Multireggeon processes in gravitation

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(Submitted 29 October 1981)
Zh. Eksp. Teor. Fiz. **82**, 991–1006 (April 1982)

The dispersion method is used to calculate the inelastic amplitudes of graviton–graviton interaction in the multireggeon kinematics of the produced particles. It is shown that the graviton is reggeized.

PACS numbers: 12.25. + e, 04.60. + n

1. INTRODUCTION

Einstein regarded the phenomenon of universal gravitation as a manifestation of the Riemannian geometry of real space-time.[1] However, one can also interpret Einstein's theory of gravitation as a variant of the theory of gauge fields in Minkowski space.[2] Particularly instructive is the analogy between the gravitational field and the Yang-Mills field.[3] For our purpose—calculation of the asymptotic behavior of scattering amplitudes in gravitation—it is helpful to dwell on this analogy in more detail.

The Yang-Mills and gravitational fields describe massless particles with spins $S=1$ and 2, respectively (gluon and graviton), which for fixed momentum p have only two degrees of freedom corresponding to helicities $\pm S$. Nevertheless, for relativistic invariance it is necessary to introduce the four-component vector potential $A^\mu(x)$ as the gluon field and the symmetric tensor $A^{\mu\nu}(x)$ with ten independent components to describe the graviton. [The field A^μ is, in addition, a Hermitian traceless matrix in the case of the group SU(n) acting on an n-dimensional complex space.] To eliminate the redundant components, it is assumed that the free (linearized) equations for the fields A^μ and $A^{\mu\nu}$ have solutions grouped in classes connected by the "truncated" gauge transformation

$$A^\mu(x)\to A'^\mu(x)=A^\mu(x)+\partial^\mu\chi(x),$$
$$A^{\mu\nu}(x)\to A'^{\mu\nu}(x)=A^{\mu\nu}(x)+\partial^\mu\chi^\nu+\partial^\nu\chi^\mu, \tag{1}$$

where $\chi(x)$ is an arbitrary $n\times n$ matrix, and $\chi^\nu(x)$ is an arbitrary vector field. Solutions belonging to the same class are assumed to be physically equivalent. It is readily verified that by means of the gauge freedom (1) solutions corresponding to free particles with momentum p ($p^2=0$) can be reduced to the form

$$A^\mu\sim e^\mu e^{-ipx},\quad A^{\mu\nu}\sim e^{\mu\nu}e^{-ipx},$$

where the polarization tensors $e^\mu(p)$ and $e^{\mu\nu}(p)$ are a certain linear combination of the independent tensors

$$e_\pm{}^\mu(p)=2^{-1/2}(e_1{}^\mu\pm ie_2{}^\mu),\quad (e_+,e_-)=-1,\quad e_\pm{}^{\mu\nu}(p)=e_\pm{}^\mu e_\pm{}^\nu, \tag{2}$$

which describe gluons and gravitions with helicities $\pm S$ [e_1 and e_2 are a pair of orthonormal spacelike vectors orthogonal to $\mathbf{p}$; $(a,b)=a_\mu b_\nu\delta^{\mu\nu}=a_0b_0-\mathbf{a}\cdot\mathbf{b}$]. In the Yang-Mills model, the vector potential A^μ for a free gluon is also proportional to a Hermitian traceless matrix, which corresponds to the internal degrees of freedom.

The projection operators onto the physical subspace spanning the tensors (2) can be written in the form

$$\Lambda^{\mu'\mu}=e_+{}^\mu e_-{}^{\mu'}+e_+{}^{\mu'}e_-{}^\mu=-\delta^{\mu'\mu}+\frac{p_0(p^{\mu'}N^\mu+p^\mu N^{\mu'})-p^{\mu'}p^\mu}{\mathbf{p}^2}-\frac{p^2}{\mathbf{p}^2}N^{\mu'}N^\mu, \tag{3}$$
$$\Lambda^{\mu'\nu',\mu\nu}=e_+{}^{\mu\nu}e_-{}^{\mu'\nu'}+e_+{}^{\mu'\nu'}e_-{}^{\mu\nu}={}^1/_2(\Lambda^{\mu'\mu}\Lambda^{\nu'\nu}+\Lambda^{\mu'\nu}\Lambda^{\nu'\mu}-\Lambda^{\mu'\nu'}\Lambda^{\mu\nu}),$$

where N^μ is the timelike vector with components (1, 0, 0, 0).

The attempt to construct Feynman propagators of the gluon and graviton with momentum p in accordance with the formula $D(p)=\Lambda/p^2$ leads to a Lorentz noninvariant result due to the dependence of $\Lambda^{\mu'\mu}$ on the vector N^μ. However, it can be seen that the contributions to the projection operators of the last term in Eq. (3) for $\Lambda^{\mu'\mu}$ do not contain singularities with respect to p_0 and can therefore be compensated by introduction into the theory of an instantaneous (Newtonian) interaction. The remaining noninvariant terms are proportional to the vector p and lead to a vanishing contribution in the sum of the Feynman diagrams provided the gluon and graviton interact with conserved tensors (a vector current for the gluon and the energy-momentum tensor for the graviton). Thus, if the above conditions are satisfied, the gluon and graviton propagators can be taken to be the tensors

$$D^{\mu'\mu}(p)=-\delta^{\mu'\mu}\Omega/p^2,\quad D^{\mu'\nu',\mu\nu}(p)=(\delta^{\mu'\mu}\delta^{\nu'\nu}+\delta^{\mu'\nu}\delta^{\nu'\mu}-\delta^{\mu'\nu'}\delta^{\mu\nu})/2p^2, \tag{4}$$

which corresponds to a definite (Feynman) gauge. The operator Ω is the projection operator onto the unitary-spin part of the gluon wave function. For the group SU(n),

$$\Omega_{b'a}^{a'b}=\delta_a{}^{a'}\delta_{b'}{}^{b}-\delta_{b'}{}^{a'}\delta_a{}^b/n. \tag{5}$$

The invariance of the linearized Euler-Lagrange equations with respect to the transformations (1) makes it possible to construct an action in a unique manner for both gauge models in the quadratic approximation:

$$S_{YM}=-\frac{1}{2}\int d^4x\,\mathrm{Sp}[(\partial_\mu A_\nu-\partial_\nu A_\mu)^2+O(A^3)],$$
$$S_G=\int d^4x\Big\{A^{\mu'\nu'}\Big[-\delta_{\mu\mu'}\overleftarrow{\partial}_\nu\overrightarrow{\partial}_{\nu'}+\frac{\delta_{\mu\nu}}{2}\overleftarrow{\partial}_{\mu'}\overrightarrow{\partial}_{\nu'}+\frac{\delta_{\mu'\nu'}}{2}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu$$
$$-\frac{1}{2}(\delta_{\mu\nu}\delta_{\mu'\nu'}-\delta_{\mu\mu'}\delta_{\nu\nu'})\delta^{\rho\sigma}\overleftarrow{\partial}_\rho\overrightarrow{\partial}_\sigma\Big]A^{\mu\nu}+O(A^3)\Big\}, \tag{6}$$

where the constant common factors are chosen in accordance with Eqs. (4); the arrow above a derivative symbol indicates which of the functions $A^{\mu\nu}$ or $A^{\mu'\nu'}$ on which the derivative acts; in a differential operator, symmetrization with respect to the substitutions $\mu\rightleftharpoons\nu$ and $\mu'\rightleftharpoons\nu'$ is assumed. The matrices A_μ can be expressed in terms of (n^2-1)-component vectors in the internal unitary SU(n) space in accordance with

 0038-5646/82/040582-09$04.00

$$A_\mu = {}^1/{}_2\lambda_a V_\mu{}^a, \quad [\lambda_a, \lambda_b] = 2if_{abc}\lambda_c, \quad \mathrm{Sp}\,(\lambda_a\lambda_b) = 2\delta_{ab}, \tag{7}$$

where λ_a is a generalization of the Gell-Mann matrices, and f_{abc} are the structure constants of the group SU(n).

