Propagation of ultrashort optical pulse in a two-level laser amplifier

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The response of a long two-level laser amplifier is obtained, i.e., the structure of the pulse at the amplifier output is determined as a function of the waveform of the firing pulse. It is shown that the output pulse is always quasi-self-similar, and that the parameters of the long-amplifier solution are determined exclusively by the rise time of the firing pulse.

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1. INTRODUCTION

The propagation of ultrashort optical pulses in a resonant medium are well described within the framework of the semiclassical Lamb model (see Ref. 1). This model deals with a classical electromagnetic field that interacts resonantly with quantum two-level objects—"two-level atoms." The model becomes even simpler if a pulse traveling in one direction is considered. In this case, neglecting all the dissipative processes, the equations of the model are written in the form[1-4]

$$E_x+E_t=2\pi i\Omega_0 p\int n(\omega)u(\omega)\bar{v}(\omega)\,d\omega, \tag{1}$$

$$u_t=i\omega u+ipEv,\quad v_t=-i\omega v+ip\bar{E}u, \tag{2}$$

which is equivalent to the known Maxwell-Bloch system. It is assumed that $\hbar=c=1$; $E(x,t)$ in (1) and (2) is the complex envelope of the electric field, so that the field in the medium is

$$\mathscr{E}(x,t)=E(x,t)e^{i\Omega_0(x-t)}+\bar{E}(x,t)e^{i\Omega_0(t-x)},$$

$u(\omega)$ and $v(\omega)$ are the amplitudes of the probabilities of the sojourn of the two-level atom in the upper (lower) states. The transition has a frequency Ω_0 and a dipole moment p. The function $n(\omega)$, which characterizes the inhomogeneous broadening, is the difference between the initial populations of the upper and lower levels: $n(\omega)=N_+(\omega)-N_-(\omega)$. The choice of the notation in (1) and (2) is such that one should assume $ue^{-i\omega t}\to 1$ and $v\to 0$ as $t\to -\infty$. Equation (1) is all that is left of Maxwell equations, while the right-hand side of (1) is proportional to the polarization of the medium at the transition frequency. Equations (2), on the other hand, are Schrödinger equations of two-level systems in an external field, written in the resonance approximation. The parameter ω in (1) and (2) is the deviation of the transition frequency of the given two-level atom from the mean frequency Ω_0 of the ensemble; the physical cause of this difference between the transition frequencies can be, for example, the Doppler shift (for gas lasers).

The pulse-propagation problem is formulated as follows. Let a medium with a population difference $n(\omega)$ occupy the half-space $x>0$ and let the field in the medium be zero at $t<0$. A pulse of specified profile $E(x,t)|_{x=0}=E_0(t)$, with $E_0(t)=0$ at $t<0$, enters the medium through the boundary $x=0$. Describe the field $E(x,t)$ in the medium. In this formulation, the boundary-value problem (1), (2) is transformed into an evolution equation, where the role of the "time" is played by x: In other words, the indicated boundary-value problem is equivalent to the Cauchy problem in terms of x.

It is clear that in principle the answer for a normal medium differ from those for a medium with inverted population. In the former case, when $n(\omega)<0$, the linear modes attenuate exponentially in the interior of the medium and solely nontrivial effects (self-induced transparency) arise only when the pulse entering the medium is strong enough and releases a finite (usually small) number of solitons (2π or 0π pulses) to which the medium is transparent.[2] A self-induced-transparency theory based on the application of the inverse-problem method to the system (1), (2) has been developed in great detail[1,3,4] and is in splendid agreement with experiment. The situation is entirely different for an amplifier, i.e., when $n(\omega)>0$ for at least a few ω. In this case the corresponding linear modes increase exponentially with x and the problem soon becomes strongly nonlinear even for a weak entering pulse. Although the inverse-problem method is also applicable to the propagation of a pulse in an amplifier, no solutions that are to any degree complete have been described heretofore. The reason is that, in contrast to an attenuator, where the evolution of the solution is described mainly by the discrete spectrum of the associated differential operator and the inverse-problem equations reduce to a system of (easily solvable) linear equations, in the case of an amplifier we must deal with a continuous spectrum and investigate the inverse problem to its full extent, i.e., solve a system of singular integral equations. Some exact solutions were obtained by Lamb[5] in the limiting case of strong inhomogeneous broadening $n(\omega)=\text{const.}$[1)]

In this article we describe the structure of the pulse at large x for the case when $\int n(\omega)d\omega>0$ as a function of the waveform of the input signal $E_0(t)$, i.e., we present the "response" of a long laser amplifier. We shall show that at large x the solution is a nonstationary π pulse whose velocity tends to that of light. With increasing x, the pulse contracts roughly speaking like $1/x$, and the field in the pulse increases like x. The inhomogeneous-broadening effects become insignificant at large x, the pulse has a constant phase, and its shape is quasi-self-similar. The solution is then de-

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termined exclusively by the behavior of $E_0(t)$ in the vicinity of $t=0$ and is insensitive to the subsequent course of this function—the system forgets the initial condition almost completely. The pulse absorbs practically all the energy and can be drawn from an inversely populated medium only by a classical electromagnetic wave. Namely, the energy removed per unit length is $\Omega_0 \int n(\omega)d\omega$, so that after the passage of the pulse the populations of the upper and lower levels become equal to $N_-(\omega)$ and $N_+(\omega)$ respectively.

The fact that at large x the pulse structure is insensitive to the form of the inhomogeneous broadening means that $E(x, t)$ coincides with the solution of the system (1), (2) in which the function $n(\omega)$ can be replaced by $n'(\omega) = N\delta(\omega)$, where $N = \int n(\omega)d\omega$. In this case, the Maxwell-Bloch equations have a family of self-similar solutions[6,7] of the form $E(x, t) = p^{-1}\partial U/\partial t$, where U is a function of the self-similar variable

$$z=4\Omega[x(t-x)]^{1/2},\quad \Omega^2=\pi p^2\Omega_0 N/2;$$

$U(z)$ satisfies the equation

$$U_{zz}+U_z/z=\sin U.$$

Each solution of (3) that is regular at zero is uniquely determined by the value $U(0)=U_0$ of U at $z=0$, so that $U=U(U_0, z)$. It turns out that the parameter U_0 can be made an arbitrary slow function of the ratio z/x; the corresponding $E(x, t)$ is then close, as before, to the true solution of Eqs. (1) and (2). We shall refer to such solutions, with "floating" parameter U, as quasi-self-similar. It is precisely in terms of these solutions that the asymptotic form of $E(x, t)$ is expressed at large x.

