The neutron diffusion velocity $\langle v_i\rangle$ and the heat flux $\langle q_i^{(n)}\rangle$ are related to the thermodynamic parameters and their derivatives by means of the transport coefficients, which are found by solving the Boltzmann equation.

The crystal formed by the heavy nuclei in the crust of a neutron star differs from an ordinary crystal in that the elastic energy of unit volume is appreciably less than the kinetic energy of the degenerate electrons. The presence of the crystal may sometimes be important in the equation of motion, which contains elastic forces that depend on the displacements. In the present paper, we calculate only the heat and neutron number transport coefficients, which depend weakly on the physical state of the heavy nuclei. We ignore the interaction of the neutrons with the electrons, and in the equation of motion we include the electron pressure P_e in the nuclear pressure P_N.

3. DERIVATION OF GENERAL EXPRESSIONS FOR THE TRANSPORT COEFFICIENTS

The Boltzmann equation can be solved by the Chapman-Enskog method of successive approximation.[8–11] The zeroth approximation to the neutron distribution function f is found by equating to zero the collision integral:

$$f_0 = \{1 + \exp[(m_n v^2 - 2\mu)/2kT]\}^{-1}, \quad B\int f_0 dv_i = n_n. \tag{5}$$

Here, μ is the chemical potential of the neutrons, k is Boltzmann's constant, T is the temperature, and $B = 2m_n^3/(2\pi\hbar)^3$.

The nuclear distribution function in the zeroth approximation, f_{N0}, is assumed to be isotropic with respect to the velocities and to depend on the local thermodynamic parameters; otherwise, it can be arbitrary with normalization

$$\int f_{N0}\, dc_{Ni} = n_N, \quad c_{Ni} = v_{Ni} + c_{0i}.$$

Using (5) in (1)–(4), we obtain the zeroth approximation for the transfer equations. In this approximation $\langle v_i\rangle = 0$, $q_i = 0$, $\Pi_{ik} = (P_n + P_N)\delta_{ik}$ (the electron pressure is included in P_N),

$$n_n = 2\left(\frac{kTm_n}{2\pi\hbar^2}\right)^{3/2} G_{3/2}(x_0), \quad P_n = 2kT\left(\frac{kTm_n}{2\pi\hbar^2}\right)^{3/2} G_{5/2}(x_0),$$
$$G_n(x_0) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}dx}{1+\exp(x - x_0)}, \quad x_0 = \mu/kT. \tag{6}$$

In what follows, instead of $G_n(x_0)$ we will everywhere write G_n, since the argument is the same.

In the first approximation, we seek the function f in the form

$$f = f_0[1 + \chi(1 - f_0)]. \tag{7}$$

We take the deviation of the nuclear distribution function from the zeroth approximation in the form

$$f_N = f_{N0}(1 + \chi_N). \tag{8}$$

We obtain the equation for χ in the standard manner.[8–11] It is linear and admits representation of the solution in the form

$$\chi = -A_i\frac{\partial \ln T}{\partial r_i} - n_n D_i d_i \frac{G_{5/2}}{G_{3/2}} - G_{ik}\frac{\partial c_{0i}}{\partial r_k}. \tag{9}$$

At the same time,

$$\chi_N = -A_{Ni}\frac{\partial \ln T}{\partial r_i} - n_n D_{Ni} d_i \frac{G_{5/2}}{G_{3/2}} - G_{Nik}\frac{\partial c_{0i}}{\partial r_k},$$
$$d_i = \frac{\rho_N}{\rho}\frac{\partial \ln P_n}{\partial r_i} - \frac{\rho_n}{P_n}\frac{1}{\rho}\frac{\partial P_N}{\partial r_i}. \tag{10}$$

