Evolution of the principal mode of density perturbations in a neutrino Universe

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The physical features of the evolution of the growing mode of density perturbations in homogeneous isotropic cosmological models are considered. The law of growth of long-wavelength density perturbations in a Friedmann neutrino Universe is obtained not only in the ultrarelativistic approximation but also in the general case.

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1. In connection with the development of a model of the Universe that includes massive collisionless particles (neutrinos), it has become very important to investigate the evolution of inhomogeneities in a medium consisting of two components, namely, a fluid that satisfies Pascal's law and a gas of collisionless massive particles.

The instability of a homogeneous collisionless gas in the nonrelativistic approximation was investigated by numerical integration in Ref. 1 and analytically in Refs. 2–4. Recently,[5] a numerical calculation has been made of the growth of inhomogeneities and the formation of structure in a neutrino-dominated Universe during the stage of derelativization of (massive) relic (primordial) neutrinos.

Although the nonrelativistic problem has been studied in great detail, in the ultrarelativistic case the problem of describing the development of gravitational perturbations in a Universe containing collisionless particles is still far from complete resolution. The qualitative analysis[6] made for an Einstein–de Sitter world (or, which is the same thing, a Friedmann world with flat comoving space) showed that density perturbations on large scales (exceeding the particle horizon scale) grow on account of the Jeans instability, while small-scale perturbations are damped. A more detailed analysis of the development of gravitational perturbations in an Einstein–de Sitter world filled with an ultrarelativistic ideal fluid and an ultrarelativistic collisionless gas was made in Ref. 7; however, the results of this analysis need to be made more accurate.

In the present paper, on the basis of the general theory of relativity, we discuss the evolution of the principal[1)] mode of density perturbations in a Friedmann Universe, including collisionless massive particles.

2. A systematic investigation of gravitational instability in a homogeneous and isotropic Universe on the basis of the general theory of relativity was presented in Lifshitz's pioneering work.[8] To describe the development of perturbations, Lifshitz proposed using a synchronous frame, which has a clear physical meaning, namely, such a system is freely falling in the gravitational field but not necessarily comoving with the matter. Lifshitz showed that in a synchronous frame the principal mode of density perturbations in an ultrarelativistic Pascal fluid increases in proportion to the physical time t. This result does not depend on the type of homogeneous isotropic cosmological model[2)] (open, closed, or flat).

After the cosmological neutrinos have been decoupled from the primordial plasma, the hydrodynamic approximation ceases to hold for the description of perturbations in the neutrino gas. The collisionless gas can be described on the basis of kinetic theory by means of a distribution function which satisfies the relativistic Vlasov equation. At the time when the neutrinos cease to interact with the hot plasma, the neutrino momentum distribution is isotropic, though anisotropy arises subsequently and, as a consequence, the energy-momentum tensor is correspondingly non-Pascal.[3)] In the present paper, we shall show that despite this anisotropy of the neutrino momentum distribution, the law of evolution of the principal mode of density perturbations in a neutrino Universe after the neutrinos have ceased to interact with the plasma remains the same as before the decoupling of the neutrinos, when there was complete thermodynamic equilibrium. This means that the validity of Lifshitz's result[8] is not restricted to the region of validity of the hydrodynamic description of perturbations.

3. The physical nature of the physical mode of density perturbations is discussed from a Newtonian point of view in the monographs Refs. 10 and 11 of Zel'dovich and Novikov.

It is well known that on large scales exceeding the critical Jeans wavelength the influence of nongravitational forces is negligibly small, and the evolution of inhomogeneities is determined solely by the gravitational interaction. It is therefore very easy to understand the results of Zel'dovich and Novikov,[10,11] who showed that in the Newtonian treatment the principal mode is coupled to inhomogeneity in the distribution of the total mechanical energy of the medium. This is usually illustrated by considering the evolution of a sphere of enhanced (or reduced) density on the background of a homogeneous isotropic Universe filled with nonrelativistic matter, but this restriction is not fundamental. It is important that for long wavelengths (exceeding the scale of the current horizon, which in order of magnitude is equal to the critical Jeans wavelength), individual regions evolve independently without interacting

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with one another. Systematic use of these arguments makes it possible to give a general method for determining the law of evolution of the principal mode if the energy density ε of the matter filling the Universe is known as a function of the scale factor a of the homogeneous isotropic cosmological model. In fact, the entire manifold of properties of the matter that influence the evolution of the principal mode of the density perturbations is contained in the dependence $\varepsilon=\varepsilon(a)$. A great advantage of this method is that it does not require the solution of differential equations; the result can be obtained directly by quadrature.

