Electromagnetic shock waves in a magnetized vacuum

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The dynamics of nonlinear electromagnetic waves in a magnetized vacuum are investigated. Equations are obtained for interacting differently polarized weakly linear fields. The solutions are obtained in the form of plane waves of *TE* polarization with an electric-field vector perpendicular to the propagation direction and to the stationary magnetic field. It is shown that long *TE* waves, whose dispersion can be neglected, evolve with formation of discontinuities. The structure of the front of a *TE* shock wave of low intensity is determined by the weak high-frequency dispersion and is oscillatory.

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1. INTRODUCTION

In strong field comparable with $B_c = m^2c^3/e\hbar$ (e and m are the charge and mass of the electron), the nonlinearity of the electrodynamics equations in vacuum becomes significant. In particular, the magnetic field influences the propagation of the electromagnetic radiation[1]: the vacuum becomes a birefringent and absorbing medium (the latter because of production of electron-positron pairs by γ quanta of energy $\hbar\omega > 2mc^2$). In the case of propagation at an angle $\theta \neq 0$ to the uniform magnetic field $\mathbf{B}_0$ there exist two normal waves (modes) with high-frequency dispersion. At low frequencies ($\hbar\omega \ll mc^2B_c/|\mathbf{B}_0|$) there is no dispersion; the phase velocity of each of the waves, being smaller than the velocity of light c, coincides with the group velocity.[2] The polarization of the normal waves is linear. In a TE wave ($\parallel$ polarization in the terminology of Refs. 1 and 2) the electric field is perpendicular to the plane containing the wave vector $\mathbf{k}$ and the vector $\mathbf{B}_0$. In a TM mode ($\perp$ polarization), the perturbation of the magnetic field is perpendicular to the indicated plane.

It should be noted that the phase velocity of a TE mode exceeds the corresponding value for a TM mode. This makes possible a parametric decay $\gamma_{\text{TE}} \to \gamma_{\text{TM}} + \gamma_{\text{TM}}$, which in this case is the only allowed three-wave interaction.[2] For this decay it is possible to satisfy exactly the photon energy and momentum conservation laws or, in different language, the synchronism conditions of the interacting quasimonochromatic waves:

$$\omega_1 = \omega_2 + \omega_3, \quad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3. \tag{1.1}$$

Here $\omega_1, \mathbf{k}_1$ and $\omega_{2,3}, \mathbf{k}_{2,3}$ are the frequencies and wave vectors of the TE and TM waves, respectively.

If the weak dispersion at frequencies $\omega \ll mc^2B_c/\hbar|\mathbf{B}_0|$ is neglected, the conservation laws (1.1) can be satisfied also for a nonlinear interaction of collinear waves of one type (e.g., $\gamma_{\text{TE}} \to \gamma_{\text{TE}} + \gamma_{\text{TE}}$). This leads to self-action of waves of finite amplitude, due to successive generation of harmonics, and consequently also to a change of the wave profile. Under certain conditions such an evolution leads to formation of stationary waves, in which the nonlinear deformation of the profile is offset by its dispersion spreading. The described evolution is possible for finite-amplitude waves under the condition that the nonlinear interaction is faster than the phase shift of the interacting waves due to deviation from the exact synchronism (1.1). It is precisely in this case that effective interaction becomes possible between those harmonics for which, strictly speaking, the synchronism conditions (1.1) cannot be satisfied because of dispersion.

We investigate in this article the dynamics of nonlinear electromagnetic waves in a magnetized vacuum. We obtain equations that describe, in first-order in the nonlinearity (in the small amplitude of the field oscillations), interacting TE and a TM waves propagating in a narrow solid angle. One-dimensional solutions of these equations are obtained in the form of simple TE waves[1)] that evolve with formation of discontinuities. The boundary conditions on the TE discontinuity and the structure of the front of low-intensity TE shock waves are considered.

2. EQUATIONS OF WEAKLY NONLINEAR WAVES

An electromagnetic field in a vacuum, with account taken of the radiative corrections, is characterized by a Lagrangian density[1] $L = L_0 + L_1$, in which

$$L_0 = (\mathfrak{E}^2 - \mathfrak{B}^2)/8\pi,$$

$$L_1 = \frac{m^4}{8\pi^2}\int_0^\infty \frac{e^{-\eta}}{\eta^3}\left\{-(\eta a \operatorname{ctg} \eta a)(\eta b \operatorname{cth} \eta b) + 1 - \frac{\eta^2}{3}(a^2 - b^2)\right\} d\eta. \tag{2.1}$$