The functions A_i, D_i, A_{Ni}, and D_{Ni} determine the diffusion and heat transfer. Substituting (9) and (10) in the equation for χ, we obtain the equations for A_i and D_i:

$$f_0(1-f_0)\left(\frac{m_n v^2}{2kT}-\frac{5}{2}\frac{G_{5/2}}{G_{3/2}}\right)v_i=I_{nn}(A_i)+I_{nN}(A_i),$$
$$\frac{1}{n_n}f_0(1-f_0)v_i=I_{nn}(D_i)+I_{nN}(D_i), \tag{11}$$

where $I_{nn}(R)$ and $I_{nN}(R)$ are the linearized collision integrals defined in Refs. 8–11. We see the solution of Eqs. (11) in the form of an expansion in polynomials $Q_n(x)$ that are orthogonal with weight $f_0(1-f_0)x^{3/2}$. These polynomials are analogous to Sonin polynomials (see Refs. 8 and 12). We use only the first two polynomials:

$$Q_0(x)=1,\quad Q_1(x)=\frac{5}{2}\frac{G_{5/2}}{G_{3/2}}-x,\quad x=u^2,\quad u_i=\left(\frac{m_n}{2kT}\right)^{1/2}v_i. \tag{12}$$

We seek A_i and D_i in the form

$$A_i=(a_0Q_0+a_1Q_1)v_i,\quad D_i=(d_0Q_0+d_1Q_1)v_i. \tag{13}$$

In A_{N1} and D_{Ni}, we take into account only the first terms of the expansion, which are determined from the condition that the corrections to the equilibrium distribution functions do not contribute to the mass-average velocity. We have

$$A_{Ni}=a_{0N}Q_0v_{Ni},\quad D_{Ni}=d_{0N}Q_0v_{Ni}, \tag{14}$$
$$n_na_0+n_Na_{0N}=0,\quad n_nd_0+n_Nd_{0N}=0.$$

Using (9), (10), (13), and (14) in (7) and (8) in the calculation of the mean values, we obtain

$$\langle v_i\rangle=-\frac{kT}{m_n}\left(a_0\frac{\partial\ln T}{\partial r_i}+d_0n_n\frac{G_{5/2}}{G_{3/2}}d_i\right),$$
$$\langle v_{Ni}\rangle=-\frac{kT}{m_N}\left(a_{0N}\frac{\partial\ln T}{\partial r_i}+d_{0N}n_n\frac{G_{5/2}}{G_{3/2}}d_i\right)=-\frac{\rho_n}{\rho_N}\langle v_i\rangle. \tag{15}$$

The distribution function f_{N0} is here assumed to satisfy the relation

$$\frac{1}{n_N}\int v_{Ni}v_{Nk}f_{N0}dc_{Ni}=\delta_{ik}\frac{kT}{m_N}$$

which is analogous to the Maxwell function. Using (14), we finally obtain

$$\langle v_i\rangle-\langle v_{Ni}\rangle=-\frac{kT}{m_n}\frac{\rho}{\rho_N}\left(a_0\frac{\partial\ln T}{\partial r_i}+d_0n_n\frac{G_{5/2}}{G_{3/2}}d_i\right). \tag{16}$$

Using (4) and (13), we obtain an expression for the heat flux transported by the neutrons:

$$q_{ni}=-\frac{5}{2}m_nn_n\left(\frac{kT}{m_n}\right)^2\frac{G_{5/2}}{G_{3/2}}$$
$$\times\left\{\left[a_0-a_1\left(\frac{7}{2}\frac{G_{7/2}}{G_{5/2}}-\frac{5}{2}\frac{G_{5/2}}{G_{3/2}}\right)\right]\frac{\partial\ln T}{\partial r_i}\right.$$
$$\left.+n_n\left[d_0-d_1\left(\frac{7}{2}\frac{G_{7/2}}{G_{5/2}}-\frac{5}{2}\frac{G_{5/2}}{G_{3/2}}\right)\right]\frac{G_{5/2}}{G_{3/2}}d_i\right\}. \tag{17}$$