Our point of departure is the relation in a Friedmann cosmological model between the Hubble constant H, the energy density ε, and the Gaussian curvature:

$$H^2=\frac{8\pi G}{3c^2}\varepsilon-\frac{kc^2}{a^2} \tag{1}$$

(G is the Newtonian gravitational constant, c is the velocity of light, and in the unperturbed Universe $k=0$, 1, or -1 for flat, closed or open cosmological models, respectively). In the Newtonian treatment, the term kc^2/a^2 describes the appropriately normalized total mechanical energy of an element of matter. If the $\varepsilon(a)$ dependence is known, it follows from (1) that

$$t=\int\limits_0^a\frac{d\xi}{\xi}\left[\frac{8\pi G}{3c^2}\varepsilon(\xi)-\frac{kc^2}{\xi^2}\right]^{-1/2}. \tag{2}$$

Using the relation[10,11]

$$\frac{\delta\varepsilon}{\varepsilon}\propto\frac{d\ln\varepsilon}{dt}\frac{dt}{dk},$$

we obtain[4)]

$$\frac{\delta\varepsilon}{\varepsilon}\propto\frac{d\ln\varepsilon}{d\ln a}H(t)\int\limits_0^a\frac{d\xi}{\xi^3}\left[\frac{8\pi G}{3c^2}\varepsilon(\xi)-\frac{kc^2}{\xi^2}\right]^{-1/2}, \tag{3}$$

which solves the problem if the evolution of the unperturbed cosmological model is known. The $\varepsilon(a)$ dependence is very simple for ultrarelativistic ($\varepsilon\propto a^{-4}$) and nonrelativistic ($\varepsilon\propto a^{-3}$) matter, but in the general case of massive particles (neutrinos) it reduces to a quadrature formula:

$$\varepsilon(a)=\frac{4\pi c}{a^4}\int\limits_0^\infty dqq^2(m^2c^2a^2+q^2)^{1/2}F_0(q), \tag{4}$$

where q is the magnitude of the conformal momentum of the particles, m is their rest mass, and $F_0(q)$ is the background distribution function. Therefore, a detailed analysis of the general solution of (3) is possible only numerically. For the ultrarelativistic distribution function $F_0(q)$, the first term in the brackets can be ignored, and (4) leads to $\varepsilon\propto a^{-4}$, which on substitution in (3) (for simplicity, we assume critical density, $k=0$, and consider the relativistic stage of expansion, $a\propto t^{1/2}$) gives $\delta\varepsilon/\varepsilon\propto t$ in complete agreement with Lifshitz's result.[8] It is interesting that in this case the collisionless nature of the particles is not manifested, and the solution is identical to the hydrodynamic solution. Therefore, the pictures of the growth of density fluctuations of the matter in a relativistic Friedmann model before and after the decoupling of the neutrinos from the primordial plasma (with allowance for the gravitational influence of the anisotropic pressure of the free particles) are asymptotically equal.

4. We demonstrate what we have said by direct calculation of the evolution of scalar perturbations in a synchronous frame that satisfy the linearized system of Einstein equations and the relativistic Vlasov equation. In the relativistic stage of expansion of the Einstein–de Sitter model ($a=a_0\eta$, where η is the conformal time), this system of equations has the form[5)]

$$\begin{gathered}
\frac{\partial\Phi(x,\eta)}{\partial\eta}+inx\Phi(x,\eta)=-\frac{\alpha}{\alpha+1}(\lambda'+\mu'-3x^2\lambda'),\\
\mu''+\frac{3}{\eta}\mu'+\frac{2}{3}n^2(\mu+\lambda)=0,\\
\lambda''+\frac{2}{\eta}\lambda'-\frac{1}{3}n^2(\mu+\lambda)=\frac{1}{\eta^2}\int\limits_{-1}^{+1}dx(1-3x^2)\Phi(x,\eta),\\
\delta v^\alpha=\left\{\frac{in\eta^2(\alpha+1)}{12}(\mu'+\lambda')-\frac{\alpha+1}{4}\int\limits_{-1}^{+1}dxx\Phi(x,\eta)\right\}\frac{n^\alpha}{n}e^{inr},\\
\frac{\delta\varepsilon_f}{\varepsilon_f}=(\alpha+1)\left\{\frac{\eta^2}{9}\left[n^2(\mu+\lambda)+\frac{3}{\eta}\mu'\right]-\frac{1}{3}\int\limits_{-1}^{+1}dx\Phi(x,\eta)\right\}e^{inr},\\
\frac{\delta\varepsilon_g}{\varepsilon_g}=\frac{1}{3}\left(\frac{\alpha+1}{\alpha}\right)\int\limits_{-1}^{+1}dx\Phi(x,\eta)e^{inr}.
\end{gathered} \tag{5}$$

Here, the prime denotes differentiation with respect to the conformal time; ε_f and ε_g are the energy density of the ideal fluid and the collisionless (neutrino) gas, respectively; δv^α is the perturbation of the fluid velocity; $\alpha=\varepsilon_g/\varepsilon_f$; and x is the cosine of the angle measured from the direction $\mathbf{n}$ in the neutrino momentum space. As usual,[8,9,12] the quantities μ and λ describe the two irreducible parts of the scalar perturbations of the metric h^α_β, namely, the spherical tracless parts:

$$h_\beta{}^\alpha=\left\{\lambda(\eta)\left[\frac{1}{3}\delta_\beta{}^\alpha-\frac{n^\alpha n_\beta}{n^2}\right]+\mu(\eta)\left[\frac{1}{3}\delta_\beta{}^\alpha\right]\right\}e^{inr}.$$