Here $\mathfrak{E}$ and $\mathfrak{B}$ are the intensities of the electric and magnetic fields,

$$\begin{aligned} a &= -\frac{i}{\sqrt{2}B_c}\left\{\left[\frac{\mathfrak{B}^2 - \mathfrak{E}^2}{2} + i(\mathfrak{B}\mathfrak{E})\right]^{1/2} - \left[\frac{\mathfrak{B}^2 - \mathfrak{E}^2}{2} - i(\mathfrak{B}\mathfrak{E})\right]^{1/2}\right\}, \\ b &= \frac{1}{\sqrt{2}B_c}\left\{\left[\frac{\mathfrak{B}^2 - \mathfrak{E}^2}{2} + i(\mathfrak{B}\mathfrak{E})\right]^{1/2} + \left[\frac{\mathfrak{B}^2 - \mathfrak{E}^2}{2} - i(\mathfrak{B}\mathfrak{E})\right]^{1/2}\right\}. \end{aligned} \tag{2.2}$$

In Eqs. (2.1) and (2.2) and hereafter we use a system of units in which $\hbar = c = 1$. We introduce the dimensionless fields

$$\mathbf{E} = \mathfrak{E}/B_c, \quad \mathbf{B} = \mathfrak{B}/B_c. \tag{2.3}$$

Expanding the correction L_1 to the Lagrangian density in powers of a and b we obtain for weak fields ($|\mathbf{E}| \ll 1$, $|\mathbf{B}| \ll 1$), accurate to $|\mathbf{E}|^6 \sim |\mathbf{B}|^6$, the expression

$$L_1 = \frac{\alpha B_c^2}{360\pi^2}\left[(\mathbf{E}^2 - \mathbf{B}^2)^2 + 7(\mathbf{E}\mathbf{B})^2 + \frac{26}{7}(\mathbf{E}^2 - \mathbf{B}^2)(\mathbf{E}\mathbf{B})^2 + \frac{4}{7}(\mathbf{E}^2 - \mathbf{B}^2)^3\right], \tag{2.4}$$

where $\alpha \approx 1/137$ is the fine-structure constant.

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The equations of the field in vacuum are of the form[1]

$$\operatorname{div}\mathbf{B}=0,\quad \operatorname{div}(\mathbf{E}+4\pi\mathbf{P})=0,$$

$$\operatorname{rot}\mathbf{E}=-\partial\mathbf{B}/\partial t,\quad \operatorname{rot}(\mathbf{B}-4\pi\mathbf{M})=\frac{\partial}{\partial t}(\mathbf{E}+4\pi\mathbf{P}),\tag{2.5}$$

where the polarization $\mathbf{P}=B_c^{-2}\partial L_1/\partial\mathbf{E}$ and the magnetization $\mathbf{M}=B_c^{-2}\partial L_1/\partial\mathbf{B}$. For weak fields we have from (2.4)

$$\mathbf{P}=\frac{\alpha}{90\pi^2}\Big[(\mathbf{E}^2-\mathbf{B}^2)\mathbf{E}+\frac{7}{2}(\mathbf{EB})\mathbf{B}+\frac{13}{7}(\mathbf{E}^2-\mathbf{B}^2)(\mathbf{EB})\mathbf{B}+\frac{13}{7}(\mathbf{EB})^2\mathbf{E}+\frac{6}{7}(\mathbf{E}^2-\mathbf{B}^2)^2\mathbf{E}\Big],$$

$$\mathbf{M}=\frac{\alpha}{90\pi^2}\Big[-(\mathbf{E}^2-\mathbf{B}^2)\mathbf{B}+\frac{7}{2}(\mathbf{EB})\mathbf{E}+\frac{13}{7}(\mathbf{E}^2-\mathbf{B}^2)(\mathbf{EB})\mathbf{E}-\frac{13}{7}(\mathbf{EB})^2\mathbf{B}-\frac{6}{7}(\mathbf{E}^2-\mathbf{B}^2)^2\mathbf{B}\Big].\tag{2.6}$$

We consider hereafter weakly nonlinear waves propagating against the background of a stationary uniform magnetic field $\mathbf{B}_0=(B_{0x},B_{0y},0)$. We can then retain in $\mathbf{P}$ and $\mathbf{M}$ only the nonlinear terms that are quadratic in the amplitude, since the higher-order terms give small (or slowly increasing) corrections to the expressions that describe the strongest nonlinear effects (parametric decay and harmonic generation).