4. FINDING OF THE MATRIX ELEMENTS a_{ij} AND b_{ij}

Multiplying Eqs. (11) by $BQ_0(x)u_i$ and $BQ_1(x)u_i$ and integrating with respect to dc_i ($c_i=c_{0i}+v_i$), we obtain a system of equations for the coefficients:

$$0=a_0(a_{00}+b_{00})+a_1(a_{01}+b_{01}),$$
$$-\frac{15}{4}n_n\left(\frac{7}{2}\frac{G_{7/2}}{G_{3/2}}-\frac{5}{2}\frac{G_{5/2}^2}{G_{3/2}^2}\right)=a_0(a_{10}+b_{10})+a_1(a_{11}+b_{11}); \tag{18}$$
$$\frac{3}{2}=d_0(a_{00}+b_{00})+d_1(a_{01}+b_{01}),$$
$$0=d_0(a_{10}+b_{10})+d_1(a_{11}+b_{11}); \tag{19}$$

$$a_{jk}=B^2\int f_0f_{01}(1-f_0')(1-f_{01}')Q_j(u^2)u_i[Q_k(u^2)u_i+Q_k(u_1^2)u_{1i}$$
$$-Q_k(u'^2)u_i'-Q_k(u_1'^2)u_{1i}']g_{nn}W_{nn}(\theta,g_{nn})\,d\Omega dc_{1i}dc_i, \tag{20}$$

$$b_{jk}=B\int f_0f_{N0}(1-f_0')Q_j(u^2)u_i[Q_k(u^2)u_i-Q_k(u'^2)u_i']$$
$$\times g_{nN}W_{nN}(\theta,g_{nN})\cdot d\Omega dc_{Ni}dc_i,\quad k\geqslant1. \tag{21}$$

Here, $W(\theta,g)d\Omega$ is the effective differential cross section for scattering of particles with relative velocity g that is deflected through angle θ and after the collision lies in the solid angle $d\Omega$. Using (14), we obtain

$$b_{j0}=B\int f_0f_{N0}(1-f_0')Q_j(u^2)u_i\Bigg[u_i-u_i'$$
$$-\frac{n_n}{n_N}\left(\frac{m_n}{m_N}\right)^{1/2}(u_{Ni}-u_{Ni}')\Bigg]g_{nN}W_{nN}(\theta,g_{nN})\,d\Omega dc_{Ni}dc_i, \tag{22}$$

where $u_{Ni}=(m_N/2kT)v_{Ni}$.

a) Calculation of b_{jk}

Assuming that the mass of the nuclei is much greater than the neutron mass, $m_N\gg m_n$, and using the normalization of f_{N0}, we obtain from (21)

$$b_{jk}=8\pi^2\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}\left(\frac{2kT}{m_n}\right)^{1/2}n_N\int_0^\infty f_0(1-f_0)$$
$$\times Q_j(x)Q_k(x)x^2(1-\cos\theta)W_{nN}(\theta,x)\sin\theta d\theta dx,\quad k\geqslant1. \tag{23}$$

Using the law of momentum conservation in a collision, we obtain

$$u_i-u_i'=-(m_N/m_n)^{1/2}(u_{Ni}-u_{Ni}').$$

From (22), we obtain

$$b_{j0}=8\pi^2\frac{\rho}{\rho_N}\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}\left(\frac{2kT}{m_n}\right)^{1/2}n_N\int_0^\infty f_0(1-f_0)$$
$$\times Q_j(x)x^2(1-\cos\theta)W_{nN}(\theta,x)\sin\theta d\theta dx. \tag{24}$$

We introduce the functions $\tilde{\Omega}_{Nn}^{(1)}(r)$, which are defined as in Ref. 8:

$$\tilde{\Omega}_{nN}^{(l)}(r)=\int_0^\infty dx\int_0^\pi d\theta f_0(1-f_0)x^{r+1}(1-\cos^l\theta)W_{nN}(\theta,x)\sin\theta. \tag{25}$$

Then the coefficients b_{jk} can be written in the form

$$b_{00}=8\pi^2\left(\frac{\rho}{\rho_N}\right)\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}\left(\frac{2kT}{m_n}\right)^{1/2} n_N\tilde{\Omega}^{(1)}_{nN}(1),$$

$$b_{01}=\frac{\rho_N}{\rho}b_{10}=8\pi^2\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}\left(\frac{2kT}{m_n}\right)^{1/2}$$

$$\times n_N\left[\frac{5}{2}\frac{G_{5/2}}{G_{3/2}}\tilde{\Omega}^{(1)}_{nN}(1)-\tilde{\Omega}^{(1)}_{nN}(2)\right], \qquad (26)$$

$$b_{11}=8\pi^2\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}\left(\frac{2kT}{m_n}\right)^{1/2} n_N\left[\frac{25}{4}\frac{G^2_{5/2}}{G^2_{3/2}}\tilde{\Omega}^{(1)}_{nN}(1)\right.$$

$$\left.-5\frac{G_{5/2}}{G_{3/2}}\tilde{\Omega}^{(1)}_{nN}(2)+\tilde{\Omega}^{(1)}_{nN}(3)\right].$$