The function $\Phi(x,\eta)$ is related to the background distribution function $F_0(q)$ of the neutrinos and its perturbation

$$\delta F(\eta,\mathbf{r};q^\alpha)=\delta f(\eta;q^\alpha)e^{inr}$$

by

$$\Phi(x,\eta)=\frac{8\pi G}{c^3a_0^2}\int\limits_0^{2\pi}d\varphi\int\limits_0^\infty dqq^3\left[\delta F-\frac{1}{2q}h_\beta{}^\alpha q_\alpha q^\beta\frac{dF_0(q)}{dq}\right]e^{-inr},$$

where φ is the azimuthal angle in the momentum space.

The solution of the system (5) corresponding to the principal mode of density perturbations in a relativistic neutrino Einstein–de Sitter model has the form ($n\eta\ll1$)

$$\begin{gathered}
\mu=C_2-{}^1/_6C_2(n\eta)^2+\dots;\\
\lambda=C_2+\frac{5}{3}C_2\frac{\alpha+1}{19\alpha+15}(n\eta)^2+\dots;\\
\Phi(x,\eta)=\frac{1}{6}C_2\frac{\alpha}{19\alpha+15}\left(\frac{9\alpha+5}{\alpha+1}+30x^2\right)(n\eta)^2\left[1-\frac{1}{3}ix(n\eta)\right]+\dots;\\
\frac{\delta\varepsilon_f}{\varepsilon_f}=\frac{\delta\varepsilon_g}{\varepsilon_g}=\frac{1}{9}C_2(n\eta)^2e^{inr}+\dots;\\
\delta v^\alpha=-\frac{i}{108}C_2(n\eta)^3\frac{n^\alpha}{n}e^{inr}+\dots.
\end{gathered}$$

Note that for $\alpha>5/27$ (which is valid in the real Universe for the ratio of the energy densities of ultrarelativistic cosmological neutrinos and photons[11]) the system (5) in the long-wavelength approximation ($n\eta\ll1$) also admits two independent solutions which correspond to oscillatory modes[6] that decrease in the metric in proportion to $t^{-1/4}$.

All proper (nonfictitious) modes of long-wavelength scalar perturbations in a relativistic neutrino Universe contain corresponding quadrupole components of the perturbed neutrino distribution function. When the neutrinos are detached from the Pascal fluid, the quadrupole distributions of the various modes cancel each other, but subsequently, because of the difference between the evolution laws of the modes, a difference between the quadrupole moments arises.

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1) By the principal mode of the density perturbations we understand the most rapidly growing (in density) mode of long-wavelength scalar perturbations. The principal mode couples, on the one hand, the initial perturbations of the metric to the fluctuations of the matter density and, on the other, the values of these fluctuations near the singularity and after the hydrogen recombination of the primordial plasma, when galaxy formation becomes possible. In a medium consisting of several components, the principal mode is adiabatic.

2) Later, Lifshitz and Khalatnikov[9] showed that the nature of the evolution of the principal mode of density perturbations also persists in the more general case of a quasi-isotropic solution with hydrodynamic ($p_{xx}=p_{yy}=p_{zz}=\varepsilon/3$) energy—momentum tensor (the quasi-isotropic solution describes an inhomogeneous anisotropic space that varies in time in accordance with a similarity law).

3) The relative measure of the extent to which the energy—momentum tensor of the collisionless particles in the principal mode is non-Pascal has the same order of smallness as the relative fluctuations of the matter density.

4) The method of solution used in Refs. 10 and 11 and the present paper is known as the method of variation of the initial data. When applied to the investigation of the stability of homogeneous isotropic cosmological models this method makes it possible to obtain the law of evolution of the principal mode of density perturbations (by variation with respect to the energy) and the law of evolution of one of the fictitious (in Lifshitz's terminology[8,9,12]) modes (by variation with respect to the physical time).

5) The system of equations (5) is obtained under the assumption that not only the background distribution of the matter (ideal fluid and collisionless gas) but also the perturbations are ultrarelativistic. A system of equations analogous to (5) was proposed in Ref. 7 but on p. 437 (of the Russian original) it is incorrectly asserted that to obtain this system it is sufficient to require only an ultrarelativistic nature of the evolution of the background cosmological model.

6) An oscillatory regime of evolution of (not the principal mode!) gravitational scalar long-wavelength perturbations in a Universe containing a gas of ultrarelativistic collisionless particles was found in Ref. 13 in connection with a study of the isotropization of weakly nonisotropic homogeneous cosmological models.

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