The quadratic forms written down in (6) are degenerate due to the symmetry (1). Nevertheless, in the momentum representation one can invert the corresponding operators in the space orthogonal to p_μ. The propagators obtained in this manner differ from (4) by the substitution

$$\delta^{\mu\nu} \to \delta^{\mu\nu} - p^\mu p^\nu / p^2,$$

i.e., by terms which vanish if the field sources are conserved:

$$\partial_\mu j^\mu(x) = \partial_\mu \theta^{\mu\nu}(x) = 0.$$

These sources arise on the right-hand sides of the classical equations as a result of variation of the terms of higher order in A^μ and $A^{\mu\nu}$ than the ones written down in (6). If the equations are to have nontrivial solutions, $A^\mu \neq 0, A^{\mu\nu} \neq 0$, they must be self-consistent (which requires conservation of j_μ and $\theta_{\mu\nu}$). For this, it is necessary to require invariance of the complete Lagrangian with respect to some transformation that generalizes (1) to the case of large fields A^μ and $A^{\mu\nu}$. For infinitesimally small parameters χ and χ^μ, we can restrict ourselves to the corrections linear in A^μ and $A^{\mu\nu}$ on the right-hand side of Eqs. (1), identifying them with global SU(n) transformations and translations in the coordinate space:

$$\begin{aligned} A^\mu(x) &\to A'^\mu(x) = A^\mu + \partial^\mu\chi + ig[\chi, A^\mu], \\ A^{\mu\nu}(x) &\to A'^{\mu\nu}(x) = A^{\mu\nu}(x) + \partial^\mu\chi^\nu + \partial^\nu\chi^\mu + 2\varkappa\chi^\rho\partial_\rho A^{\mu\nu}, \end{aligned} \tag{8}$$

where g and $\varkappa$ are certain constants. In Eqs. (8), it is possible to change the coefficients of the derivatives $\partial^\mu\chi$ and $\partial^\rho\chi^\sigma$, introducing in them a dependence on the fields. This corresponds to a possible reparametrization of the potentials: $B^\mu = f(A^\mu), B^{\mu\nu} = \varphi(A^{\mu\nu})$. If the symmetry transformations are chosen in the form (8), then on the basis of the expressions (6) it is possible to construct the action uniquely in the form of an expansion in the couplings constants g and $\varkappa$ for both gauge theories:

$$\begin{aligned} S_{YM} &= -\frac{1}{2}\int d^4x\, \mathrm{Sp}(\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu])^2, \\ S_G &= \frac{1}{2\varkappa^2}\int d^4x g^{1/2} R, \quad g = \det{}^{-1}(g^{\mu\nu}), \end{aligned} \tag{9}$$

where R is the scalar curvature,[1] which depends on the metric tensor $g^{\mu\nu}(x)$, which itself is expressed in terms of the field $A^{\mu\nu}(x)$ by[1)]

$$g^{\mu\nu}(x) = \delta^{\mu\nu} - 2\varkappa A^{\mu\nu}(x). \tag{10}$$

In the presence of other fields, the Lagrangians for them are constructed in such a way that the theory is invariant under the transformations (8) augmented by corresponding SU(n) rotations and Lorentz shifts of the new fields by the small quantities $g\chi$ and $2\varkappa\chi^\rho$. In particular, the corrections to Eqs. (9) for small gA^μ and $\varkappa A^{\mu\nu}$ due to the interaction with the other fields have the form

$$\begin{aligned} \Delta S_{YM} &= -g\int d^4x\, \mathrm{Sp}(A^\mu J_\mu{}^M), \quad gA^\mu \to 0, \\ \Delta S_G &= -\varkappa\int d^4x A^{\mu\nu} T_{\mu\nu}{}^M, \quad \varkappa A^{\mu\nu} \to 0, \end{aligned} \tag{11}$$

where J^M_μ and $T^M_{\mu\nu}$ are the unitary current and the energy-momentum tensor calculated in accordance with Noether's theorem in the absence of Yang-Mills and gravitational fields.

Quantization of the gauge theories in the framework of the noncovariant Hamiltonian approach[4] leads to a very cumbersome perturbation theory. A manifestly Lorentz invariant approach to the construction of a Feynman diagram technique was proposed in Ref. 5. It is well known that gravitation is a nonrenormalizable theory. Formally, this is due to the nonvanishing dimensionality of the coupling constant $\varkappa$ and the possibility of constructing generally covariant (gauge invariant) counter terms with high derivatives of the fields $A^{\mu\nu}$. Supergravity apparently has the same shortcoming.[6] Therefore, it could be that Einstein's theory of gravitation is the low-energy limit of a future (possibly, nonlocal) yield theory. Investigation of the high-energy asymptotic behavior in gravitation could assist in the construction of this theory.

In the present paper, we calculate the amplitude of particle production in multireggeon kinematics due to the gravitational interaction in the tree approximation, and also with allowance for the infrared-divergent radiative corrections, which lead to reggeization of the graviton. The method of calculation is based on considerable use of analyticity and unitarity and is a further development of the dispersion approach used earlier[7] to find the high-energy behavior in Yang-Mills theory. As we shall show below, to find the asymptotic contributions in the framework of perturbation theory it is not essential to know the actual form of the Lagrangian (9) in gravitation but is sufficient to use only the general principles on the basis of which it is constructed. It will be found that the analogy noted above with the Yang-Mills model is helpful

2. ELASTIC AMPLITUDES IN THE BORN APPROXIMATION

The use of the expressions (4) and (11) leads to the following expressions for the scattering amplitudes of a $AB \to A'B'$ process realized through exchange in the t channel ($t = (p_A - p_{A'})^2$) of a gluon and a graviton, respectively:

$$\begin{aligned} F_{YM} &= -\frac{\delta^{\mu\mu'}}{t}(-g) j_\mu(p_A, p_{A'})(-g) j^a_{\mu'}(p_B, p_{B'}), \\ F_G &= \frac{1}{2t}(\delta^{\mu\mu'}\delta^{\nu\nu'} + \delta^{\mu\nu'}\delta^{\nu\mu'} - \delta^{\mu\nu}\delta^{\mu'\nu'})(-\varkappa)^2 T_{\mu\nu}(p_A, p_{A'}) T_{\mu'\nu'}(p_B, p_{B'}), \end{aligned} \tag{12}$$

where $j^a_\mu(p_A, p_{A'})$ is the unitary current in the momentum representation [a labels the $n^2 - 1$ components of the gluon wave function in the internal space of the group SU(n)]; $T_{\mu\nu}(p_A, p'_A)$ is the matrix element of the energy-momentum tensor in the same representation. For a point scalar particle with mass m,

$$\begin{aligned} (j_\mu{}^a)_{A'A} &= (T^a)_{A'A}(p_{A'} + p_A)_\mu, \\ T_{\mu\nu}(p_A, p_{A'}) &= (p_A)_\mu (p_{A'})_\nu + (p_{A'})_\mu (p_A)_\nu - \delta_{\mu\nu}((p_A, p_{A'}) - m^2). \end{aligned} \tag{13}$$

Here, $T^a_{A'A}$ is a generator of the group SU(n); A and A' label the unitary indices of the initial and the final particle. Thus, in the case of scattering of structureless scalar particles the expression (12) can be rewritten in the form

$$F_{YM}=-\frac{s-u}{t}g^2T^a_{A'A}T^a_{B'B},\quad s=(p_A+p_B)^2,\quad u=(p_A-p_{B'})^2,$$
$$F_G=\frac{1}{t}\varkappa^2\left[\frac{1}{2}(s^2+u^2-t^2)+2tm^2-6m^4\right].\tag{14}$$

Note that there are corrections to these expressions corresponding to diagrams with gluon (and graviton) exchange in other channels (s and u), but they do not contain a singularity $1/t$ as $t\to 0$. The residue at the pole $1/t$ is universal: it does not depend on the internal structure of the colliding objects. In particular, in the presence of spin the s-channel helicities of each of the scattered particles is conserved in the limit $t\to 0$:

$$F_{YM}=-\frac{s-u}{t}g^2T^a_{A'A}T^a_{B'B}\delta_{\lambda_A\lambda_{A'}}\delta_{\lambda_B\lambda_{B'}},$$
$$F_G=\frac{1}{t}\left[\frac{1}{2}(s^2+u^2)-6m^4\right]\varkappa^2\delta_{\lambda_A\lambda_{A'}}\delta_{\lambda_B\lambda_{B'}}.\tag{15}$$

Comparing the expressions (15) for the scattering amplitude F_G in the nonrelativistic limit $s\to 4m^2$ $(u\to 0)$ with the well-known quantum-mechanical formula

$$F_G=4\int e^{iqr}V(r)d^3r,\quad q^2=-t,$$

where $V(r)=-Gm^2/r$ is the Newtonian potential (G is the constant of gravitation), we can obtain Einstein's relation between G and $\varkappa$:

$$\varkappa=(8\pi G)^{1/2}.\tag{16}$$

For gluons and gravitons, which are massless and structureless objects in the framework of perturbation theory, the expressions (15) are valid in the Regge limit

$$s\gg t.\tag{17}$$

The antisymmetry of F_{YM} and the symmetry of F_G with respect to the interchange $s\rightleftarrows n$ corresponds to the well-known dependence of the signature P_j on the spin of the exchanged particle: $P_j=(-1)^S$.