The function $U_0(z/x)$ can be explicitly expressed in terms of $E_0(t)$, and furthermore quite simply, avoiding the inverse-problem technique, by matching the solution obtained by the linear approximation to the quasi-self-similar solution, since the regions where both are applicable overlap at large x. This is a very important circumstance for the following region: The main shortcoming of the model (1), (2) as applied to real laser amplifier is the requirement that the amplifier operate on a transition between nondegenerate states, whereas in real systems these states are degenerate.[2)] A system with degeneracy can no longer be integrated by using the inverse-problem method. However, so crude a property as quasi-self-similarity of the asymptotic form should obtain in this case, too: Since the integrable system forgets the initial condition, one need not expect the nonintegrable system to have a better memory. Therefore, accepting the self-similarity hypothesis, we can calculate also the characteristics of an amplifier with degeneracy (this will be done elsewhere). The model (1), (2), however, being integrable provides a unique opportunity of verifying the self-similarity hypothesis, and this can be done only by the inverse-problem method.

The plan of the article is the following. In Sec. 2 we present the needed inverse-problem information in a form convenient for our purpose. In Sec. 3 the inverse-problem equations are solved explicitly for very large x, $\ln\ln\Omega x>1$, when the pulse has a particularly simple form—it consists of a sequence of 2π and -2π pulses. The technique used in this section shows that the solution is quasi-self-similar even at $\ln\Omega x<1$. In Sec. 4 the connection between the parameters of the quasi-self-similar solution and the initial condition $E_0(t)$ is obtained.

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2. NEEDED INVERSE-PROBLEM METHOD INFORMATION

We describe here the general procedure for integrating the system (1), (2) within the framework of the inverse-problem method. Since the formally analytic aspect of the problem is the same for an amplifier as for an attenuator, our description will be as brief as possible. The standard details of the method can be found in Refs. 3 and 9.

The inverse-problem method can be applied to the Maxwell-Bloch system because Maxwell's equation (1) can be replaced by a supplementary integrodifferential equation for the amplitudes $u(\omega, x, t)$ and $v(\omega, x, t)$, which are the solutions of the system (2). Namely, we can write in lieu of (1)

$$\frac{\partial}{\partial x}\begin{pmatrix}u\\v\end{pmatrix}=-i\left\{\omega\begin{pmatrix}1&0\\0&-1\end{pmatrix}+p\begin{pmatrix}0&E\\\bar E&0\end{pmatrix}+\frac{\pi p^2\Omega_0}{2}\int\frac{n(\omega')\hat\rho(\omega')d\omega'}{\omega'-\omega}\right\}\begin{pmatrix}u\\v\end{pmatrix}\qquad(3)$$

where the matrix $\hat\rho$ is given by

$$\hat\rho(\omega, x, t)=\begin{pmatrix}|u|^2-|v|^2, & 2u\bar v\\ 2\bar u v, & -|u|^2+|v|^2\end{pmatrix}.\qquad(4)$$

In fact, the Schrödinger equations (2), which we rewrite in the form

$$\frac{\partial}{\partial t}\begin{pmatrix}u\\v\end{pmatrix}=i\left\{\omega\begin{pmatrix}1&0\\0&-1\end{pmatrix}+p\begin{pmatrix}0&E\\\bar E&0\end{pmatrix}\right\}\begin{pmatrix}u\\v\end{pmatrix},\qquad(5)$$

and Eq. (3) constitute an overdefined system for u and v. The condition for the compatibility of this system at all ω (equality of the second derivatives $(\partial/\partial t)/(\partial/\partial x)\binom{u}{v}$ and $(\partial/\partial x)/(\partial/\partial t)\binom{u}{v}$ is

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\begin{pmatrix}0&pE\\p\bar E&0\end{pmatrix}=i\frac{\pi\Omega_0 p^2}{2}\int n(\omega)[\sigma_3,\hat\rho(\omega)]d\omega,$$

$$\hat\rho_t=i\left[\omega\sigma_3+p\begin{pmatrix}0&E\\\bar E&0\end{pmatrix},\hat\rho\right],\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

By choosing $\hat\rho$ in the form (4), we satisfy the last of these equations [as soon as u and v solve (5)]; the first equation goes over in this case into (1).

Once the representation (3), (5) is established for the system (1), (2), the standard technique of the inverse-problem method becomes applicable. We introduce a special solution of the system (5), fixed by the condition

$$u=e^{i\omega t},\quad v=0\quad\text{at}\quad t<0.\qquad(6)$$

[We recall that $E(x, t)$ in (5) is zero in our problem at all $x>0$ and negative t, while the solution (6) is precisely the one used when Eqs. (1) and (2) were written down.] We designate hereafter the column u, v by φ:

$$\varphi=\begin{pmatrix} u \\ v \end{pmatrix}.$$

The condition (6), however, is not the solution of (3) at $E=0$. Wishing to preserve the definition (6) of φ for all x, we add to the right-hand side of (3) the term

$$i\left(\omega+\frac{\pi p^2\Omega_0}{2}\int\frac{n(\omega')}{\omega'-\omega}d\omega'\right)\varphi,$$

which obviously does not change the conditions for the compatibility of the systems (3) and (5); in place of (3) we have then

$$\varphi_x=-\varphi_t+i\left(\omega-\frac{\pi p^2\Omega_0}{2}\int\frac{\hat\rho(\omega')-1}{\omega'-\omega}n(\omega')d\omega'\right)\varphi. \qquad (7)$$

We shall indicate now how the integral in (7) is to be understood at real ω. The introduced solution φ is analytic in the lower ω half-plane; we therefore define the integral in (7) at $\operatorname{Im}\omega=0$ as the limit when ω tends to the real axis from below, i.e., we replace ω by $\omega-i0$.