Substituting the cross section $W_{nN}(x,\theta)$ in the form[1)]

$$W_{nN}(x,\theta)=\sum_{l=0}^{\infty}W^{(l)}_{nN}(x)P_l(\cos\theta), \qquad (27)$$

we obtain from (25)

$$\tilde{\Omega}^{(1)}_{nN}(r)=2\int_0^{\infty}\overline{W}_{nN}(x)f_0(1-f_0)x^{r+1}dx,$$

$$\overline{W}_{nN}(x)=W^{(0)}_{nN}(x)-{}^1/_3W^{(1)}_{nN}(x). \qquad (27a)$$

In the crust of a neutron star the density is much less than the nuclear density, the energy of the neutrons does not exceed a few MeV, and they can be regarded as having a "low energy" for nuclear reactions. In this case, S-wave scattering is predominant, and its cross section $W^{(0)}_{nN}(x)$ does not depend on the angle.[13]

b) Calculation of a_{jk}, weak degeneracy

In the case of weak degeneracy ($e^{-x_0}\gg 1$), the quantities a_{jk} were calculated in Ref. 10. We go over to the variables G_i and g_i:

$$\tilde{G}_i={}^1/_2(v_i+v_{1i}),\quad \tilde{g}_i=v_{1i}-v_i. \qquad (28)$$

We introduce the dimensionless velocities

$$G_i=\left(\frac{m_n}{kT}\right)^{1/2}\tilde{G}_i,\quad g_i=\frac{1}{2}\left(\frac{m_n}{kT}\right)^{1/2}\tilde{g}_i,$$

$$dc_idc_{1i}=\left(\frac{2kT}{m_n}\right)^{3}dG_idg_i. \qquad (29)$$

Using (12) and (28)–(29) in (20), we obtain $a_{00}=a_{01}=a_{10}=0$. This is also true for the case of arbitrary degeneracy. For a_{11} in the general case we obtain the expression

$$a_{11}=8\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^3\left(\frac{kT}{m_n}\right)^{1/2}\int f_0f_{01}(1-f_{01}')(1-f_0')$$

$$\times\left[\frac{1}{2}(G^2+g^2)-G_ig_i\right][G_kg_k(G_jg_j-g^2)$$

$$+G_kg_k'(g_jg_j'-G_jg_j')]gW_{nn}(g,\theta)d\Omega dg_idG_i. \qquad (30)$$

Expanding with respect to the small parameter e^{x_0} and using (5), (12), (28), and (29), we obtain

$$f_0f_{01}(1-f_0')(1-f_{01}')=\exp[-(G^2+g^2)+2x_0]\cdot$$

$$\times\{1-2\exp[x_0-{}^1/_2(G^2+g^2)][\cosh(G_ig_i)+\cosh(G_ig_i')]\}. \qquad (31)$$

Calculating a_{11} with allowance for (30) and (31), we obtain[2)]

$$a_{11}=32\pi^3\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^3\left(\frac{kT}{m_n}\right)^{1/2}\{e^{2x_0}\Omega^{(2)}_{nn}(2,1)$$

$$-{}^8/_3\sqrt{{}^2/_3}e^{3x_0}[{}^2/_3\Omega^{(2)}_{nn}(2,{}^4/_3)+{}^2/_{27}\Omega^{(2)}(3,{}^4/_3)]\}, \qquad (32)$$

$$\Omega^{(l)}_{nn}(r,d)=\sqrt{\pi}\int_0^{\infty}e^{-dg^2}g^{2r+3}dg\int_0^{\pi}(1-\cos^l\theta)W_{nn}(\theta,g)\sin\theta d\theta. \qquad (33)$$