In what follows, we need expressions for the gluon and graviton scattering amplitudes, not in the helicity basis (15) but in the Lorentz covariant form

$$F_{YM}=-\frac{2s}{t}g^2T_{A'A}T_{B'B}\Gamma^{\mu\mu'}(p_A,p_{A'})\Gamma^{\nu\nu'}(p_B,p_{B'}),$$
$$F_G=\frac{s^2}{t}\varkappa^2\Gamma^{\mu\nu,\mu'\nu'}(p_A,p_{A'})\Gamma^{\rho\sigma,\rho'\sigma'}(p_B,p_{B'}),\tag{18}$$

where the matrices Γ are given by

$$\Gamma^{\mu\mu'}(p_A,p_{A'})=-\sum_{\lambda=\pm}e_\lambda^{\mu'}(p_{A'})(e_\lambda^{\mu}(p_A))^*,$$
$$\Gamma^{\mu\nu,\mu'\nu'}(p_A,p_{A'})=\sum_{\lambda=\pm}e_\lambda^{\mu'\nu'}(p_{A'})(e_\lambda^{\mu\nu}(p_A))^*.\tag{19}$$

Here, $e^\mu_\pm$ and $e^{\mu\nu}_\pm$ are the polarization tensors of particles with helicities $\pm S$ [see (2)].

The matrices $\Gamma^{\mu\mu'}$ and $\Gamma^{\mu\nu,\mu'\nu'}$ describe a rotation of the subspace spanned by $e^\mu_\pm(p)$ and $e^{\mu\nu}_\pm(p)$ in the plane formed by the vectors p_B and $q=p_A-p_{A'}$. Indeed, by choice of the gauge one can make the polarization tensors orthogonal to the vector p_B; in addition, the vector p_A must be carried by this rotation to the vector $p_{A'}=p_A-q$. For small q [see (17)], the matrix of the rotation of the entire 4-space with these properties can be readily found in the form of the expansion

$$U^{\mu\mu'}(p_A,p_{A'})=\delta^{\mu\mu'}-q^{\mu'}\frac{p_B^\mu}{p_Ap_B}+q^\mu\frac{p_B^{\mu'}}{p_Ap_B}-\frac{1}{2}\frac{q^2p_B^\mu p_B^{\mu'}}{(p_Ap_B)^2}\tag{20}$$

The relations

$$U^{\mu\mu'}(p_A)_\mu=(p_{A'})^{\mu'},\quad U^{\mu\sigma}U_{\nu\sigma}=\delta_\nu^{\mu}+O(q^3),\tag{21}$$

which guarantee the fulfillment of the above properties for small q, are readily verified.

Projecting $U^{\mu\mu'}$ onto the subspace orthogonal to the vectors p^μ_A and p^μ_B, we obtain the required expression for $\Gamma^{\mu\mu'}$ (cf. Ref. 7):

$$\Gamma^{\mu\mu'}(p_A,p_{A'})=\delta^{\perp\mu\mu'}+q^{\perp\mu}\frac{p_B^{\mu'}}{pp}$$
$$=\delta^{\mu\mu'}-\frac{p_{A'}^{\mu}p_B^{\mu'}+p_B^\mu p_A^{\mu'}}{p_Ap_B}-q^2\frac{p_B^\mu p_B^{\mu'}}{2(p_Ap_B)^2}\tag{22}$$

which, in particular, has the transversality properties

$$\Gamma^{\mu\mu'}(p_A)_\mu=\Gamma^{\mu\mu'}(p_{A'})_{\mu'}=0,\quad \Gamma^{\mu\mu'}\Gamma^{\nu\nu'}\delta_{\mu'\nu'}=\delta^{\perp\mu\nu}.\tag{23}$$

For a symmetric second-rank tensor, the rotation matrix corresponding to the transformation (20) has the form

$$U^{\mu\nu,\mu'\nu'}(p_A,p_{A'})=U^{\mu\mu'}(p_A,p_{A'})U^{\nu\nu'}(p_A,p_{A'}).\tag{24}$$

Projecting $U^{\mu\nu,\mu'\nu'}$ onto the physical subspace spanned by the vectors $e^{\mu\nu}_\pm$ (this amounts to separation of the tensor components with indices μ and ν transversal to the p_A, p_B plane, symmetrization with respect to them, and the addition of terms $\sim\delta^{\perp\mu\nu}$ make the trace vanish), we obtain for the vertices $\Gamma^{\mu\nu,\mu'\nu'}$ [see (18), (19)] the expression

$$\Gamma^{\mu\nu,\mu'\nu'}(p_A,p_{A'})=\frac{1}{2}\Gamma^{\mu\mu'}\Gamma^{\nu\nu'}+\frac{1}{2}\Gamma^{\mu\nu'}\Gamma^{\nu\mu'}$$
$$-\frac{1}{2}\left(\delta^{\mu\nu}-\frac{p_A^\mu p_B^\nu+p_A^\nu p_B^\mu}{p_Ap_B}\right)\left(\delta^{\mu'\nu'}-\frac{p_{A'}^{\mu'}p_B^{\nu'}+p_{A'}^{\nu'}p_B^{\mu'}}{p_{A'}p_B}\right).\tag{25}$$

The tensor $\Gamma^{\mu\nu,\mu'\nu'}$ has the properties

$$\Gamma^{\mu\nu,\mu'\nu'}(p_A)_\mu=\Gamma^{\mu\nu,\mu'\nu'}(p_{A'})_{\mu'}=0,\quad \Gamma_\sigma^{\sigma,\mu'\nu'}=\Gamma_\sigma^{\mu\nu,\sigma}=0,$$
$$\Gamma^{\mu\nu,\mu'\nu'}\Gamma^{\rho\sigma,}_{\mu'\nu'}=\frac{1}{2}(\delta^{\perp\mu\rho}\delta^{\perp\nu\sigma}+\delta^{\perp\mu\sigma}\delta^{\perp\nu\rho}-\delta^{\perp\mu\nu}\delta^{\perp\rho\sigma}).\tag{26}$$

Below, we shall use Eqs. (22) and (25) to calculate the inelastic amplitudes.

3. INELASTIC AMPLITUDES IN MULTIREGGEON KINEMATICS

In renormalizable field theories at high energies, $s\gg m^2$, the main contribution to the total cross section of the inelastic process $2\to 2+n$ comes from the kinematic region[7]

$$s_{i+1}\equiv(p_{D_i}+p_{D_{i+1}})^2\gg q_1^2\sim q_2^2\sim\ldots\sim q_{n+1}^2,\tag{27}$$

where $p_{D_0}=p_{A'}, p_{D_1},\ldots,p_{D_{n+1}}=p_{B'}$ are the momenta of the particles in the final state, and $q_i=p_A-p_{D_0}-p_{D_1}-\ldots-p_{D_{i-1}}$ is the set of momentum transfers that in the case of chromodynamics are equal in order of magnitude to the transverse momenta of the quarks in the colliding hadrons.[8] From the conditions of reality of the final particles, we obtain a restriction on the product of the invariants s_i:

$$s_1 s_2 \ldots s_{n+1} = s \prod_{i=1}^{n} (m_i^2 - p_{D_i}^{\perp 2}) \sim s q^{2n}, \tag{28}$$

where m_i are the masses and $p_{D_i} = q_i - q_{i+1}$ the momenta of the produced particles, and q is the characteristic transverse momentum.

When the total cross section is calculated in the framework of the tree approximation of perturbation theory in the case of gravitation, the integrals over q_i^2 diverge in the ultraviolet region. (This is due to the fact that the coupling constant $\varkappa$ has a nonzero dimensionality.) Nevertheless, in the present paper we shall restrict ourselves to calculating the inelastic amplitudes in the multireggeon kinematics (27), assuming that the radiative corrections result in a rapid decrease of the amplitudes with growth of the momentum transfers q_i, this leading to convergence of the corresponding integrals over q_i^2. As will be shown in the following sections, this assumption is to a large degree justified.