At fixed x and as $t\to\infty$ the quantity $E(x,t)$ vanishes and the asymptotic form of the solution (6) is

$$\varphi\to\begin{pmatrix} A(x,\omega)e^{i\omega t} \\ B(x,\omega)e^{-i\omega t}\end{pmatrix},\quad t\to\infty.$$

The dependence of the functions A and B on x is obtained from (7) explicitly. We note first that by virtue of (4) the asymptotic form of $\hat\rho(\omega)$ is

$$\hat\rho(\omega,x,t)|_{t\to\infty}\equiv\rho^+(\omega)=\begin{pmatrix}|A|^2-|B|^2, & 2A\bar B e^{2i\omega t} \\ 2\bar A B e^{-2i\omega t}, & |B|^2-|A|^2\end{pmatrix}.$$

Using next the known formula

$$\lim_{t\to\infty}\frac{e^{i\gamma\omega' t}}{\omega'-\omega+i0}=\begin{cases} 0 & \text{at} \quad \gamma>0 \\ -2\pi i e^{i\gamma\omega t}\delta(\omega-\omega') & \text{at} \quad \gamma<0\end{cases}$$

we find that

$$\int\frac{n(\omega')\rho^+(\omega')}{\omega'-\omega+i0}d\omega'=\begin{pmatrix} Q(\omega), & 0 \\ -4\pi i n(\omega)\bar A B e^{-2i\omega t}, & -Q(\omega)\end{pmatrix},$$

$$Q(\omega)=\int\frac{n(\omega')(|A|^2-|B|^2)(\omega')d\omega'}{\omega'-\omega+i0}.$$

Substituting this expression in (7), we obtain after simple calculations

$$\frac{\partial}{\partial x}\frac{B}{\bar A}(\omega)=\left[2i\omega-i\pi\Omega_0 p^2\int\frac{n(\omega')d\omega'}{\omega'-\omega-i0}\right]\frac{B}{\bar A}(\omega).$$

In other words, putting $B/\bar A=R(\omega,x)$, we have

$$R(\omega,x)=R(\omega,0)\exp\left[2ix\left(\omega-\frac{\pi\Omega_0 p^2}{2}\int\frac{n(\omega')d\omega'}{\omega'-\omega-i0}\right)\right]. \qquad (8)$$

The function $R(\omega,x)$ has the meaning of the reflection coefficient for the system (5) and is one of the principal objects in the inverse-problem method. The transformation $E(x,t)\to R(\omega,x)$ is uniquely reversible subject to certain limitations.[3)] Formula (8) solves therefore in principle the Cauchy problem with respect to x for the system (1), (2): $R(\omega,0)$ is calculated given $E_0(t)=E(x,t)|_{x=0}$; $E(x,t)$ is reconstructed then from the known $R(\omega,x)$ (8). The problem of reconstructing E from R is the subject of the inverse scattering problem. Its solution reduces to a solution of a system of singular integral equations for the function $\chi(\omega,x,t)=\binom{\chi_1}{\chi_2}=\varphi(\omega,x,t)e^{-i\omega t}$ (see Ref. 9, Chap. 1, Secs. 9 and 10):

$$\chi(\omega)=\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{R(\omega',x)e^{-2i\omega' t}}{\omega'-\omega+i0}\tilde\chi(\omega')d\omega', \qquad (9)$$

where

$$\tilde\chi(\omega)=\begin{pmatrix}-\bar\chi_2\\ \bar\chi_1\end{pmatrix}=\begin{pmatrix}-\bar v(\omega)\\ \bar u(\omega)\end{pmatrix}e^{i\omega t}.$$

Solution of this system yields the functions $u(\omega)$ and $v(\omega)$, knowledge of which allows us obviously to determine $E(x,t)$ from Eqs. (5). There exists, however, a very convenient linear expression for E in terms of φ. It follows from a comparison of the first terms of the asymptotic expansion of in powers of $1/\omega$, calculated by starting directly from the problem (5), and on the other hand from Eqs. (9). Putting

$$\chi=\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{i\omega}\chi^{(1)}+O\left(\frac{1}{\omega^2}\right),$$

we have[9]

$$\chi^{(1)}=\begin{pmatrix} -\dfrac{p^2}{2}\displaystyle\int_{-\infty}^{t}|E|^2dt' \\ i\dfrac{p}{2}\bar E\end{pmatrix}=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(\omega,x)e^{-2i\omega t}\tilde\chi(\omega)d\omega. \qquad (10)$$

We shall investigate the system (9) in the next section. We note now only that even (8) leads directly to nontrivial physical consequences. The solution introduced for the system (5) with the aid of the condition (6) yields the amplitudes u and v for finding an initially excited atom in the upper and lower states, respectively. For an atom initially in the lower state, the solutions of the Schrödinger Eq. (5) are respectively $-\bar v$ and $\bar u$. Therefore the average number of atoms on the upper level at the instant of time t is

$$N_+(\omega,t)=N_+(\omega)|u(\omega,t)|^2+N_-(\omega)|v(\omega,t)|^2,$$

and on the lower level

$$N_-(\omega,t)=N_+(\omega)|v(\omega,t)|^2+N_-(\omega)|u(\omega,t)|^2,$$

where N_+ and N_- are the initial populations of the upper and lower levels. It follows from these equations that the population difference at the point x after the passage of the pulse ($t\to\infty$) is

$$N_+(\omega,\infty,x)-N_-(\omega,\infty,x)=n(\omega)\frac{1-|R(\omega,x)|^2}{1+|R(\omega,x)|^2}$$

(since $|u|^2+|v|^2=|A|^2+|B|^2=1$). But it follows from (8) that the dependence of the modulus of the reflection coefficient on x is of the form

$$|R(\omega,x)|=|R(\omega,0)|\exp(\pi^2\Omega_0 p^2 n(\omega)x).$$

Thus, if $n(\omega)>0$, we have at large x

$$N_+(\omega,\infty,x)=N_-(\omega),\quad N_-(\omega,\infty,x)=N_+(\omega)$$

with exponential accuracy. Thus, after passage of the pulse, all the excited atoms go over to the ground state and conversely. This means that the pulse energy increases linearly with x, meaning that the pulse draws an energy $\Omega_0\int n(\omega)d\omega$ per unit length.