Using the expansion of the cross section $W_{nn}(g,\theta)$ with respect to Legendre polynomials,

$$W_{nn}(g,\theta)=\sum_{l=0}^{\infty}W^{(l)}_{nn}(g)P_l(\cos\theta), \qquad (34)$$

we obtain

$$\Omega^{(2)}_{nn}(r,d)=\sqrt{\pi}\int_0^{\infty}e^{-dg^2}g^{2r+3}\left[\frac{4}{3}W^{(0)}_{nn}(g)-\frac{4}{15}W^{(2)}_{nn}(g)\right]dg. \qquad (35)$$

c) Calculation of a_{ij}, strong degeneracy

We make the calculation as in Ref. 11. The dependence on the cross section can be expressed in terms of the function [see (34)]

$$W_{nn}(g)=\sum_{l=0}^{\infty}W^{(l)}_{nn}(g)[P_l(0)]^2. \qquad (36)$$

As a result of integration, we obtain[11]

$$a_{11}=\frac{1024\sqrt{2}}{63}\pi^7\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^3\left(\frac{kT}{m_n}\right)^{1/2}x_0^{3/2}I; \qquad (37)$$

$$I=\int_0^1\frac{y^3W_{nn}(y)dy}{(1-y^2)^{1/2}},\quad y=\frac{g}{(2x_0)^{1/2}}.$$

5. INTERPOLATION FORMULAS FOR THE TRANSPORT COEFFICIENTS

For arbitrary degree of degeneracy of the neutrons, it is not possible to obtain an analytic expression for a_{11}. Therefore, we construct an interpolation formula for a_{11} that gives correct results in the limiting cases of weak and strong degeneracy. For simplicity, we retain only the first term in Eq. (32). Expressing x_0 in terms of n_n and T by means of (6) and substituting in (32) and (37) in the two limiting cases $e^{x_0}\gg 1$ and $x^{x_0}\ll 1$, we obtain

$$a_{11}=8n_n^2\left(\frac{kT}{m_n}\right)^{1/2}\Omega_{nn}^{(2)}(2)\quad \text{for}\quad e^{x_0}\ll 1,$$
$$a_{11}=\frac{256}{21}\sqrt{2}\pi^6 n_n\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}\left(\frac{kT}{m_n}\right)^{1/2} I\quad \text{for}\quad e^{x_0}\gg 1. \tag{38}$$

Introducing the parameter

$$\varepsilon=\frac{21\sqrt{2}}{8\pi^{13/2}}\left(\frac{2\pi^2\hbar^2}{kTm_n}\right)^{3/2} n_n\frac{\sqrt{\pi}\Omega_{nn}^{(2)}(2)}{8I}\begin{cases}\gg 1 & \text{for}\quad e^{x_0}\gg 1,\\ \ll 1 & \text{for}\quad e^{x_0}\ll 1,\end{cases} \tag{39}$$

where $\Omega_{nn}^{(l)}(2)\equiv\Omega_{nn}^{(l)}(2,1)$, see (33) and (35), we obtain an interpolation formula for a_{11} which holds approximately for arbitrary x_0:

$$a_{11}=\frac{256\sqrt{2}}{21}\pi^6 n_n\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}\left(\frac{kT}{m_n}\right)^{1/2} I\frac{\varepsilon}{1+\varepsilon}. \tag{40}$$

a) Diffusion and thermal diffusion

We write the difference between the mean velocities $(\langle v_i\rangle-\langle v_{Ni}\rangle)$ in the following standard form (Ref. 8, p. 176 of the Russian translation):

$$\langle v_i\rangle-\langle v_{N_i}\rangle=-\frac{n^2}{n_n n_N}D_n\left(\frac{n_n}{n}d_i+k_T\frac{\partial\ln T}{\partial r_i}\right),\quad n=n_n+n_N. \tag{41}$$