We begin by considering the simplest inelastic process, $AB \to A'DB'$, and we shall assume that in the case of Yang-Mills theory all the particles participating in the reaction are gluons, and in the case of gravitation gravitons. In the multireggeon kinematics (27), (28), we have the following Sudakov expansion for the momentum of a produced particle:

$$p_D = \frac{s_2}{s} p_A + \frac{s_1}{s} p_B + p_D^{\perp}, \quad (p_D^{\perp}, p_A) = (p_D^{\perp}, p_B) = 0,$$
$$(p_D^{\perp})^2 = (q_1^{\perp} - q_2^{\perp})^2 \sim q_1^2 \sim q_2^2 \ll s, \quad s_1 s_2 = s(-p_D^{\perp 2}). \tag{29}$$

We shall recover the inelastic amplitude $A_{2\to 2+1}$ in the tree approximation by the dispersion method, using the properties of analyticity and unitarity in the q_1^2 and q_2^2 channels. We find first the pole singularity of $A_{2\to 3}$ at $q_1^2 = 0$. In the Regge region $s_1 \gg q_1^2$ (27), we can represent the propagators (4) for the gluon and graviton in the q_1^2 channel in the factorized form

$$D^{\rho\mu}(q_1) = -\frac{\delta^{ic_1}}{q_1^2} \frac{(p_B)^\rho (p_A)^\mu}{p_A p_B}, \quad D^{\rho\sigma,\mu\nu}(q_1) = \frac{1}{q_1^2} \frac{p_B^\rho p_B^\sigma p_A^\mu p_A^\nu}{(p_A p_B)^2}, \tag{30}$$

which from the point of view of a partial-wave expansion in the q_1^2 channel corresponds to the contribution from the state with "nonsense" helicity.[9] The indices ρ and σ in Eq. (30) refer to the end which is joined to the lines corresponding to the particles A and A'. The corresponding vertices are tensors of the type (13), which give factors s and s^2 when contracted with p_B^ρ and $p_B^\rho p_B^\sigma$:

$$\Gamma_\rho^i(p_A, p_{A'}) p_B^\rho \approx s(-g) T^i \delta_{\lambda_A \lambda_A}, \quad \Gamma_{\rho\sigma}(p_A, p_{A'}) p_B^\rho p_B^\sigma \approx -{}^1/{}_2 \varkappa s^2 \delta_{\lambda_A \lambda_{A'}}, \tag{31}$$

the result (31) in the considered limit $q_1^2 \to 0$ being, as we have noted above, universal [i.e., it is valid for a particle in any representation of the group SU(n) and with any spin S]. Note that using Eqs. (30) and (31) we can derive the expression (15) independently for the elastic amplitude in the Regge limit ($s \approx -u \gg -t$). To calculate the pole contribution in q_1^2 from the amplitude $A_{2\to 3}$, we must know the elastic amplitudes for the channel s_2 in the mixed tensor-helicity representation. They can be obtained by comparing the expressions (15) and (18):

$$F_{YM} = -\frac{2s_2}{q_2^2} T^{c_2}_{Dc_1} T^{c_2}_{B'B} \Gamma^{\mu\rho}(q_1, p_D) \delta_{\lambda_B \lambda_{B'}},$$
$$F_G = \frac{s_2^2}{q_2^2} \varkappa^2 \Gamma^{\mu\nu,\rho\delta}(q_1, p_D) \delta_{\lambda_B \lambda_{B'}}, \tag{32}$$

where c_1 and c_2 are the indices of the unitary components of the gluons in the q_1^2 and q_2^2 channels, respectively, and D and ρ, δ are used for the unitary and Lorentz components of the wave function of the produced particle.

Using (30)–(32), we obtain for the pole singularity of $A_{2\to 3}$ in q_1^2 the expressions

$$A^{YM}_{2\to 3}|_{q_1^2 \to 0} = -2sg\delta_{\lambda_A \lambda_{A'}} T^{c_1}_{A'A} q_1^{-2} g C_{\rm I}^\rho(q_2, q_1) T^D_{c_2 c_1} q_2^{-2} g \delta_{\lambda_B \lambda_{B'}} T^{c_2}_{B'B},$$
$$A^G_{2\to 3}|_{q_1^2 \to 0} = s^2 \varkappa \delta_{\lambda_A \lambda_{A'}} q_1^{-2} (-\varkappa) C_{\rm I}^{\rho\delta}(q_2, q_1) q_2^{-2} \varkappa \delta_{\lambda_B \lambda_{B'}}, \tag{33}$$

where the tensors $C_{\rm I}^\rho$ and $C_{\rm I}^{\rho\delta}$, calculated using Eqs. (22) and (25) have the form

$$C_{\rm I}^\rho(q_2, q_1) = \frac{2s_2}{s} (p_A)_\mu \Gamma^{\mu\rho}(q_1, p_D)|_{q_1^2 \to 0} = -2q_{1\perp}^\rho - 2p_B^\rho \left(\frac{q_2^2}{s} + \frac{s_1}{s}\right),$$
$$C_{\rm I}^{\rho\delta}(q_2, q_1) = \frac{2s_2^2}{s^2} (p_A)_\mu (p_A)_\nu \Gamma^{\mu\nu,\rho\delta}(q_1, p_D) = \frac{1}{2} C_{\rm I}^\rho(q_2, q_1) C_{\rm I}^\delta(q_2, q_1). \tag{34}$$

A similar calculation of the pole singularity of $A_{2\to 3}$ in q_2^2 leads to the expressions (33) with the substitution $C_{\rm I} \to C_{\rm II}$, where

$$C_{\rm II}^\rho(q_2, q_1) = \frac{2s_1 (p_B)_\mu}{s} \Gamma^{\rho\mu}(p_D, q_2)|_{q_2^2 \to 0} = -2q_{2\perp}^\rho + 2p_A^\rho \left(\frac{q_1^2}{s_1} + \frac{s_2}{s}\right),$$
$$C_{\rm II}^{\rho\delta}(q_2, q_1) = \frac{2s_1^2}{s^2} (p_B)_\mu (p_B)_\nu \Gamma^{\rho\delta,\mu\nu}(p_D, q_2)|_{q_2^2 \to 0} = \frac{1}{2} C_{\rm II}^\rho(q_2, q_1) C_{\rm II}^\delta(q_2, q_1). \tag{35}$$

Our next task is to construct the inelastic amplitudes $A_{2\to 3}$ from the residues at the poles at $q_1^2 = 0$ and $q_2^2 = 0$ found above using the analyticity properties with respect to the invariants $s_1, s_2, s, q_1^2, q_2^2$ inherent in the Feynman diagrams. We shall seek these amplitudes in the multiperipheral form [cf. (33)]:

$$A^{YM}_{2\to 3} = -2sg\delta_{\lambda_A \lambda_{A'}} T^{c_1}_{A'A} q_1^{-2} g C^\rho(q_2, q_1) T^D_{c_2 c_1} q_2^{-2} \delta_{\lambda_B \lambda_{B'}} g T^{c_2}_{B'B},$$
$$A^G_{2\to 3} = s^2 \varkappa \delta_{\lambda_A \lambda_{A'}} q_1^{-2} (-\varkappa) C^{\rho\delta}(q_2, q_1) q_2^{-2} \varkappa \delta_{\lambda_B \lambda_{B'}}, \tag{36}$$

where C^ρ and $C^{\rho\delta}$ may differ from $C_{\rm I}$(34) and $C_{\rm II}$(35) by terms which vanish at $q_1^2 = 0$ and $q_2^2 = 0$, respectively. In addition, we can add to $C_{\rm I}$ and $C_{\rm II}$ tensors that vanish on the physical polarizations of the particle D. It is easy to find a particular solution for the effective vertices C^ρ and $C^{\rho\delta}$ with the given properties:

$$\tilde{C}^\rho(q_2, q_1) = C_{\rm I}^\rho(q_2, q_1) + p_D^\rho + \frac{2q_1^2}{s_1} p_A^\rho = C_{\rm II}^\rho(q_2, q_1) - p_D^\rho - \frac{2q_2^2}{s_2} p_B^\rho,$$
$$\tilde{C}^{\rho\delta}(q_2, q_1) = {}^1/{}_2 \tilde{C}^\rho(q_2, q_1) \tilde{C}^\delta(q_2, q_1). \tag{37}$$