The coefficients in the linear terms will be obtained in the first nonvanishing approximation in the parameters $\alpha\ll 1$, $|\mathbf{B}_0|\ll 1$. We note that in the linear approximation the correction to the phase velocity of the waves is of the order of $\alpha|\mathbf{B}_0|^2\ll 1$. The nonlinear correction to the phase velocity, on the other hand, will be shown below to be of the order $\alpha|\mathbf{B}_0|^3|\mathbf{B}|$, where $\mathbf{B}$ is the perturbation of the magnetic field.

Taking into account the possibility of the decay of a TE wave into TM waves that travel at a small angle to it, we shall assume also that the wave field comprises a beam of waves propagating at small angles ($\sim\alpha^{1/2}|\mathbf{B}_0|$) to the x axis. According to Ref. 1 we can use in this case, when calculating the values of $\mathbf{P}$ and $\mathbf{M}$ that enter in the equations of the field with the small parameter $\alpha|\mathbf{B}_0|^2$, a collinear approximation within the framework of which $\partial/\partial y=\partial/\partial z=0$. In addition, it is necessary to put in this approximation

$$B_x=0,\quad E_x=0,\quad \partial E_z/\partial x=\partial B_y/\partial t,\quad \partial E_y/\partial x=-\partial B_z/\partial t.\tag{2.7}$$

These relations follow from the field equations (2.5) if we neglect in them the terms $\lesssim\alpha^{1/2}|\mathbf{B}_0|$.

From Eqs. (2.5) we easily obtain

$$\Delta\mathbf{B}-\partial^2\mathbf{B}/\partial t^2=4\pi\{\Delta\mathbf{M}-\nabla(\nabla\mathbf{M})-\operatorname{rot}(\partial\mathbf{P}/\partial t)\}.$$

Introducing the operator $\hat{L}$ of linearization in terms of the amplitude of the perturbation of the field $|\mathbf{B}|\sim|\mathbf{E}|$ and the operator $\hat{N}_2$ that leaves only terms quadratic in the amplitude, we obtain for the components of the perturbation of the magnetic field $\mathbf{B}$ the following equations:

$$\partial^2B_z/\partial x^2-\partial^2B_z/\partial t^2+\Delta_\perp B_z-4\pi\hat{L}(\partial^2M_z/\partial x^2-\partial^2P_y/\partial x\,\partial t)=4\pi\frac{\partial}{\partial x}\hat{N}_2\Big(\frac{\partial M_z}{\partial x}-\frac{\partial P_y}{\partial t}\Big);\tag{2.8}$$

$$\frac{\partial^2B_y}{\partial x^2}-\frac{\partial^2B_y}{\partial t^2}+\Delta_\perp B_y-4\pi\hat{L}\Big(\frac{\partial^2M_y}{\partial x^2}+\frac{\partial^2P_z}{\partial x\partial t}\Big)=4\pi\frac{\partial}{\partial x}\hat{N}_2\Big(\frac{\partial M_y}{\partial x}+\frac{\partial P_z}{\partial t}\Big),\tag{2.9}$$

where $\Delta_\perp=\partial^2/\partial y^2+\partial^2/\partial z^2$. From (2.6), taking (2.7) into account, we easily find that

$$\hat{L}P_y=\frac{\alpha}{90\pi^2}\Big[B_{0y}^2\Big(\frac{7}{2}-\frac{13}{7}\mathbf{B}_0^2\Big)-\mathbf{B}_0^2\Big(1-\frac{6}{7}\mathbf{B}_0^2\Big)\Big]E_y,$$

$$\hat{L}P_z=\frac{\alpha}{90\pi^2}\Big[-\mathbf{B}_0^2+\frac{6}{7}\mathbf{B}_0^4\Big]E_z,$$

$$\hat{L}M_y=\frac{\alpha}{90\pi^2}\Big[\mathbf{B}_0^2\Big(1-\frac{6}{7}\mathbf{B}_0^2\Big)+B_{0y}^2\Big(2-\frac{24}{7}\mathbf{B}_0^2\Big)\Big]B_y,\tag{2.10}$$

$$\hat{L}M_z=\frac{\alpha}{90\pi^2}\Big[\mathbf{B}_0^2-\frac{6}{7}\mathbf{B}_0^4\Big]B_z;$$

$$\hat{N}_2P_y=\frac{\alpha B_{0y}}{90\pi^2}\Big[\Big(5-\frac{26}{7}B_{0y}^2-\frac{2}{7}\mathbf{B}_0^2\Big)B_yE_y+\Big(\frac{7}{2}-\frac{13}{7}\mathbf{B}_0^2\Big)B_zE_z\Big]$$