As already mentioned, $R(\omega,0)$ is calculated from the input pulse $E_0(t)$. To this end, generally speaking, we must solve the system (5) for all ω. This can be done in the general case, of course, only for very special forms of the function $E_0(t)$. If, however, the firing pulse is weak enough, an expression for R can be obtained in explicit form. Namely, if $E(x,t)$ is small, the solution of (5) is given by the first Born approximation. In this case

$$R(\omega,x)=ip\int_{-\infty}^{\infty}\bar{E}(x,t)e^{2i\omega t}dt,$$

i.e., the reflection coefficient for weak pulses is simply the Fourier transform of E. [For just this reason, and also because the function $R(x)$ (8) is the result of the linear approximation of the system (1), (2), the transform $E\to R$ is frequently called the nonlinear Fourier transform also in the general case.] In our problem, $E_0(t)$ is concentrated near $t\geqslant 0$, therefore

$$R(\omega,0)=ip\int_{0}^{\infty}\bar{E}_0(t)e^{2i\omega t}dt. \tag{11}$$

This expression is analytic in the upper half-plane. We shall need hereafter the asymptotic form of $R(\omega,0)$ as $|\omega|\to\infty$, $\mathrm{Im}\,\omega>0$. This asymptotic form is determined by the behavior of $E_0(t)$ at zero t. We confine ourselves to the following two cases: 1) $E_0(t)$ has a δ-function singularity at zero, and 2) $E_0(t)$ has a power-law behavior at zero. In the former case, when $E_0(t)=E_{-1}\delta(t)$ + terms that are regular at zero, we have

$$R(\omega,0)=ipE_{-1}+O(1/\omega). \tag{12}$$

In the latter case, when

$$E_0(t)=E_\nu\left(\frac{t}{\tau}\right)^\nu\left(1+O\left(\frac{t}{\tau}\right)\right),\quad \nu>0$$

we have

$$R(\omega,0)=c_\nu\omega^{-(\nu+1)}(1+O(1/\omega)), \tag{13}$$

where

$$c_\nu=-i^\nu 2^{-(\nu+1)}\Gamma(\nu+1)p\bar{E}_\nu/\tau^\nu. \tag{14}$$

Without loss of generality, the quantities E_{-1} and E_ν can be regarded as real. We have introduced the notation E_{-1} in the first case, since formally (12) is obtained from (13) at $\nu=-1$.

To conclude this section, we call attention to the following circumstance. As already noted, $R(\omega,0)$ (11) is analytic in the upper ω half-plane. This holds true also in the general case of (not small) E_0 concentrated about $t\geqslant 0$. This circumstance together with the analytic properties of the function $\chi(\omega)$ guarantees causality of the solutions of our problem. Indeed, analyticity of $\chi(\omega)$ in the lower ω half-plane implies analyticity of the function $\tilde{\chi}(\omega)$ in the upper half-plane. It can then be easily seen from (8) and (9) that the integrand in (9) is analytic in the upper ω half-plane and vanishes as $|\omega|\to\infty$ at $x>t$. For this reason, the integral in the right-hand side of (9) vanishes at $x>t$ and $\chi(\omega)=\binom{1}{0}$ in this region. The latter means in turn that $E(x,t)=0$ at $x>t$, meaning that the solution does not "set out" behind the light cone and is causal (in contrast, e.g., to the spontaneous solutions discussed in Ref. 7).

3. ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS AT LARGE x

It is obviously impossible to obtain a solution of the inverse-problem Eqs. (9) in the general case. It is possible, however, to describe these solutions in sufficient detail at large x, i.e., in the case of "long" lasers. A similar situation obtains also for other equations that are integrable by the inverse-problem method—the asymptotics of the solutions at long times has been explicitly described for such problems as the nonlinear Schrödinger equation, the Korteweg–de Vries and sine-Gordon equations,[10] as well as for some non-one-dimensional systems.[11,12] Our problem, however, differs from those previously studied. The main difference is that in all the mentioned systems the temporal asymptotics are "quasi-linear," i.e., they coincide, roughly speaking, with the solutions of the corresponding linearized equations. The latter circumstance is certainly excluded in our problem, since the corresponding linear modes grow exponentially with the "time" x. The long-time solution becomes therefore essentially (and in a certain sense, extremely) nonlinear. Accordingly the formally mathematical aspect of the problem has likewise nothing in common with the technique developed in Refs. 10–12.

We shall study the solution of the system (9) at large x in a narrow region (that becomes narrower with increasing x) near the light cone $t=x$. It is precisely in this region, of approximate size $\ln\Omega x/x$, where the bulk of the solution is localized (all the energy drawn by the pulse from the medium is concentrated here). Recalling that the reflection coefficient $R(\omega,x)$ has all the qualitative properties of a Fourier transform, we can state that the behavior in a narrow region of size $\sim 1/x$ is governed by $R(\omega,x)$ with $\omega\gtrsim x$. At large ω the function $R(\omega,x)$ (8) is greatly simplified:

$$R(\omega,x)=R(\omega,0)\exp\left[2ix\left(\omega+\frac{\Omega^2}{\omega+i0}\right)\right], \tag{15}$$

where

$$\Omega^2=\frac{\pi\Omega_0p^2}{2}\int n(\omega)d\omega. \tag{16}$$

Equation (9) takes then the form

$$\chi(\omega)=\begin{pmatrix}1\\0\end{pmatrix}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{R(\omega',0)}{\omega'-\omega+i0}\exp\left\{2i\left[(x-t)\omega'+\frac{\Omega^2x}{\omega'+i0}\right]\right\}\tilde{\chi}(\omega')d\omega'. \tag{17}$$

We are interested in solutions of this equation at $t>x$ (causality leads to zero $E(x,t)$ at $t<x$).

We introduce the variable

$$z=4\Omega[x(t-x)]^{1/2} \tag{18}$$

and make the substitution $\omega\to\Lambda$, where

$$\Lambda=z\omega/4\Omega^2x. \tag{19}$$

In the new variables, Eq. (17) takes, with allowance with the asymptotic form (13) of $R(\omega,0)$, the form

$$\chi(\Lambda)=\begin{pmatrix}1\\0\end{pmatrix}-\frac{f_\nu(z/x)}{2\pi i}\int_C\frac{\mathscr{E}(\Lambda')}{\Lambda'-\Lambda}\frac{\tilde{\chi}(\Lambda')}{\Lambda'^{\nu+1}}d\Lambda', \tag{20}$$

where

$$f_\nu\left(\frac{z}{x}\right)=c_\nu\left(\frac{z}{4\Omega^2x}\right)^{\nu+1},\quad \mathscr{E}(\Lambda')=\exp\left[-i\frac{z}{2}\left(\Lambda'-\frac{1}{\Lambda'}\right)\right]. \tag{21}$$

The integration contour C in (20) goes from $-\infty$ to $+\infty$ and circles around the singularity of the integrand ($\Lambda'=\Lambda$, $\Lambda'=0$) from above.