For the case of Yang-Mills theory (see Ref. 7), the amplitude $A^{YM}_{2\to 3}$ (36) with emission vertex

$$C(q_2, q_1) = \tilde{C}(q_2, q_1) = -q_{1\perp} - q_{2\perp} + p_A \left(\frac{q_1^2}{p_D p_A} + \frac{p_D p_B}{p_A p_B}\right) - p_B \left(\frac{q_2^2}{p_D p_B} + \frac{p_D p_A}{p_A p_B}\right) \tag{38}$$

satisfies the required analyticity properties (there are Feynman diagrams with the same singularities), and the value of the amplitude does not depend on the gauge of the polarization vector of the produced particle by virtue of the relation

$$(p_D, C) = 0. \tag{39}$$

It also follows from (39) that in calculating the cross section we can sum over all four polarization vectors of the particle D, obtaining thereby a manifestly relativistically invariant result. A more general quantity—the imaginary part of the elastic amplitude with momentum transfer $q \neq 0$—can be expressed by virtue of (39) in terms of a contraction of the vertices C:

$$K^{YM}(q_2, q_1) = -C_\mu(q_2, q_1) C^\mu(q-q_2, q-q_1) = -2q^2 + 2[(q-q_1)_\perp^2 q_{2\perp}^2 + (q-q_2)_\perp^2 q_{1\perp}^2]/(q_1-q_2)_\perp^2. \quad (40)$$

The nonuniqueness of the construction of the amplitude $A_{2\to3}$ by the dispersion method is due to the possibility of adding to the vertex C (38) a term proportional to $q_1^2 q_2^2$. On dimensional grounds and because of analyticity, the maximally possible contribution can come from corrections $\sim q_1^2 q_2^2 p_A/s_1^2$ and $\sim q_1^2 q_2^2 p_B/s_2^2$, which are small in the considered kinematic region (27) compared with the terms in formula (38). Note that a change in $C(q_2, q_1)$ by terms proportional to the momentum p_D of the particle leads by virtue of the relation (39) to the same physical results.

In the case of gravitation, the vertex $C^{\rho\delta}$ in the formula for $A_{2\to3}^G$ cannot be equal to the tensor $\tilde{C}^{\rho\delta}$ (37), since the term $-2q_1^2 q_2^2 (p_A^\rho p_B^\delta + p_A^\delta p_B^\rho)/s_1 s_2$ in this tensor contradicts the requirement of there being no simultaneous singularities in the overlapping channels[2] s_1 and s_2. To avoid this contradiction, we use the nonuniqueness in the recovery of the amplitude from the poles in the q_1^2 and q_2^2 channels. We add to $\tilde{C}^{\rho\delta}$ a term chosen in a particular manner and proportional to the product of q_1^2 and q_2^2 in order to eliminate the above term:

$$C^{\rho\delta}(q_2, q_1) = {}^1/_2 C^\rho(q_2, q_1) C^\delta(q_2, q_1) - {}^1/_2 N^\rho(q_2, q_1) N^\delta(q_2, q_1). \quad (41)$$

The vector N introduced here is determined by the formula

$$N(q_2, q_1) = (q_1^2 q_2^2)^{1/2} (p_A/p_D p_A - p_B/p_D p_B). \quad (42)$$

The term we have added is chosen to make the vertex $C^{\rho\delta}(q_2, q_1)$ satisfy the transversality property

$$C^{\rho\delta}(q_2, q_1)(p_D)_\rho = C^{\rho\delta}(q_2, q_1)(p_D)_\delta = 0, \quad (43)$$

which follows from the gauge invariance of the theory.

The nonuniqueness remaining after fulfillment of the requirement (43) is associated with the possibility of adding to $C^{\rho\delta}$ terms of the form

$$\Delta C^{\rho\delta} = (-\delta^{\rho\delta}(p_D, Q) + Q^\rho p_D^\delta + Q^\delta p_D^\delta) C, \quad (44)$$

which do not change the physical content of the theory because the graviton polarization tensor $e_{\rho\delta}$ is transversal and traceless. We fixed $C^{\rho\delta}$ by imposing on the effective vertex the additional requirement

$$\delta_{\rho\delta} C^{\rho\delta} = 0, \quad (45)$$

which makes it possible to use the tensor $\delta^{\rho\rho'}\delta^{\delta\delta'}$ instead of the projection operator $\Lambda^{\rho'\delta',\rho\delta}$ (3) in the summation over the graviton helicities. Note that in the calculation of the inelastic amplitude $A_{2\to3}$ by means of the Feynman diagram technique a term of the form (44) is added to the tensor $C^{\rho\delta}$.

Using the relation (40) and the equations

$$N_\rho(q_2, q_1) N^\rho(q-q_2, q-q_1) = 4[q_1^2 q_2^2 (q-q_1)_\perp^2 (q-q_2)_\perp^2]^{1/2}/(q_1-q_2)_\perp^2,$$
$$N_\rho(q_2, q_1) C^\rho(q-q_2, q-q_1) = -2(q_1^2 q_2^2)^{1/2}\{1 - [(q-q_1)_\perp^2 + (q-q_2)_\perp^2]/(q_1-q_2)_\perp^2\}, \quad (46)$$

we obtain an expression for the contraction of the two tensors $C^{\rho\delta}$ (41):

$$K^G(q_2, q_1) = C^{\rho\delta}(q_2, q_1) C_{\rho\delta}(q-q_2, q-q_1)$$
$$= \{q^2 - [(q-q_1)_\perp^2 q_{2\perp}^2 + (q-q_2)_\perp^2 q_{1\perp}^2]/(q_1-q_2)_\perp^2\}^2$$
$$+ 4(q_1-q_2)_\perp^{-4} [q_1^2 q_2^2 (q-q_1)_\perp^2 (q-q_2)_\perp^2$$
$$- q_1^2 q_2^2 (q^\perp - q_1^\perp, q^\perp - q_2^\perp)^2 - (q-q_1)_\perp^2 (q-q_2)_\perp^2 (q_1^\perp, q_2^\perp)^2]. \quad (47)$$

We shall need this expression later when discussing the imaginary part of the elastic amplitude with momentum transfer $q \neq 0$. The following limiting expressions for the vertices C^ρ and $C^{\rho\delta}$ are helpful:

$$C^\rho(q_2, q_1)|_{q_2^2 \gg q_1^2} \approx p_D^\rho + C_I^\rho(q_2, q_1)|_{q_2^2 \gg q_1^2},$$
$$C_I^\rho(q_2, q_1)|_{q_2^2 \gg q_1^2} = -2q_1^{\perp\rho} - 4\frac{p_B^\rho}{s_2}(q_1^\perp, q_2^\perp),$$
$$C^{\rho\delta}(q_2, q_1)|_{q_2^2 \gg q_1^2} \approx {}^1/_2 [p_D^\rho + C_I^\rho(q_2, q_1)][p_D^\delta + C_I^\delta(q_2, q_1)] - {}^1/_2 N^\rho N^\delta|_{q_2^2 \gg q_1^2}, \quad (48)$$

from which by virtue of the transversality of the polarization tensors of particle D it follows that for fixed q_1^2 the differential cross section decreases rapidly with respect to q_2^2 in the ultraviolet region $q_2^2 \gg q_1^2$ (as in the $\lambda\varphi^3$ theory).

In the general case of the inelastic amplitude $A_{2\to2+n}$ in the kinematic region (27) the calculations are completely analogous. In the tree approximation, the result has the multiperipheral form [cf. (36)]

$$A_{2\to2+n}^{YM} = -2sg\delta_{\lambda_A\lambda_{A'}} T_{A'A}^{c_1} q_1^{-2} g C^{\rho_1}(q_2, q_1)$$
$$\times T_{c_2c_1}^{D_1} q_2^{-2} g C^{\rho_2}(q_3, q_2) T_{c_3c_2}^{D_2} \ldots g\delta_{\lambda_B\lambda_{B'}} T_{B'B}^{c_{n+1}},$$
$$A_{2\to2+n}^G = s^2 \varkappa \delta_{\lambda_A\lambda_{A'}} q_1^{-2}(-\varkappa) C^{\rho_1\delta_1}(q_2, q_1) q_2^{-2}(-\varkappa) C^{\rho_2\delta_2}(q_3, q_2) \ldots q_{n+1}^{-2} \varkappa \delta_{\lambda_B\lambda_{B'}}. \quad (49)$$

The proof of the expressions (49) is based on the method of mathematical induction (cf. Ref. 7). Suppose the expressions (49) are valid for all $n < n_1$; then the singular part of the amplitude $A_{2\to2+n_1}$ in the channel q_i^2 can be expressed in terms of the product of the known amplitudes $A_{2\to i}$ and $A_{2\to2+n_1-i}$, in which the initial particles are, respectively, A, c_i and c_i, B. In the amplitude $A_{2\to i}$ at the vertices (38) and (41) it is necessary to make the substitution $p_B \to -q_i$ (when s is replaced by $-2p_A q_i \approx 2p_A p_{D_{i-1}}$); in addition, the symbol $\perp$ here denotes orthogonality to the p_B, q_i plane. However, it is readily verified that in the region (27) the difference between the old and new vertices C is unimportant. Similarly, in the amplitude $A_{2\to2+n_1-i}$ it is possible to use the vector p_A instead of the vector q_i in the vertices C^ρ and $C^{\rho\delta}$. Vertices corresponding to the particles D_{i-1} and D_i arise in the form (34) and (35), but at the pole as $q_i^2 \to 0$ they can be replaced by the expressions (38) and (41). Thus, formulas (49) have the correct analytic properties in the direct channels s_i and satisfy the single-particle "unitarity conditions" in the crossed channels q_i^2. The nonuniqueness of the vertices in the amplitude $A_{2\to2+n}$ reduces, as above, to unimportant corrections of the type (44).