$$\hat{N}_2P_z=\frac{\alpha B_{0y}}{90\pi^2}\Big[\Big(-2+\frac{24}{7}\mathbf{B}_0^2\Big)B_yE_z+\Big(\frac{7}{2}-\frac{13}{7}\mathbf{B}_0^2\Big)B_zE_y\Big],$$

$$\hat{N}_2M_y=\frac{\alpha B_{0y}}{90\pi^2}\Big[\Big(1-\frac{12}{7}\mathbf{B}_0^2\Big)(B_z^2-E_z^2)+\Big(3-\frac{36}{7}\mathbf{B}_0^2-\frac{24}{7}B_{0y}^2\Big)B_y^2+\Big(\frac{5}{2}-\frac{1}{7}\mathbf{B}_0^2-\frac{13}{7}B_{0y}^2\Big)E_y^2\Big],\tag{2.11}$$

$$\hat{N}_2M_z=\frac{\alpha B_{0y}}{90\pi^2}\Big[\Big(2-\frac{24}{7}\mathbf{B}_0^2\Big)B_yB_z+\Big(\frac{7}{2}-\frac{13}{7}\mathbf{B}_0^2\Big)E_yE_z\Big].$$

In the linear approximation we have from (2.8) and (2.9), for waves propagating along the x axis, accurate to $|\alpha\mathbf{B}_0|^4$,

$$\Big[1-\frac{2\alpha}{45\pi}\mathbf{B}_0^2\Big(1-\frac{6}{7}\mathbf{B}_0^2\Big)\Big]\frac{\partial^2B_z}{\partial x^2}-\Big[1-\frac{2\alpha}{45\pi}\mathbf{B}_0^2\Big(1-\frac{6}{7}\mathbf{B}_0^2\Big)+\frac{2\alpha}{45\pi}B_{0y}^2\Big(\frac{7}{2}-\frac{13}{7}\mathbf{B}_0^2\Big)\Big]\frac{\partial^2B_z}{\partial t^2}=0,\tag{2.12}$$

$$\Big[1-\frac{2\alpha}{45\pi}\mathbf{B}_0^2\Big(1-\frac{6}{7}\mathbf{B}_0^2\Big)-\frac{2\alpha}{45\pi}B_{0y}^2\Big(2-\frac{24}{7}\mathbf{B}_0^2\Big)\Big]\frac{\partial^2B_y}{\partial x^2}-\Big[1-\frac{2\alpha}{45\pi}\mathbf{B}_0^2\Big(1-\frac{6}{7}\mathbf{B}_0^2\Big)\Big]\frac{\partial^2B_y}{\partial t^2}=0.\tag{2.13}$$

From this we easily obtain, with the same accuracy, the phase velocities of the TE and TM waves (cf. Ref. 1):

$$V_{TE}=1-\frac{2\alpha}{45\pi}B_{0y}^2\Big(1-\frac{12}{7}\mathbf{B}_0^2\Big),\tag{2.14}$$

$$V_{TM}=1-\frac{7\alpha}{90\pi}B_{0y}^2\Big(1-\frac{26}{49}\mathbf{B}_0^2\Big).\tag{2.15}$$

We consider now waves that travel at small angles to the x axis in the positive direction. We seek a weakly nonlinear solution close to a superposition of linear waves whose shape varies slowly as a result of the nonlinear effects. Introducing the "running" coordinate $\xi=x-t$ and recognizing that

$$U_{TE}=(V_{TE}-1)\sim\alpha\mathbf{B}_0^2,\quad U_{TM}=(V_{TM}-1)\sim\alpha\mathbf{B}_0^2,\tag{2.16}$$

we put $\partial^2/\partial t^2\approx\partial^2/\partial\xi^2-2\partial^2/\partial\xi\partial t$ in the linear terms of the left-hand sides of (2.8) and (2.9), and $\partial/\partial t=-\partial/\partial\xi$ in the nonlinear terms of the right-hand sides, accurate to $\alpha\mathbf{B}_0^4$. Next, taking (2.7) into account, we put, with the same accuracy,

$$E_y=B_z,\quad E_z=-B_y\tag{2.17}$$

for the calculation of the nonlinear terms. As a result we obtain from (2.8) and (2.9)

$$\frac{\partial}{\partial\xi}\Big(\frac{\partial B_y}{\partial t}+U_{TE}\frac{\partial B_y}{\partial\xi}\Big)+\frac{1}{2}\Delta_\perp B_y=2\pi\frac{\partial^2}{\partial\xi^2}\hat{N}_2(M_y-P_z)=-\frac{2\alpha}{315\pi}B_{0y}^3\frac{\partial^2}{\partial\xi^2}\Big(12B_y^2+\frac{13}{2}B_z^2\Big),\tag{2.18}$$