We note that even at this stage the inhomogeneous broadening has dropped out of the problem. It is easy to verify that the transition from the system (9) to (20) is legitimate if the following inequality is satisfied:

$$\Delta\omega \ll x\Omega^2/z,$$

where $\Delta\omega$ is the characteristic width of the function $n(\omega)$. This condition has a clear meaning: In the vicinity of the light cone the field is determined by the high-frequency harmonics of $E_0(t)$, whose dispersion law (15) is insensitive to the form of the inhomogeneous broadening.

Equation (20) is the main object of the investigation that follows. It is convenient at the same time to have the equation obtained from (20) by applying to the latter the operation designated by the tilde symbol:

$$\tilde{\chi}(\Lambda)=\begin{pmatrix}0\\1\end{pmatrix}-\frac{\tilde{f}_\nu(z/x)}{2\pi i}\int_{\tilde{C}}\frac{\mathscr{E}(\Lambda')}{\Lambda'-\Lambda}\frac{\chi(\Lambda')}{\Lambda'^{\nu+1}}d\Lambda'. \quad (22)$$

The integration contour $\tilde{C}$ in (22) passes from $-\infty$ to $+\infty$ in the lower half-plane.

We consider first the solution of the system (20)–(22) when $f_\nu(z/x)$ is small and z is not too large, so that the solution of this system can be obtained by simple iteration:

$$\chi(\Lambda)=\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\frac{f_\nu(z/x)}{2\pi i}\int_C\frac{\mathscr{E}(\Lambda')}{\Lambda'-\Lambda}\frac{d\Lambda'}{\Lambda'^{\nu+1}}.$$

The asymptotic expression for the second component of $\chi(\Lambda)$ is here

$$\chi_2(\Lambda)=\frac{1}{\Lambda}\frac{f_\nu(z/x)}{2\pi i}\int_C\frac{\mathscr{E}(\Lambda')}{\Lambda'^{\nu+1}}d\Lambda'.$$

Substitution of this expression in (19), with (19) taken into account, yields

$$p\bar{E}(x,t)=\frac{4\Omega^2 x}{i\pi z}f_\nu\left(\frac{z}{x}\right)\int_C\frac{\mathscr{E}(\Lambda)}{\Lambda^{\nu+1}}d\Lambda.$$

The integral in the right-hand side of this equation is a Bessel function of imaginary argument (the integration contour C can be deformed into the unit circle), therefore

$$p\bar{E}(x,t)=-\frac{8\Omega^2 x}{z}f_\nu\left(\frac{z}{x}\right)J_\nu(-iz). \quad (23)$$

At small z, when J_ν can be replaced by the first term of its expansion, Eq. (23), with allowance for (21), (14), and (18), leads to

$$\bar{E}(x,t)=\bar{E}_\nu\left(\frac{t-x}{\tau}\right)^\nu, \quad (23a)$$

i.e., the pulse preserves its initial shape in the vicinity of the front. However, the region of applicability of (23a), as seen from (23), becomes rapidly narrower with increasing x.

Equation (23) remains valid also at large z if f_ν is at the same time small enough (larger than x). In this case $J_\nu(-iz)$ can be replaced by the first term of asymptotic expansion, so that

$$p\bar{E}(x,t)=-\frac{8\Omega^2 x}{z}f_\nu\left(\frac{z}{x}\right)\frac{e^z}{(2\pi z)^{1/2}}e^{-i\pi\nu/2}. \quad (23b)$$

The region of validity of this expression is given by the inequalities

$$z\gg 1,\quad f_\nu\left(\frac{z}{x}\right)\frac{e^z}{z^{1/2}}\ll 1. \quad (24)$$

The solution in this region is still given by the linear theory.

Expression (23b) shows that at small f_ν the main interest attaches to the region of large z. It is just in this region that the "entire" solution is concentrated. We shall therefore assume hereafter, as before, that $z\gg 1$ and will gradually relax the second inequality of (24).

The integration contours C and $\tilde{C}$ in Eqs. (20) and (22) can be arbitrarily deformed in the upper or lower Λ half-planes, respectively. The exponentials in the integrands, however, have saddle points, namely $\Lambda'=i$ in (20) and $\Lambda'=-i$ in (22). The vicinities of the saddle points give the main (exponentially large) contributions to the integrals (20) and (22) at large z. It is therefore natural to use the saddle-point method when calculating these integrals at $z\gg 1$. Drawing in suitable manner the contours C and $\tilde{C}$ through the saddle points and formally calculating the first term of the asymptotic expansion of the integrals (20) and (22), we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_C\frac{\mathscr{E}(\Lambda')}{\Lambda'-\Lambda}\frac{\chi(\Lambda')d\Lambda'}{\Lambda'^{\nu+1}}=e^{-i\pi\nu/2}\frac{e^z}{(2\pi z)^{1/2}}\frac{\tilde{\chi}(i)}{\Lambda-i},\\&\frac{1}{2\pi i}\int_{\tilde{C}}\frac{\mathscr{E}(\Lambda')}{\Lambda'-\Lambda}\frac{\chi(\Lambda')}{\Lambda'^{\nu+1}}d\Lambda'=-e^{i\pi\nu/2}\frac{e^z}{(2\pi z)^{1/2}}\frac{\chi(-i)}{\Lambda+i}.\end{aligned} \quad (25)$$

Equations (20) and (22) take thus the form

$$\begin{aligned}&\chi(\Lambda)=\begin{pmatrix}1\\0\end{pmatrix}-\beta_\nu e^z\frac{\chi(i)}{\Lambda-i},\\&\chi(\Lambda)=\begin{pmatrix}0\\1\end{pmatrix}+\beta_\nu e^z\frac{\chi(-i)}{\Lambda+i},\end{aligned} \quad (26)$$

where

$$\beta_\nu(z,x)=\frac{f_\nu(z/x)}{(2\pi z)^{1/2}}e^{-i\pi\nu/2}. \quad (27)$$

The solution of the system (26) is trivial: Putting $\Lambda=-i$ and $\Lambda=i$ in the first and second of these equations, respectively, we obtain a closed system of linear algebraic equations for $\chi(-i)$ and $\tilde{\chi}(i)$; the solution of this system is

$$\tilde{\chi}(i)=\left[\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\0\end{pmatrix}\frac{\beta_\nu}{2i}e^z\right]\left(1+\frac{|\beta_\nu|^2}{4}e^{2z}\right)^{-1}.$$