Comparing (41) with (16), we find that the diffusion coefficient D_n and the thermal diffusion ratio k_T are determined by

$$D_n=\frac{kT}{m_n}\frac{\rho}{\rho_N}\frac{n_n n_N}{n}\frac{G_{5/2}}{G_{3/2}}d_0,\quad k_T=\frac{1}{n}\frac{G_{3/2}}{G_{5/2}}\frac{a_0}{d_0}. \tag{42}$$

a.1) Diffusion coefficient

Substituting the solution of the system (19) in D_n (42) and using (26) and (40), we obtain

$$D_n=\frac{3}{16\pi^2\sqrt{2}}\frac{n_n}{n}\left(\frac{kT}{m_n}\right)^{1/2}\left(\frac{2\pi^2\hbar^2}{kTm_n}\right)^{3/2}$$
$$\times\frac{G_{5/2}}{G_{3/2}}\left[\frac{25}{4}\frac{G_{5/2}^2}{G_{3/2}^2}\tilde{\Omega}_{nN}^{(1)}(1)-5\frac{G_{5/2}}{G_{3/2}}\tilde{\Omega}_{nN}^{(1)}(2)+\tilde{\Omega}_{nN}^{(1)}(3)\right.$$
$$\left.+\frac{32}{21}\pi^4\frac{n_n}{n_N}I\frac{\varepsilon}{1+\varepsilon}\right]\frac{1}{\Delta}. \tag{43}$$

$$\Delta=\tilde{\Omega}_{nN}^{(1)}(1)\tilde{\Omega}_{nN}^{(1)}(3)-\tilde{\Omega}_{nN}^{(1)2}(2)+\frac{32}{21}\pi^4\tilde{\Omega}_{nN}^{(1)}(1)\frac{n_n}{n_N}I\frac{\varepsilon}{1+\varepsilon}, \tag{44}$$

We consider the limit of a nondegenerate gas: $e^{x_0}\ll 1$, $e\ll 1$. From (6) and (25), we obtain

$$G_i/G_k=1,\quad n_n=2\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}\pi^{3/2}e^{x_0},$$
$$\tilde{\Omega}_{nN}^{(l)}(r)=\frac{2}{\sqrt{\pi}}e^{x_0}\Omega_{nN}^{(l)}(r), \tag{45}$$

where [see (27)]

$$\Omega_{nN}^{(l)}(r)=\sqrt{\pi}\int_0^\infty e^{-g^2}g^{2r+3}\,dg\int_0^\pi(1-\cos^l\theta)W_{nN}(g,\theta)\sin\theta\,d\theta,$$
$$\Omega_{nN}^{(1)}(r)=2\sqrt{\pi}\int_0^\infty e^{-g^2}\overline{W}_{nN}(g)\,g^{2r+3}\,dg. \tag{46}$$

For D_n, we obtain the expression

$$D_n^{(\text{nondeg})}=\frac{3}{16\sqrt{2}}\frac{1}{n}\left(\frac{kT}{m_n}\right)^{1/2}\left[\frac{25}{4}\Omega_{nN}^{(1)}(1)-5\Omega_{nN}^{(1)}(2)\right.$$
$$\left.+\Omega_{nN}^{(1)}(3)+\Omega_{nn}^{(2)}(2)\frac{n_n}{n_N\sqrt{2}}\right]\Delta_{(\text{nondeg})}^{-1}, \tag{47}$$

$$\Delta_{(\text{nondeg})}=\Omega_{nN}^{(1)}(1)\Omega_{nN}^{(1)}(3)-\Omega_{nN}^{(1)2}(2)+\Omega_{nN}^{(1)}(1)\Omega_{nn}^{(2)}(2)n_n/n_N\sqrt{2}. \tag{48}$$

In the limit of strong neutron degeneracy, $e^{x_0}\gg 1$, $e\gg 1$, we obtain from (6), (25), and (27), using the asymptotic expressions in Ref. 14 (p. 188 of the Russian original),

$$G_n=\frac{1}{\Gamma(n)}\left[\frac{x_0^n}{n}+\frac{\pi^2}{6}(n-1)x_0^{n-2}\right],\quad n_n=\frac{8\pi}{3}\left(\frac{kTm_n}{2\pi^2\hbar^2}\right)^{3/2}x_0^{3/2},$$
$$\tilde{\Omega}_{nN}^{(1)}(r)=2\left\{x_0^{r+1}\overline{W}_{nN}(x_0)+\frac{\pi^2}{6}[r(r+1)x_0^{r-1}\overline{W}_{nN}(x_0)\right.$$
$$\left.+2(r+1)x_0^r\overline{W}_{nN}'(x_0)+x_0^{r+1}\overline{W}_{nN}''(x_0)]\right\}, \tag{49}$$