4. RADIATIVE CORRECTIONS TO THE INELASTIC AMPLITUDES

The formulas for the production amplitudes of soft gluons and gravitons and the radiative corrections to the amplitudes of the Regge processes due to virtual particles with small transverse momenta have a universal form. We recall that the amplitude of accompanying emission of a soft photon in the two-particle process

$AB\to A'B'$ at a high energies $s^{1/2}$ and fixed $t\sim m^2_{\text{char}}$ is

$$A_{2\to 3}=\left(Q_{A'}\frac{p_{A'}}{p_{A'}k}+Q_{B'}\frac{p_{B'}}{p_{B'}k}-Q_A\frac{p_A}{p_Ak}-Q_B\frac{p_B}{p_Bk}\right)_\mu e^\mu(k)A(s,t), \qquad (50)$$

where Q_i are the charges of the particles, $e^\mu(k)$ is the polarization vector of the γ photon with momentum k, and $A(s,t)$ is the amplitude of the main process. The region of applicability of the expression (50) is[10]

$$kp_A/p_Bp_A\ll 1,\quad kp_B/p_Bp_A\ll 1,\quad |k_\perp|\ll m_{\text{char}}. \qquad (51)$$

In the case of Yang-Mills theory the amplitude of gluon production in the two-particle process in the kinematic region (51) has the form (50) with the substitution $Q_i\to gT_i^D$, where T_i^D is the generator of the group SU(n) in the representation corresponding to particle i. In what follows in this section, we shall consider only gravitation, since for Yang-Mills theory the generalization of the expressions found below is trivial (see Ref. 7). For the amplitude for emission of an additional soft graviton in an arbitrary process $m\to n$ we have the representation

$$A_{m\to n+1}=-\varkappa\sum_r\eta_r\frac{p_r{}^\rho p_r{}^\delta}{(p_rk)}e_{\rho\delta}(k)A_{m\to n}, \qquad (52)$$

where $A_{m\to n}$ is the amplitude of the main process on the mass shell, and $\eta_r=\pm 1$ depending on whether the final or initial particle emits the graviton. The expression (52) is gauge invariant by virtue of the energy–momentum conservation law $\Sigma\eta_r p_r=0$.

Obviously, for the validity of the expression (52) it is necessary that the invariants on which the amplitude $A_{m\to n}$ depends essentially should vary little as a result of emission of the graviton. This condition is also sufficient, as can be seen in each concrete case by considering the many-particle imaginary parts of the amplitudes $A_{m\to n+1}$ with respect to the invariants $(p_r+\eta_r k)^2$. These imaginary parts are small because the graviton interacts with the conserved energy–momentum tensor (cf. Ref. 10).

We consider now the special case when the main process is an inelastic collision with the production of n additional particles in the multireggeon kinematics (27). The emission of a graviton does not change the kinematics of the main process if the transverse component of its momentum k is sufficiently small. Moreover, if its Sudakov parameters in the expansion $k=\alpha p_B+\beta p_A+k_\perp$ satisfy the condition [see (29)]

$$\begin{gathered}1\approx\beta_{D_0}\gg\beta_{D_1}\gg\ldots\gg\beta_{D_{i-1}}\gg\beta\gg\beta_{D_i}\gg\ldots\gg\beta_{D_{n+1}}\sim q_{n+1}^2/s,\\ q_1^2/s\sim\alpha_{D_0}\ll\alpha_{D_1}\ll\ldots\ll\alpha_{D_{i-1}}\ll\alpha\ll\alpha_{D_i}\ll\ldots\ll\alpha_{D_{n+1}}\approx 1,\quad |k_\perp^2|\ll q_i^2,\end{gathered} \qquad (53)$$

it is readily verified that the expression (52) simplifies as follows:

$$\begin{gathered}A_{2\to 2+n+1}=-\varkappa A_{2\to 2+n}\gamma^{\rho\delta}(q_i-k,q_i)e_{\rho\delta}(k),\\ \gamma^{\rho\delta}(q_i-k,q_i)=2[p_A{}^\rho p_A{}^\delta/(s\alpha)^2-p_B{}^\rho p_B{}^\delta/(s\beta)^2](k_\perp q_i{}^\perp)\\ -2q_{i\perp}{}^\rho[p_A{}^\delta/s\alpha-p_B{}^\delta/s\beta]+(\rho\rightleftharpoons\delta),\end{gathered} \qquad (54)$$

where $(\rho\rightleftharpoons\delta)$ denotes the terms obtained from the given terms by means of the indicated substitution.

Note that for an inelastic graviton–graviton collision formula (54) in the region (53) follows from the expression (49), since the effective graviton emission vertex (41) has the property

$$C^{\rho\delta}(q_i-k,q_i)=q_i{}^2\gamma^{\rho\delta}(q_i-k,q_i),\quad |k_\perp{}^2|\ll q_i{}^2. \qquad (55)$$

The representation (54) can be readily generalized to the emission of any number r of gravitons:

$$A_{2\to 2+n+r}=\prod_{i=1}^r(-\varkappa)\gamma_i{}^{\rho\delta}e_{\rho\delta}{}^iA_{m\to 2+n}. \qquad (56)$$

For a virtual graviton in the region of integration (53), we can modify the propagator as follows,

$$i\delta_{\rho\rho'}\delta_{\delta\delta'}/(k^2+i\varepsilon)\to i\delta_{\rho\rho'}\delta_{\delta\delta'}(-i\pi)\delta(k^2), \qquad (57)$$

since in the integral in the principal value sense the large contribution $\sim\ln s$ is lost. Then for the lowest radiative correction to the process $2\to 2+n$ we obtain from Eq. (56) (for $r=2$) the factorized expression

$$A^{(1)}_{m\to n}=\rho(q_i{}^2)A_{m\to n}, \qquad (58)$$

$$\rho(q_i{}^2)\,|_{k_\perp{}^2\ll q_i^2}=\frac{1}{2}\frac{\varkappa^2}{(2\pi)^4}\int d^4k\pi\delta(k^2)\gamma_{\rho\delta}(q_i-k,q_i)\gamma^{\rho\delta}(q_i,q_i-k). \qquad (59)$$

Here, the factor $1/2!$ is needed to compensate the double counting of the same Feynman diagrams in our procedure, in which the emission and absorption of a graviton are taken into account independently.

Substituting $\gamma_{\rho\delta}$ in Eq. (59) and introducing the Sudakov variables, we obtain

$$\rho(q_i{}^2)\,|_{k_\perp{}^2\ll q_i{}^2}=\frac{\varkappa^2}{(2\pi)^4}\frac{|s|}{2}\int d\alpha d\beta d^2k_\perp\left[\frac{4s^2(k_\perp q_i{}^\perp)^2}{(-s\alpha+i\varepsilon)^2(s\beta+i\varepsilon)^2}-\frac{4sq_{i\perp}{}^2}{(-s\alpha+i\varepsilon)(s\beta+i\varepsilon)}\right]\pi\delta(k^2). \qquad (60)$$

Using the δ function to calculate the integral over α and calculating the logarithmic integral over β in the interval $\beta_{D_{i-1}}\ll|\beta|\ll\beta_{D_i}$ [see (53)] and averaging over the angles in the $k_\perp$ plane, we obtain

$$\rho(q_i^2)=\omega(q_i^2)\ln\frac{\beta_{D_{i-1}}}{\beta_{D_i}}=\omega(q_i^2)\ln\frac{s_i}{q^2},\quad q^2=\max(q_i^2,q_{i-1}^2,q_{i+1}^2), \qquad (61)$$

$$\omega(q_i{}^2)\,|_{-k_\perp{}^2\ll -q_i{}^2}=\frac{\varkappa^2}{(2\pi)^3}q_i{}^2\int\frac{d^2k_\perp}{-k_\perp{}^2}\Bigg|_{-k_\perp^2\ll q^2_i}. \qquad (62)$$

Integrals of the type (62) must be truncated below by a fictitious quantity such as a graviton mass, this being replaced after calculation of the cross section by the energy resolution $\Delta\omega$.