$$\frac{\partial}{\partial\xi}\Big(\frac{\partial B_z}{\partial t}+U_{TM}\frac{\partial B_z}{\partial\xi}\Big)+\frac{1}{2}\Delta_\perp B_z=2\pi\frac{\partial^2}{\partial\xi^2}\hat{N}_2(M_z+P_y)=-\frac{26\alpha}{315\pi}B_{0y}^3\frac{\partial^2}{\partial\xi^2}(B_yB_z).\tag{2.19}$$

These equations describe, in the collinear approximation, nonlinearly interacting waves those spatial spectrum is concentrated in a narrow cone near the positive x axis.

No really complete investigation of the three-dimensional system of nonlinear partial differential equations (2.18) and (2.19) is possible. We confine ourselves therefore to a study of particular solutions corresponding to one-dimensional waves in which $\partial/\partial y = \partial/\partial z = 0$. In this case we obtain nonlinearly coupled TE and TM waves traveling along the x axis. Since their phase velocities are different, no synchronism or effective interaction between waves of different polarization is possible. As for self-action of the waves, according to (2.19) there is no intrinsic nonlinearity for the TM mode, since Eq. (2.19) becomes linear at $B_y = 0$. On the contrary, self action is possible for TE waves in which $B_z = 0$. In this case Eq. (2.19) is satisfied automatically, so that a solution in the form of TE waves is realized if $B_z = 0$ in the initial (boundary) conditions.

3. PLANE TE WAVES

We consider a nonlinear TE wave traveling along the x axis. Putting $B_z = 0$ in (2.18) and introducing the coordinate $\chi = \xi - U_{TE}t$, we obtain an equation for a simple TE wave:

$$\frac{\partial B_y}{\partial t} + \frac{48\alpha}{315\pi} B_{0y}^3 B_y \frac{\partial B_y}{\partial \chi} = 0. \tag{3.1}$$

The solution of this equation is of the form[4]

$$B_y = f\left(\chi - \frac{48\alpha B_{0y}^3}{315\pi} B_y t\right), \tag{3.2}$$

where $f(z)$ is an arbitrary function determined by the initial condition. The smooth initial profile of the field B_y becomes distorted in the course of time, since the profile points with large B_y move more rapidly. The result is a discontinuity—a jump in the magnitude and direction of the magnetic field (see Fig. 1).

The velocity of such a TE discontinuity is

$$V_0 = V_{TE} + \frac{48\alpha B_{0y}^3}{315\pi} \frac{B_{y-} + B_{y+}}{2}, \tag{3.3}$$

where B_{y-} and B_{y+} are the values of the field behind and ahead of the discontinuity.

We note that whereas the field perturbation ahead of the discontinuity is $B_{y+} = 0$, i.e., there exists only a uniform magnetic field $\mathbf{B}_0$, behind the discontinuity there appears both a magnetic field $\mathbf{B}_0 + \mathbf{B}$ and a perpendicularly directed electric field $\mathbf{E}$, the value of which is determined from (2.17). Such a field configuration ensures an increased energy density of the field behind the discontinuity: to produce this density we need a constant energy flux determined by a nonzero Poynting vector.

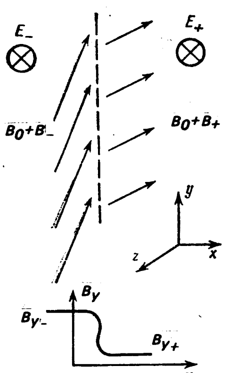

FIG. 1. TE shock wave.

The increase in the slope of the TE wave leads to a redistribution of the energy over the spectrum in the direction of the higher harmonics. For harmonics with sufficiently large wave numbers it is necessary to take into account the dispersion (the dependence of the phase velocity on the frequency). The refractive index for the TE wave can be represented as an expansion in powers of the small parameter $\omega B_{0y}/2m$ (Ref. 2):

$$n_{TE} = 1 + \frac{\alpha}{\pi}\left(\frac{B_{0y}}{2}\right)^2 \left[0.18 + 0.24\left(\frac{\omega B_{0y}}{2m}\right)^2 + \ldots\right]. \tag{3.4}$$