where the prime denotes the derivative with respect to x_0. In this limit, the diffusion coefficient

$$D_n^{(\text{deg})}=\frac{1}{20}\left(\frac{3}{\pi}\right)^{1/3}\frac{\hbar}{nm_n}\frac{n_n^{1/3}}{\overline{W}_{nN}(x_0)} \tag{50}$$

does not contain terms with neutron-neutron interaction. If in (13) for D_i we retain only one term of the expansion, with d_0, then in the nondegenerate case we obtain [like Eq. (47), see Ref. 8, p. 200 of the Russian translation]

$$[D_n^{(\text{nondeg})}]_1=\frac{3}{16\sqrt{2}}\frac{1}{n}\left(\frac{kT}{m_n}\right)^{1/2}\frac{1}{\Omega_{nN}^{(1)}(1)}. \tag{51}$$

For the classical interaction of elastic spheres,

$$\Omega_{nN}^{(1)}(1)=\frac{\sqrt{\pi}}{2}R_{nN}^2,\quad \overline{W}_{nN}=\frac{R_{nN}^2}{4}, \tag{52}$$

where $R_{nN}=R_n+R_N$ is the sum of the radii of a neutron and a nucleus. Using (52) in (50) and (51), we see immediately that D_n^{deg} (50) can be obtained, apart from a numerical coefficient, from (51) by replacement of the characteristic velocity $(kT/m_n)^{1/2}$ by the characteristic velocity in a degenerate gas, $(kTx_0/m_n)^{1/2}$, where x_0 is determined in (49). Such an approximation was used in the calculations in Ref. 4.

a.2) Thermal diffusion ratio

Substituting the solutions of the systems (18) and (19) in k_T (42) and using (26) and (40), we obtain

$$k_T=\frac{5}{2}\frac{n_n}{n}\left(\frac{7}{2}\frac{G_{7/2}}{G_{5/2}}-\frac{5}{2}\frac{G_{5/2}}{G_{3/2}}\right)\left[\frac{5}{2}\frac{G_{5/2}}{G_{3/2}}\tilde{\Omega}_{nN}^{(1)}(1)-\tilde{\Omega}_{nN}^{(1)}(2)\right]$$

$$\times\left[\frac{25}{4}\frac{G_{5/2}^2}{G_{3/2}^2}\tilde{\Omega}_{nN}^{(1)}(1)-5\frac{G_{5/2}}{G_{3/2}}\tilde{\Omega}_{nN}^{(1)}(2)+\tilde{\Omega}_{nN}^{(1)}(3)+\frac{32\pi^4}{21}\frac{n_n}{n_N}I\frac{\varepsilon}{1+\varepsilon}\right]^{-1}. \quad (53)$$

For nondegenerate neutrons, using (45) and (46), we obtain

$$k_T^{(\text{nondeg})}=\frac{5}{2}\frac{n_n}{n}\left[\frac{5}{2}\Omega_{nN}^{(1)}(1)-\Omega_{nN}^{(1)}(2)\right]$$

$$\times\left[\frac{25}{4}\Omega_{nN}^{(1)}(1)-5\Omega_{nN}^{(1)}(2)+\Omega_{nN}^{(1)}(3)+\Omega_{nn}^{(2)}(2)\frac{n_n}{n_N\sqrt{2}}\right]^{-1}. \quad (54)$$