The contribution of an arbitrary number of virtual gravitons can be calculated similarly. At the same time, it is easy to show that the expression (61) is taken into the argument of the exponential, as a result of which we obtain

$$A^{\infty}_{2\to 2+n}\,|_{-k^2_{\perp i}\ll -q_i^2}=A_{2\to n+2}\prod_{i=1}^{n+1}\left(\frac{s_i}{q^2}\right)^{\omega(q_i^2)}\Bigg|_{-k^2_{\perp i}\ll q_i^2}, \qquad (63)$$

i.e., if $A_{2\to 2+n}$ has the multireggeon form

$$A_{2\to 2+n}\sim\prod_{i=1}^{n+1}s_i^{\alpha(q_i^2)},$$

then as a result of integration over the region of small transverse momenta of the virtual gravitons there arises a universal correction to the Regge trajectory: $\Delta\alpha(q_i^2)=\omega(q_i^2)$, $-k_\perp^2\ll -q_i^2$.

We now show that the expression (58) also holds for virtual gravitons with momenta $-k_\perp^2 \sim q_i^2$ when the main process is inelastic gravitational interaction in the multireggeon kinematics (27). We must here use the expression (49) for the amplitude $A^G_{2\to 2+n+2}$ in the kinematics when two additional gravitons with momenta k and $-k$ are emitted from the line with momentum q_i [see the inequalities (53) for β and α]. Instead of the expression (59), we obtain in this case

$$\rho(q_i^2) = -\frac{1}{2}\cdot\frac{1}{2}\frac{\varkappa^2 i}{(2\pi)^4}\int \frac{d^4k q_i^{-2}}{k^2(q_i-k)^2} C^{\rho\delta}(q_i-k, q_i) C_{\rho\delta}(q_i, q_i-k), \quad (64)$$

where $C_{\rho\delta}$ is determined by Eq. (41), and it is assumed that $C_{\rho\delta}C^{\rho\delta}$ is continued in an appropriate manner off the mass shell $k^2=0$. The additional factor $\frac{1}{2}$ in Eq. (64) compared with (59) is due to the need to compensate the doubled contribution which arises as a result of replacement of one of the propagators $1/k^2$ and $1/(q_i-k)^2$ in accordance with (57). In the product $C_{\rho\delta}C^{\rho\delta}$, we retain only the terms that have a singularity in $s\alpha$ or $s\beta$, since the contribution of the remaining terms does not have a singularity in s and therefore does not contain a large factor $\ln s$. In addition, we add terms proportional to k^2 to achieve symmetry under the substitution $k \to q_i - k$:

$$\frac{1}{q_i^2} C^{\rho\delta}(q_i-k, q_i) C_{\rho\delta}(q_i, q_i-k)$$
$$= q_i^2\left[\frac{4s^2(k_\perp, q_i-k_\perp)^2}{(-s\alpha+i\varepsilon)^2(s\beta+i\varepsilon)^2} - \frac{4sq_i^2}{(-s\alpha+i\varepsilon)(s\beta+i\varepsilon)}\right]. \quad (65)$$

The corrections $i\varepsilon$ in the denominators are introduced to ensure the correct analytic properties of the amplitude (a more accurate form of expression requires additional symmetrization of (65) with respect to the substitutions $\alpha \to -\alpha$ and $\beta \to -\beta$, and also allowance for the terms $s\alpha\beta$ in the denominator).

Calculating the integral over α in the expression (64) from the residues for $\beta_{D_{i-1}} \ll \beta \ll \beta_{D_i}$, we obtain for the graviton trajectory determined by the relation (61) the expression

$$\omega(q^2) = q^2 \frac{\varkappa^2}{(2\pi)^3}\int \frac{d^2k_\perp}{k_\perp^2(q-k_\perp)^2}\left[(k_\perp, q-k_\perp)^2\left(\frac{1}{k_\perp^2}+\frac{1}{(q-k_\perp)^2}\right) - q^2\right]. \quad (66)$$

Note that the contribution from the domains of integration $k_\perp^2 \to 0$ and $(q-k_\perp)^2 \to 0$ is equal to the contribution (62) obtained above.

In the case of an arbitrary number of virtual gravitons, a formula of the type (63) arises when allowance is made for the ordering of their Sudakov components α_i and β_i. Taking into account the expression (49), which corresponds to the amplitude in the tree approximation, we obtain the following expression for the inelastic amplitude $A_{2\to 2+n}$ in the multireggeon kinematics (27):

$$A_{2\to 2+n} = s^2\varkappa\delta_{\lambda_A\lambda_{A'}}\frac{s_1^{\omega(q_1^2)}}{q_1^2}(-\varkappa)C^{\rho_1\delta_1}(q_2, q_1)\frac{s_2^{\omega(q_2^2)}}{q_2^2}(-\varkappa)C^{\rho_2\delta_2}(q_3, q_2)$$
$$\ldots\frac{s_{n+1}^{\omega(q_{n+1}^2)}}{q_{n+1}^2}\varkappa\delta_{\lambda_B\lambda_{B'}}. \quad (67)$$

Thus, due to the radiative corrections the graviton is reggeized, i.e., in the j plane there arises a Regge pole with trajectory

$$j(q^2) = 2 + \omega(q^2), \quad (68)$$

which for $q^2=0$ passes through the physical point equal to the graviton spin:

$$j(q^2)|_{q^2=0} = 2. \quad (69)$$

Note that to a good accuracy the trajectory (68) is linear,

$$j \approx 2 + Cq^2 \ln q^2,$$

but it contains an infrared and an ultraviolet logarithmic divergence. As usual, the infrared divergence cancels in the calculation of inclusive cross sections due to the emission of soft particles. Here, the fictitious graviton mass which must be introduced into the propagators of the expression (66) is replaced by the energy resolution $\Delta\omega$ for the final particles. With regard to the ultraviolet divergence of the trajectory for $k^2 \gg q^2$, its appearance indicates the occurrence of doubly logarithmic terms $\sim\varkappa^2 q^2 \ln^2 s$, since it is known that in the single-loop approximation gravitation is renormalizable, i.e., all divergences can be included in the renormalized coupling constant $\varkappa$. It could well be that in supergravity too there is no such divergence.

To conclude this section, we discuss the occurrence of graviton reggization in the language of Feynman diagrams. We consider the scattering of two particles A and B due to the exchange of two gravitons. At high energies, the maximal contribution is due to the nonsense polarizations of the virtual gravitons, which leads to the approximation representation (30) for the propagators $D^{\rho\delta,\mu\nu}$. Introducing the Sudakov variables, we can represent the scattering amplitude in the form

$$A_{2\to 2} = -\frac{i}{2!}\frac{1}{(2\pi)^4}\int\frac{1}{2|s|}d(\alpha s)\,d(\beta s)\,d^2k_\perp D^{\rho_1\sigma_1,\mu_1\nu_1}(k)D^{\rho_2\sigma_2,\mu_2\nu_2}(q-k)$$
$$\times A_{\rho_1\sigma_1,\rho_2\sigma_2}(-s\alpha, k_\perp, q-k_\perp)A_{\mu_1\nu_1,\mu_2\nu_2}(s\beta, k_\perp, q-k_\perp), \quad (70)$$

where $A(-s\alpha)$ and $A(s\beta)$ are the amplitudes for scattering of a virtual graviton by the initial particles A and $B[(p_A-k)^2 \approx -s\alpha, (p_B+k)^2 \approx s\beta]$.