This determines also $\chi(\Lambda)$ (26). The asymptotic form of $\chi(\Lambda)$, which determines $E(x,t)$ in accord with (10), is

$$\chi_2(\Lambda)=-\frac{1}{\Lambda}\beta_\nu e^z\left(1+\frac{|\beta_\nu|^2}{4}e^{2z}\right)^{-1}+O\left(\frac{1}{\Lambda^2}\right).$$

Thus,

$$p\bar{E}(x,t)=-\frac{8\Omega^2 x}{z}\beta_\nu e^z\left(1+\frac{|\beta_\nu|^2}{4}e^{2z}\right)^{-1}. \quad (28)$$

The region of validity of the obtained expression will be discussed somewhat later. At present we consider (28) in detail. We recall the explicit expression for $\beta_\nu(z,x)$:

$$\beta_\nu(z,x)=-\frac{p\bar{E}_\nu}{\tau^\nu}\frac{\Gamma(\nu+1)}{(2\pi z)^{1/2}}\left(\frac{z}{8\Omega^2 x}\right)^{\nu+1}.$$

We introduce the notation

$$a(z,x)=\ln\frac{2}{|\beta_\nu(z,x)|}. \quad (29)$$

Assuming E_ν to be real and positive, we rewrite (28) in the form

$$pE(x,t)=8\Omega^2\frac{x}{z}\frac{1}{\operatorname{ch}(z-a(z,x))}. \quad (30)$$

At large x the maximum of this expression lies at the point z_1, where the argument of the hyperbolic cosine vanishes,

$$z_1=a(z_1,x).$$

The solution of this equation is

$$z_1(x)=\lambda_\nu(x)-\left(\nu+\frac{1}{2}\right)\ln\lambda_\nu(x)+O\left(\frac{\ln\lambda_\nu}{\lambda_\nu}\right), \quad (31)$$

where

$$\lambda_\nu(x)=\ln\left[\frac{(8\Omega^2x)^{\nu+1}}{pE_\nu}\frac{2^{1/2}\pi\tau^\nu}{\Gamma(\nu+1)}\right]. \quad (32)$$

We assume that $\lambda_\nu(x)\gg1$. (We note that λ_ν can be increased either by increasing x or by decreasing E_ν, i.e., by weakening the entering pulse.)

In the vicinity of its maximum, Eq. (30) expressed in terms of the initial variables x and t takes the form

$$pE(x,t)=\frac{8\Omega^2x}{z_1(x)}\operatorname{ch}^{-1}\left[\frac{8\Omega^2x}{z_1(x)}(t-x-\xi_1)\right],\quad \xi_1=\left(\frac{z_1(x)}{4\Omega}\right)^2\frac{1}{x}. \quad (33)$$

Equation (33) is none other than a soliton (2π pulse) with increasing amplitude and decreasing width. The soliton velocity barely exceeds that of light, so that the soliton overtakes the front $t=x$ of the pulse, and the distance from the front is ξ_1.

We consider now the region where the solution obtained is applicable. In the calculation of the integrals in (25) we have confined ourselves to the principal term of the asymptotic expansion in powers of $1/z$. This approximation is valid if the function is not small compared with its derivatives at the saddle point. From the obtained solution, however, it follows that, say at large z, the quantity $\chi_1(-i)$ decreases exponentially:

$$\chi_1(-i)\approx\frac{4}{|\beta_\nu|^2}e^{-2z}.$$

On the other hand $\partial\chi/\partial\Lambda|_{\Lambda=-i}$ calculated from the first formula of (26) is of the order of unity (more accurately, equal to $i/2$) at large z. The solution obtained is therefore valid so long as the inequality

$$\chi_1(-i)>\frac{1}{z}\frac{\partial\chi_1}{\partial\Lambda}\bigg|_{\Lambda=-i}$$

is satisfied, i.e., if z is such that

$$|\beta_\nu|e^z<z^{1/2}. \quad (34)$$

It is useful to compare (34) with the region of applicability of the linear approximation (24); expressed in terms of β_ν, this region is given by the inequality

$$|\beta_\nu|e^z<1.$$

Solving (34) with respect to z, we obtain

$$z<z_1+{}^1/_2\ln\lambda_\nu(x), \quad (35)$$

where z_1 is the coordinate of the center of the soliton.

The soliton (30) is wholly located in the applicability region of the approximation employed if

$$\ln\lambda_\nu(x)\gg1.$$

Thus, if this condition is satisfied, a clearly pronounced 2π pulse is present in the region (35). In the region supplementary to (35), however, the approximation employed yields no information whatever on the true solution. It is clear, however, how our approximation should be improved: It is necessary to take into account in the expansions of the integrals (25) the terms of next order in $1/z$. Thus, formal calculation of these integrals by the saddle-point method accurate to $1/z^{N+1/2}$ inclusive yields an expression for the function $\chi(\Lambda)$ in terms of $\tilde\chi(\Lambda)$ and its first $2N$ derivatives taken at $\Lambda=i$; $\tilde\chi(\Lambda)$ is expressed in terms of $\chi(\Lambda),\chi'(\Lambda)\ldots\chi^{(2N)}(\Lambda)|_{\Lambda=-i}$. Just as above, these expressions allow us to write a closed system of linear algebraic equations for $\tilde\chi,\ldots,\tilde\chi^{(2N)}|_{\Lambda=i}$ and $\chi,\ldots,\chi^{(2N)}|_{\Lambda=-i}$, and the solution of these equations gives an expression for $E(x,t)$. The calculations, however, become very cumbersome even at $N=1$. We therefore do not present them here, and confine ourselves to a statement of the answer, all the more since this answer requires much less labor when obtained on the basis of the results of the next section.

The procedure indicated yields an expression for the solution $E(x,t)$ in the region

$$z<z_1+(N+{}^1/_2)\ln\lambda_\nu(x).$$

If $\ln\lambda_\nu(x)\gg1$, the solution in this region is represented by a superposition of $N+1$ 2π pulses (more accurately, a sequence of 2π and -2π pulses):

$$pE(x,t)=\sum_{k=0}^{N}(-1)^k\frac{8\Omega^2x}{z_{k+1}(x)}\operatorname{ch}^{-1}\left[\frac{8\Omega^2x}{z_{k+1}(x)}(t-x-\xi_{k+1})\right], \quad (36)$$

where $z_{k+1}=z_1(x)+k\ln\lambda_\nu(x)$, and the coordinate of the center of the $(k+1)$st soliton is $\xi_{k+1}=z_{k+1}^2/16\Omega^2x$. The soliton amplitudes decrease with increasing k, and their widths increase. The ratio of the pulse width to their spacing is $[\ln\lambda_\nu(x)]^{-1}$, i.e., the solitons are well separated. Thus, at $\ln\lambda_\nu\gg1$ the soliton front consists of a sequence of solitons; expression (36) is applicable at $N<N_{max}\sim\lambda_\nu/\ln\lambda_\nu$, so that a different approach is necessary to obtain the solution at $z\gtrsim z_1+N_{max}\ln\lambda_\nu$.