At a sufficiently large characteristic scale of the field perturbations, such that the following inequality is satisfied

$$k_x \sim \omega \ll m/B_{0y}, \tag{3.5}$$

the wave dissipation, determined by the production of electron-positron pairs, is exponentially small. At the same time, under the condition (3.5) one can neglect also the higher-order terms of the series (3.4) and represent the dispersion dependence $\omega(k_x)$ in the form

$$\omega = V_{TE}k_x + \frac{k_x^3}{6}\frac{d^3\omega}{dk_x^3}\bigg|_{k_x=0} = V_{TE}k_x - 0.24\frac{\alpha}{\pi}\left(\frac{B_{0y}^2}{4m}\right)^2 k_x^3.$$

The equation (3.1) of the simple wave can be easily modified to take into account the weak high-frequency dispersion $d^3\omega/dk_x^3 \neq 0$:

$$\frac{\partial B_y}{\partial t} + \frac{48\alpha B_{0y}^3}{315\pi} B_y \frac{\partial B_y}{\partial \chi} + 0.24\frac{\alpha}{\pi}\left(\frac{B_{0y}^2}{4m}\right)^2 \frac{\partial^3 B_y}{\partial \chi^3} = 0. \tag{3.6}$$

Relation (3.6) is the Korteweg-de Vries equation, whose solutions have been thoroughly investigated.[4] It is known, in particular, that an arbitrary initial pulse $B_y(x, t=0)$ decays in accord with (3.6) into a finite number of isolated stationary waves (solitons). The latter take in this case the form

$$B_y = B_{y\max}\,\mathrm{ch}^{-2}\left[\frac{8m}{3}(x - x_0) B_{y\max}^{1/2} (8.4 B_{0y})^{-1/2}\right]. \tag{3.7}$$

If $B_{y\max} \ll B_{0y}$, the characteristic width of the soliton

$$l \sim m^{-1}(B_{0y}/B_{y\max})^{1/2} \tag{3.8}$$

is large compared with the Compton wavelength $\Lambda = m^{-1}$. In the case of a strong nonlinearity, however, the dispersion equation (3.6) also becomes incorrect.

In the theory of nonlinear waves, Eq. (3.6) has also a known solution that describes the evolution of a drop or "step" in the initial conditions (see Fig. 2). In the

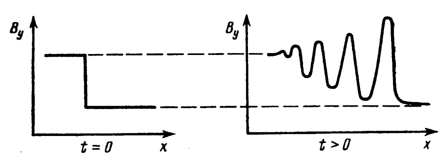

FIG. 2. Evolution of "step" in the initial conditions for a TE wave and formation of an oscillatory front structure.

course of time solitons of equal amplitude are detached from the step and move ahead. As a result, the "step" breaks up into a sequence of solitons, whose number increases without limit (in the absence of dissipation). The front of the drop becomes oscillatory.[5]

We note that the one-dimensional soliton (3.7) is stable to self-focusing (bending or pulsations of the wave front), since $d^3\omega/dk^3 < 0$ (see Ref. 5). In principle, however, it is possible to have an instability of the decay type, similar to the process $\gamma_{TE} \to \gamma_{TM} + \gamma_{TM}$ for quasi-monochromatic waves.[2] This instability of nonlinear TE waves of finite amplitude with respect to decay into TM waves calls for a special investigation. The study of nonlinear regimes, however, is made complicated by the fact that with increasing amplitude of the TM wave the decay can give way to the inverse process, coalescence: $\gamma_{TM} + \gamma_{TM} \to \gamma_{TE}$ (Ref. 5). The result should be nonlinear waves of mixed polarization, described by the system of coupled equations (2.18) and (2.19).

4. TE WAVES OF FINITE INTENSITY

Solutions in the form of simple TE waves can be obtained without imposing the restriction $|\mathbf{B}| \ll |\mathbf{B}_0|$ that the perturbation be small. If only the condition $|\mathbf{B}| \ll 1$, $|\mathbf{B}_0| \ll 1$ are satisfied, then accurate to $\alpha|\mathbf{B}|^4$ we have $(\mathbf{E}\cdot\mathbf{B}) = 0$ and $(\mathbf{E}\cdot\mathbf{B}_0) = 0$ for TE polarization. We then obtain from (2.6), taking relations (2.7) into account,

$$P_z = \frac{\alpha}{90\pi^2} E_z [E_z^2 - (\mathbf{B}_0 + \mathbf{B})^2] \left\{ 1 + \frac{6}{7} [E_z^2 - (\mathbf{B}_0 + \mathbf{B})^2] \right\},$$
$$M_y = -\frac{\alpha}{90\pi^2} (B_{0y} + B_y) [E_z^2 - (\mathbf{B}_0 + \mathbf{B})^2] \left\{ 1 + \frac{6}{7} [E_z^2 - (\mathbf{B}_0 + \mathbf{B})^2] \right\}. \tag{4.1}$$