For strong degeneracy, using (49), we obtain

$$k_T^{(\text{deg})}=\frac{1}{2}\frac{n_n}{n}\left[1+\pi^2\left(\frac{3}{\pi}\right)^{2/3}\frac{\hbar^2}{m_n}\frac{n_n^{2/3}}{kT}\frac{\overline{W}_{nN}'(x_0)}{\overline{W}_{nN}(x_0)}\right]$$

$$\times\left[1+\frac{64}{21\pi}\left(\frac{\pi}{3}\right)^{1/3}\frac{m_n^2}{\hbar^4}\frac{(kT)^2}{n_n^{1/3}n_N}\frac{I}{\overline{W}_{nN}(x_0)}\right]^{-1}. \quad (55)$$

b) Neutron heat transfer

We find d_i using (16) and substitute it in (15). Then, using k_T from (53) and (15), we obtain an expression for the neutron heat flux in the form

$$q_{ni}=-\lambda_n\frac{\partial T}{\partial r_i}+\frac{5}{2}n_nkT\frac{G_{5/2}}{G_{3/2}}\langle v_i\rangle+nkT\frac{G_{5/2}}{G_{3/2}}k_T(\langle v_i\rangle-\langle v_{Ni}\rangle), \quad (56)$$

which is a generalization of the classical definition (see Ref. 8, p. 176 of the Russian translation) taking into account degeneracy. For the neutron coefficient of thermal conductivity λ_n, we obtain

$$\lambda_n=\frac{75}{64\pi^2}\frac{k}{\sqrt{2}}\left(\frac{kT}{m_n}\right)^{1/2}\frac{n_n^2}{n_N}\left(\frac{2\pi^2\hbar^2}{kTm_n}\right)^{3/2}\frac{G_{5/2}^2}{G_{3/2}^2}$$

$$\times\left(\frac{7}{2}\frac{G_{7/2}}{G_{5/2}}-\frac{5}{2}\frac{G_{5/2}}{G_{3/2}}\right)^2\cdot\left[\frac{25}{4}\frac{G_{5/2}^2}{G_{3/2}^2}\tilde{\Omega}_{nN}^{(1)}(1)\right.$$

$$\left.-5\frac{G_{5/2}}{G_{3/2}}\Omega_{nN}^{(1)}(2)+\tilde{\Omega}_{nN}^{(1)}(3)+\frac{32}{21}\pi^4\frac{n_n}{n_N}I\frac{\varepsilon}{1+\varepsilon}\right]^{-1}. \quad (57)$$

In the limit of nondegenerate neutrons, we obtain, using (45) and (46),

$$\lambda_n^{(\text{nondeg})}=\frac{75}{64}\frac{k}{\sqrt{2}}\left(\frac{kT}{m_n}\right)^{1/2}\frac{n_n}{n_N}\left[\frac{25}{4}\Omega_{nN}^{(1)}(1)\right.$$

$$\left.-5\Omega_{nN}^{(1)}(2)+\Omega_{nN}^{(1)}(3)+\Omega_{nn}^{(2)}(2)\frac{n_n}{n_N\sqrt{2}}\right]^{-1}. \quad (58)$$

From (58) in the limit $n_N/n_n\to 0$ we obtain the well-known (Ref. 8, p. 198 of the Russian translation) expression for the coefficient of thermal conductivity of a nondegenerate neutron gas:

$$\lambda_{nn}^{(\text{nondeg})}=\frac{75}{64}k\left(\frac{kT}{m_n}\right)^{1/2}\frac{1}{\Omega_{nn}^{(2)}(2)}.$$

In the case of strong degeneracy, using (49), we obtain

$$\lambda_n^{(\text{deg})}=\frac{1}{12}\left(\frac{\pi}{3}\right)^{1/3}\frac{k^2T}{\hbar}\frac{n_n^{2/3}}{n_N}\frac{1}{\overline{W}_{nN}(x_0)}$$

$$\times\left[1+\frac{64}{21\pi}\left(\frac{\pi}{3}\right)^{1/3}\frac{m_n^2}{\hbar^4}\frac{(kT)^2}{n_n^{1/3}n_N}\frac{I}{\overline{W}_{nN}(x_0)}\right]^{-1}.$$

From this in the limit $n_N\to 0$ we obtain the coefficient for a degenerate neutron gas (see Ref. 11):

$$\lambda_{nn}^{(\text{deg})}=7\pi\hbar^3n_n/256Tm_n^2I.$$

[1)]The expansion (27) is not suitable for the Coulomb interaction because of the strong divergence at small angles θ.

[2)]Note that in Ref. 10 the coefficient of $\Omega^{(2)}(3, 4/3)$ is 1.5 times greater.

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