4. QUASI-SELF-SIMILAR SOLUTIONS

As already noted, the dependence of the reflection coefficient on x is significantly simplified in the vicinity of the light cone: Equation (8) is replaced by (15). The equations of the inverse-problem are correspondingly simpler: if the inequality $x/z\gg\Delta\omega/\Omega^2$ holds, the solutions of the system (5) satisfy Eq. (17). It is known[3,9] that in this case

$$U=p\int_{-\infty}^{t}E(x,t')dt' \quad (37)$$

is the solution of the sine-Gordon equation[4)]

$$U_{xt}+U_{tt}=4\Omega^2\sin U. \quad (38)$$

Since $R(\omega,x)|_{\omega=0}=\infty$ [see (15)], the boundary conditions for $U(x,t)$ take here the form[7]

$$U(x,t)=\pi \text{ at } t<x, \quad U(x,\infty)=0 \pmod{2\pi}.$$

Equation (38) has self-similar solutions that depend only on $z=4\Omega[x(t-x)]^{1/2}$, such that

$$U_{zz}+U_z/z=\sin U. \tag{39}$$

Those solutions of this equation which are regular at zero make up a one-parameter family; it is convenient to choose the parameter that numbers these solutions to be the value of $U(z)$ at $z=0$,

$$U(U_0,z)|_{z=0}=U_0.$$

The field $E(x,t)$ corresponding to the self-similar solution is given by

$$pE(x,t)=8\Omega^2\frac{x}{z}\frac{\partial U}{\partial z}, \quad z>0. \tag{40}$$

From among the solutions described in the preceding section, as seen from (20), only those for which (21) is independent of x are self-similar. In this case the $\chi(\Lambda)$ depend only on z and the corresponding expression for $E(x,t)$, as follows from (19) and (10), coincides in form with (40). But f_ν [Eq. (21)] is independent of x only at $\nu=-1$, i.e., the self-similar solutions correspond to a δ-like firing pulse $E_0(t)=E_{-1}\delta(t)$. It can be easily seen that the complete expression for $E(x,t)$ is

$$E(x,t)=E_{-1}\delta(t-x)+8\Omega^2\frac{x}{z}\theta(t-x)\frac{\partial U}{\partial z}(U_0,z), \tag{41}$$

where θ is the usual step function, $\theta(\xi)=0$ at $\xi<0$ and $\theta(\xi)=1$ at $\xi>0$. The parameter U_0 is then a function of E_{-1}. The function $U_0(E_{-1})$ is particularly simple at small E_{-1}, when the solution of Eq. (20) at small z is obtained by simple iteration [see (23)]. In this case, comparing (12), (23), and (40), we obtain $U_0=pE_{-1}$.

As for the remaining solutions of the system (1), (2), they are also functions that depend mainly on z at large x and at sufficiently large z. This follows directly from the system (20), (22), inasmuch as at large x and sufficiently large z they are also functions that depend mainly on z. This follows directly from the system (20), (22), inasmuch as at large z the main contributions to the integrals in the right-hand sides of (20) and (22) are made by the vicinities of the saddle points. These contributions are proportional to e^z, and the dependence of $\chi(\Lambda)$ on x via $f_\nu(z/x)$ is thus merely logarithmic [see, e.g., (29) and (30)]. Accordingly, the expressions for $E(x,t)$ will likewise have the structure of (40), but U must now be regarded as a slow function of x. It is important to note that such solutions of (38), which are weakly dependent on the non-self-similar variables, can be constructed by starting from the family of the self-similar solutions (39). Namely, the parameter U_0, which is constant for the self-similar solution, can be made an arbitrary function of the ratio z/x. The function $U(U_0(z/x),z)$ will then differ little from the true solution of Eq. (38).

In fact, in terms of the coordinates x and z, Eq. (38) takes the form

$$2\frac{x}{z}\frac{\partial}{\partial z}\left(\frac{\partial U}{\partial x}+\frac{z}{2x}\frac{\partial U}{\partial z}\right)-\sin U=0. \tag{42}$$

Let $U(\gamma,z)$ satisfy Eq. (39) at all γ and let furthermore the parameter γ be a slow function of x and z. More accurately, we assume that

$$\frac{\partial\gamma}{\partial x}=O\left(\frac{\varepsilon}{x}\right), \quad \frac{\partial\gamma}{\partial z}=O\left(\frac{\varepsilon}{z}\right), \tag{43}$$

where ε is a small parameter. (Obviously, γ is an arbitrary function of the paremeter U_0 introduced above.) The derivatives U_z and U_{zz} in (39) must then be taken to mean the partial derivatives of the function $U(\gamma,z)$ with respect to the second argument. We calculate now the left-hand side of (42) with U as indicated. By virtue of (40) and (43), this left-hand side becomes

$$\frac{2}{z}\frac{\partial U_z}{\partial\gamma}\left(\frac{\partial\gamma}{\partial\ln x}+\frac{\partial\gamma}{\partial\ln z}\right)+O(\varepsilon^2).$$

This expression is of the order of ε^2 if γ satisfies the equation

$$\frac{\partial\gamma}{\partial\ln x}+\frac{\partial\gamma}{\partial\ln z}=0,$$

i.e., if it is a function of the ratio z/x, $\gamma=\gamma(z/x)$. In this case $U(\gamma,z)$ satisfies (42) accurate to terms of order ε^2.

The function $U(U_0(z/x),z)$ is thus a good approximation of the true solution (38). The "slowness" of the dependence of U on z/x is ensured by the fact that at small U_0 the solution $U(U_0,z)$ (and only such solutions will be needed hereafter) depends on U_0 only logarithmically (in fact, even more weakly).