We consider Eqs. (2.5) for the field components E_z and B_y

$$\frac{\partial B_y}{\partial t} - \frac{\partial E_z}{\partial x} = 0, \quad \frac{\partial B_y}{\partial x} - \frac{\partial E_z}{\partial t} = 4\pi \left(\frac{\partial M_y}{\partial x} + \frac{\partial P_z}{\partial t} \right). \tag{4.2}$$

The system (4.2) is obviously hyperbolic and has solutions in the form of simple waves whose velocity depends on the field. Consider a simple wave traveling with velocity $V(B_y)$ in the positive direction along the x axis. In this wave

$$E_z \approx E_z(B_y), \quad \frac{\partial B_y}{\partial t} \Big/ \frac{\partial B_y}{\partial x} = -\left(\frac{\partial x}{\partial t} \right)_{B_y} = -V(B_y).$$

Recognizing that $(V-1) \sim \alpha |\mathbf{B}_0 + \mathbf{B}|^2$, we put $\partial/\partial t = -\partial/\partial x$ and $E_z = -B_y$ in the left-hand sides of (4.2). We then obtain from (4.1) and (4.2)

$$\frac{\partial E_z}{\partial B_y} \frac{\partial B_y}{\partial x} - \frac{\partial B_y}{\partial t} = 0,$$
$$\frac{\partial B_y}{\partial x} - \frac{\partial E_z}{\partial B_y} \frac{\partial B_y}{\partial t} = \frac{2\alpha}{45\pi} B_{0y} \frac{\partial}{\partial x} \left\{ (\mathbf{B}_0^2 + 2B_{0y}B_y) \left[1 - \frac{6}{7} (B_0^2 + 2B_{0y}B_y) \right] \right\}. \tag{4.3}$$

From this we easily obtain, accurate to $\alpha|\mathbf{B}|^4 \sim \alpha|\mathbf{B}_0|^4$, the velocity

$$V = 1 - \frac{2\alpha B_{0y}^2}{45\pi} \left(1 - \frac{12}{7} B_0^2 - \frac{24}{7} B_{0y} B_y \right). \tag{4.4}$$

This expression is a sum of the linear phase velocity (2.14) and its nonlinear correction [see (3.2)], both obtained in the approximation $|\mathbf{B}| \ll |\mathbf{B}_0|$. Now, however, we can state that in the region $|\mathbf{B}| \ll 1$, $|\mathbf{B}_0| \ll 1$ it is valid for any relation between $|\mathbf{B}|$ and $|\mathbf{B}_0|$.

The velocity of the discontinuity produced as a result of the evolution of a simple wave is determined by Eq. (3.3). At the same time, the quantity can be obtained by using the boundary conditions[6] on the moving discontinuity:

$$\{\mathbf{E} + [\mathbf{V}_0 \times \mathbf{B}]\}_\tau = 0, \quad \{\mathbf{B}\}_n = 0, \tag{4.5}$$
$$\{\mathbf{B} - 4\pi\mathbf{M} - [\mathbf{V}_0 (\mathbf{E} + 4\pi\mathbf{P})]\}_\tau = 0. \tag{4.6}$$

Assuming for simplicity that only a magnetic field $\mathbf{B}_0$ is present ahead of the shockwave front, and that the fields behind the front ($\mathbf{B}_0 + \mathbf{B}$ and $\mathbf{E}$) correspond to TE polarization, we obtain

$$E_z + V_0 B_y = 0, \quad B_x = 0,$$
$$B_y - 4\pi [M_y(\mathbf{B}, \mathbf{B}_0) - M_y(0, \mathbf{B}_0)] + V_0 (E_z + 4\pi P_z) = 0, \tag{4.7}$$

where P_z and $M_y(\mathbf{B}, \mathbf{B}_0)$ are given by Eqs. (4.1). From this we readily obtain, accurate to $\alpha|\mathbf{B}|^4$,

$$V_0 = 1 - \frac{2\alpha B_{0y}^2}{45\pi} \left(1 - \frac{12}{7} B_0^2 - \frac{12}{7} B_{0y} B_y \right). \tag{4.8}$$

The last term in (4.8) determines the difference between the velocity of a discontinuity of finite amplitude and the phase velocity of the linear waves (2.14). This term is equal to half the nonlinear correction to the velocity of a simple TE wave [see the last term of (4.4)], in agreement with Eq. (3.3).