It remains for us to establish the connection between $U_0(z/x)$ and the entering pulse $E_0(t)$. This can be done in the following manner. At small U, Eq. (39) can be replaced by its linear part

$$U_{zz}+U_z/z=U.$$

The solution, regular at zero, of this equation is a Bessel function of imaginary argument:

$$U=U_0I_0(z).$$

If $\ln(1/U_0)\gg1$, this expression is a suitable solution of (39) also at sufficiently large z, when I_0 can be replaced by its asymptotic form [which is the same for all $I_\nu(z)$]:

$$U=U_0e^z(2\pi z)^{-1/2}.$$

The corresponding expression for the field E takes the form (40)

$$pE(x,t)=8\Omega^2\frac{xU_0}{z}I_1(z)\approx8\Omega^2\frac{x}{z}\frac{e^z}{(2\pi z)^{1/2}}U_0. \tag{44}$$

On the other hand, we have in the same region the answer calculated in the linear approximation [Eqs. (23) and (23b)]. The last of these equations, written directly in the notation used for the firing pulse, takes the form

$$pE(x,t)=8\Omega^2\frac{x}{z}I_\nu(z)\Gamma(\nu+1)\frac{pE_\nu}{\tau^\nu}\left(\frac{z}{8\Omega^2x}\right)^{\nu+1}. \tag{45}$$

Comparison of (45) and (44) yields in fact that sought expression for $U_0(z/x)$:

$$U_0\left(\frac{z}{x}\right)=\Gamma(\nu+1)\frac{pE_\nu}{\tau^\nu}\left(\frac{z}{8\Omega^2x}\right)^{\nu+1}. \tag{46}$$

We present now the final expression for $E(x,t)$ (40),

$$pE(x,t)=8\Omega^2\frac{x}{z}U_z\left(\Gamma(\nu+1)\frac{pE_\nu}{\tau^\nu}\left(\frac{z}{8\Omega^2 x}\right)^{\nu+1},z\right). \quad (47)$$

We recall that the firing pulse $E_0(t)$ is given at zero t by $E_0(t)=E_\nu(t/\tau)^\nu+O(t/\tau)^\nu$. It is only in terms of the parameters of this "sprout" of $E_0(t)$ that the field in the pulse at the output of a long amplifier is given. The solution (47) does not remember at all the subsequent behavior of $E_0(t)$.

It must be emphasized once more that Eq. (23), which plays a very important role in our approach, follows directly from the linear theory and need not be verified by resorting to the inverse-problem equations used to derive it in the preceding sections merely to save space.

We indicate, finally, the limits of applicability of the solution (47). It is limited, first, by the inequality $x/z \gg \Omega^{-2}\Delta\omega$, which determines the possibility of neglecting the inhomogeneous broadening. This limitation, however, is not burdensome. A more serious restriction is the following. With increasing x, Eq. (47) becomes first applicable when Eq. (45) becomes suitable at sufficiently large z that ensure the possibility of replacing the ratio $I_\nu(z)/I_1(z)$ by unity, i.e., when

$$z\gg 1,\quad I_\nu(z)\Gamma(\nu+1)\frac{pE_\nu}{\tau^\nu}\left(\frac{z}{8\Omega^2 x}\right)^{\nu+1}\ll 1.$$

Solving the latter inequality, we obtain

$$1\ll z<\ln\frac{(8\Omega^2 x)^{\nu+1}\tau^\nu}{\Gamma(\nu+1)pE_\nu}-\left(\nu+\frac{1}{2}\right)\ln\ln\frac{(8\Omega^2 x)^{\nu+1}\tau^\nu}{\Gamma(\nu+1)pE_\nu}=Z(x),$$

i.e., $Z(x)\gg 1$. Let the length x_0 be such that $Z(x_0)$ is large at the accuracy convenient for us (the ratio $I_\nu(z)/I_0(z)-1$ is of the order of $1/z$ at large z). At $x>x_0$ Eq. (47) is then suitable so long as z/x does not exceed $Z(x_0)/x_0$, i.e., $z<xZ(x_0)/x_0$. [We simply do not know the function $U_0(z/x)$ for larger z.] Thus, the region of applicability of (47) with respect to z increases linearly with x. The maximum characteristics scale of the solution, on the other hand, is of the order of

$$\ln\ln\left[(8\Omega^2 x)^{\nu+1}\tau^\nu/pE_\nu\right]$$

[the distance between neighboring maxima of $E(z,x)$]. Therefore almost the entire solution is contained in the region of validity of (47) even at small ratios x/x_0. Finally the last restriction is due to the power-law approximation of $E_0(t)$ at zero. It is clear that we can confine ourselves to such a growth of E_0 if the characteristic time scale of E (47) is small compared with the pulse turning-on time, i.e., if

$$t(x)=\ln\frac{(8\Omega^2 x)^{\nu+1}\tau^\nu}{pE_\nu}\ln\ln\frac{(8\Omega^2 x)^{\nu+1}\tau^\nu}{pE_\nu}\Big/8\Omega^2 x\ll\tau.$$

As for the degree to which (47) is explicit and effective, even though in the general case the solution of (39) is not expressed in terms of any tabulated function (these are so-called Painlevê transcendentals of type III), in the limit of interest to use $\ln(1/U_0)\gg 1U(U_0,z)$, can be approximated by elliptic functions. This can be easily understood by noting that Eq. (39) can be interpreted as the equation of motion of a Newtonian particle in a potential $\cos U$ with a friction inversely proportional to the time. This mechanical analogy provides an exhaustive qualitative picture of the behavior of the solutions $U(U_0,z)$ which will not be described here.

[1)] Lamb[5] considered also the case of a narrow line $[n(\omega)\sim\delta(\omega)]$. His analysis of this case, however, is incorrect.

[2)] Another shortcoming of the model is the neglect of relaxation processes. It can be shown, however, that, e.g., allowance for the so-called transverse relaxation does not alter the qualitative picture of the solution, and that the only dissipative mechanism that limits the growth and contraction of the pulse is connected with the conductivity of the medium at the transition frequency. It is precisely the allowance for conductivity that imposes the upper bound on the value of x at which the obtained solutions are valid.

[3)] We confine ourselves to a soliton-free sector, when $A(\omega)$ has no zeros in the lower ω half-plane.

[4)] To this end it is necessary to satisfy, besides (15), also the condition $R(-\omega,0)=-\bar{R}(\omega,0)$ that guarantees that $E(x,t)$ is real. Without loss of generality, however, we can assume this condition to be satisfied, since the phase of $E(x,t)$ in the vicinity of the cone coincides with the phase of $E_0(t)$ at the point $t=0$ and is thus constant. We shall take hereafter this phase to be zero.

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