We note that the boundary condition (4.6) is correct only in the absence of surface currents on the wave front. This approximation is justified if the production of electron-positron pairs on the discontinuity can be neglected. The same condition limits the possibility of using a Lagrangian in the form (2.1) (Ref. 1). We have shown above that the width Δ of the front of a low-intensity shock wave ($|\mathbf{B}| \sim |\mathbf{E}| \ll |\mathbf{B}_0|$) is large compared with the Compton wavelength Λ. In this case $\hbar\omega \sim \hbar c/\Delta \ll \hbar c/\Lambda = mc^2$ (in the usual units) and pair production is impossible.

For an intense TE wave ($|\mathbf{B}| \sim |\mathbf{E}| \sim |\mathbf{B}_0|$) the linear dispersion (3.4) becomes comparable with the nonlinear correction to the velocity (4.4), if the characteristic scale of variation of the field is $\Delta \sim \Lambda$. In this case production of electron-positron pairs on the wave front is possible. The foregoing analysis does not make it possible to determine the width of the shock-wave front at $|\mathbf{B}| \gtrsim |\mathbf{B}_0|$. At $\Delta \sim \Lambda$ and $|\mathbf{B}| \sim |\mathbf{B}_0| \ll 1$, however, the probability of pair photoproduction is exponentially small[1] and the boundary condition of the discontinuity (4.6) remains in force. Effective production of electron-positron pairs on a shock wave front can be expected only at $|\mathbf{B}| \sim |\mathbf{B}_0| \sim 1$; but then the theory developed above is patently inapplicable.

5. DISCUSSION

It follows from the content of the present article that electromagnetic TE waves evolve in a magnetized vacuum with formation of discontinuities. The characteristic time of such a process can be easily obtained from (3.2):

$$\tau \sim 315\pi B_c^4 \lambda / 192 \alpha c \mathscr{B}_{0y}^3 \mathscr{B}_y, \tag{5.1}$$

where $\mathscr{B}_y$ and λ are the characteristic amplitude and wavelength at the instant of time $t=0$ (we use here the universally accepted system of units). The restriction which leads to the discontinuity, on the successive generation of harmonics is determined by the high-frequency dispersion of the electromagnetic fields in the magnetic vacuum. The result is a shock wave with an oscillating front, similar to the collisionless shock waves in a plasma[2)] (Ref. 7).

The nonlinear TE waves were investigated under the assumption that there is no field with TM polarization. A monochromatic TE wave, however, is unstable to parametric decay into TM waves.[2] Since the interactions of the TM harmonics with one another is forbidden, the process of parametric decay is reversible: it should give way to coalescence of TM harmonics with formation of a TE wave, followed by the interactions of the TE harmonics with one another, etc. The result is a wave of mixed polarization.

It is very important that the characteristic times of the evolution of parametric instability and formation of a TE shock wave are comparable, so that at appropriate initial and boundary conditions (relatively small amplitude of the TM waves compared with the TE-polarization wave) a TE shock wave is produced before the decay into TM waves sets in. An investigation of the last process involves the question of parametric instability of nonlinear nonsinusoidal TE waves [in particular, the solitons (3.7)]. This question is still moot and calls for a special investigation.

In conclusion, we estimate the conditions for the formation of a discontinuity in a magnetized vacuum surrounding a neutron star with a magnetic field $\mathscr{B}_0 \sim 0.1 B_c \approx 4 \cdot 10^{12}$ G. The characteristic scale of the magnetic field in this case is of the order of the radius of the star, $r_* \approx 10^6$ cm. According to (5.1), at such a distance a light wave with characteristic wavelength $\lambda \sim 10^5$ cm forms a discontinuity if its amplitude $\mathscr{B}_y \gtrsim 3 \cdot 10^8$ G. The width of the solitons on the front is then $l \gtrsim 100\Lambda \sim 4 \cdot 10^{-9}$ cm [see Eq. (3.8)]. The foregoing estimates take no account of the effect of the plasma on the character of the evolution of the nonlinear wave.

[1)]After this article went to press, we learned of Ref. 3, where it is shown that the field equations in a magnetized vacuum have solutions in the form of simple waves.

[2)]We note that such a structure of the front is not stationary: the number of oscillations increases without limit in the approximation considered. A stationary structure can be realized only in the presence of dissipation, the magnitude of which determined the steady-state number of oscillations on the front. For intense shock waves ($|\mathbf{B}| \gtrsim |\mathbf{B}_0|$) the dissipation mechanism may be connected with production of electron-positron pairs.

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