In the important region of integration

$$-s\alpha \sim s\beta \sim -q^2,\quad k_\perp^2 \sim q^2 \quad (71)$$

the expression (70) can, after use of formula (30), be simplified as follows (cf. Ref. 8):

$$A_{2\to 2} = \frac{i}{4}|s|^3\int\frac{d^2k_\perp}{k_\perp^2(q-k_\perp)^2}\rho_A(k_\perp, q-k_\perp)\rho_B(k_\perp, q-k_\perp), \quad (72)$$

where the factors ρ_A and ρ_B do not depend on s,

$$\rho_A(k_\perp, q-k_\perp) = 4\int\frac{p_B^{\rho_1}p_B^{\sigma_1}p_B^{\rho_2}p_B^{\sigma_2}}{s^4}A_{\rho_1\sigma_1,\rho_2\sigma_2}d(\alpha s), \quad (73)$$

since the tensor $A_{\rho_1\sigma_1,\rho_2\sigma_2}$ contains terms proportional to $(p_A)_{\rho_1}(p_A)_{\sigma_1}\cdot(p_A)_{\rho_2}(p_A)_{\sigma_2}$. The asymptotic formula (72) has no bearing on the reggeization of the graviton, but is the contribution of the Pomeranchuk singularity in the lowest approximation of perturbation theory (this singularity is near the point $j=3$). To obtain the expression (58), which is proportional to $s^2\ln s$, it is necessary to consider in the integral (70) the region of large values of the invariants $s\alpha$ and $s\beta$ [cf. (71)]:

$$s\alpha \gg q^2,\quad s\beta \gg q^2,\quad s\alpha\beta \sim -k_\perp^2 \sim -q^2. \quad (74)$$

At the same time, if we use the following representation for the amplitude $A(s\alpha)$ [see (11)]:

$$A_{\rho_1\sigma_1,\rho_2\sigma_2}(-s\alpha, k_\perp, q-k_\perp)=i\int d^4x e^{ikx}\langle A'|T(T^M_{\rho_1\sigma_1}(x)T^M_{\rho_2\sigma_2}(0))|A\rangle, \quad (75)$$

then in the region (74) it is small x^2 which are important, which makes it possible to use a Wilson expansion. In the matrix element of the product of operators $T^M_{\rho_1\sigma_1}(x)$ and $T^M_{\rho_2\sigma_2}(0)$ there is a contribution proportional to $\langle A'|T^M_{\mu\nu}(0)|A\rangle$, the graviton–matter interaction vertex. The coefficient of proportionality between $A(s\alpha)$ and $\langle A'|T^M_{\mu\nu}(0)|A\rangle$ is universal (it does not depend on the structure of the theory or the species of the particles A and A'), and therefore it can be found from the amplitude for graviton scattering by a structureless scalar particle in the Born approximation, for which formula (13) holds for the energy–momentum tensor. For the scattering of point scalar particles, there are two Feynman graphs corresponding to two-graviton exchange; they are "squares" in the s and u channels (the contribution of the remaining diagrams does not contain singularities in one of the channels s, u, or t). In the calculation of the logarithmic contribution $\sim s^2\ln s$, it can be assumed that the scattering amplitudes of scalar particles due to single-graviton exchange, from the product of which our amplitude is constructed, are on the mass shell. Using formula (14) for F_G with $m^2=0$, we obtain for the amplitude $A^{(1)}_{2\to 2}$ (70) the expression

$$\begin{aligned}A^{(1)}_{2\to 2}=&-\frac{i\varkappa^4}{2l(2\pi)^4}\int\frac{d^4k}{(k^2+i\varepsilon)((q-k)^2+i\varepsilon)}\\&\times\Big[\Big(\frac{s^2+u^2}{2}\Big)^2\Big(\frac{1}{s_1+i\varepsilon}+\frac{1}{u_1+i\varepsilon}\Big)\Big(\frac{1}{s_2+i\varepsilon}+\frac{1}{u_2+i\varepsilon}\Big)\\&+\frac{s^2+u^2}{2}\frac{s-u}{2}(k^2+(q-k)^2-2q^2)\Big(\frac{1}{s_1+i\varepsilon}-\frac{1}{u_1+i\varepsilon}\Big)\Big(\frac{1}{s_2+i\varepsilon}-\frac{1}{u_2+i\varepsilon}\Big)\Big]\\&s_1=(p_A-k)^2,\quad u_1=(p_{A'}+k)^2,\quad s_2=(p_B+k)^2,\quad u_2=(p_{B'}-k)^2.\end{aligned}\quad (76)$$

In the region of integration (74), we can use the approximate formula

$$1/s_i+1/u_i\approx 2[(q^\perp-k^\perp, k^\perp)-s\alpha\beta]/s_i^2. \quad (77)$$

Subsequent integration over α and β leads to the expression (58), and $\rho(q^2)$ is determined by formulas (60) and (66).

In conclusion, it should be noted that it is difficult to generalize the approach based on the calculation of the contributions of Feynman diagrams to the case of the inelastic amplitude $A_{2\to 2+n}$ with an arbitrary number of virtual gravitons.

5. TOTAL CROSS SECTIONS OF INELASTIC GRAVITON-GRAVITON INTERACTION

To calculate the elastic amplitude $A_{2\to 2}$ for the scattering of two gravitons, we can use the s- and u-channel unitarity condition. The contribution to the s-channel imaginary part from the $(n+2)$-particle intermediate state in the multireggeon region (27) can be expressed in terms of the product of the inelastic amplitudes (67) in accordance with the formula (cf. Ref. 7)

$$\begin{aligned}A_{2\to n+2\to 2}(s,t)\approx i\,\mathrm{Im}\,A_{2\to n+2\to 2}(s,t)=\frac{\pi i}{s}\int\prod_{i=1}^{n+1}\frac{d^2k_\perp{}^i}{2(2\pi)^3}\prod_{i=1}^{n+1}ds_i\\ \times\delta\Big(sq^{2n}-\prod_{i=1}^{n+1}s_i\Big)A_{2\to n+2}(s_i,k_\perp{}^i)A_{2\to n+2}(s_i,q-k_\perp{}^i),\quad q^2=t.\end{aligned}\quad (78)$$

Going over to the ω representation

$$A_{2\to 2}(s,t)=s^2\int\frac{d\omega}{2i}s^\omega\frac{1-e^{-i\pi\omega}}{\sin\pi\omega}f_\omega(t), \quad (79)$$

we can write down for the elastic amplitude

$$f_\omega(t)=f_\omega(k,k-q)\,|_{k^2=(q-k)^2=0}$$

with arbitrary number of particles in the intermediate state the Bethe-Salpeter equation

$$\begin{aligned}&[\omega-\omega(k_\perp{}^2)-\omega((q-k)_\perp{}^2)]f_\omega(k,k-q)\\&=\frac{\varkappa^2\omega}{q^4}+\varkappa^2\int\frac{d^2k_\perp'}{2(2\pi)^3}\frac{1}{k_\perp'^2(q-k')_\perp{}^2}K(k,k')f_\omega(k',k'-q),\end{aligned}\quad (80)$$

where the kernel of the integral equation $K(k,k')$ is determined by formula (47).

The total cross section of graviton–graviton scattering can be expressed in terms of the imaginary part of the elastic amplitude (79) at $t=0$:

$$\sigma(s)=s^2\int\frac{d\omega}{2i}s^\omega f_\omega(k)\,|_{k^2=0,q^2\to 0}, \quad (81)$$

where for the amplitude $f_\omega(k)$ we have the integral equation

$$[\omega-2\omega(k_\perp{}^2)]f_\omega(k)=\frac{\varkappa^2\omega}{q^4}+\varkappa^2\int\frac{d^2k_\perp'}{2(2\pi)^3}\frac{1}{(k_\perp')^4}K(k,k')\,|_{q^2\to 0}f_\omega(k'). \quad (82)$$

Thus, we have shown above that in Einstein's theory the graviton is reggized, and the pomeron is a bound state of reggeons (in our approximation, there are two reggeons). For the scattering amplitudes at high energies in the general case there arises a reggeon diagram technique in which the interaction vertices of the reggeized gravitons and their trajectories can, in principle, be calculated by perturbation theory. It is possible that the infinite number of counter terms needed to eliminate the ultraviolet divergences in gravitation could be fixed by analytic continuation of the partial waves $f_j(t)$ from large j and negative t.

I thank A. A. Ansel'm for helpful discussions.

[1)] Note that the indices of the derivatives in the expression (8) are raised by means of the tensor $g^{\mu\nu}$.

[2)] Note that the poles of second order in s_1 and s_2 in $C^{\rho\delta}$ do not contradict the analytic properties of the inelastic amplitude $A_{2\to 3}$, since these poles arise as a result of almost complete cancellation of the contributions of the poles of first order in s_i and u_i in the asymptotic region (29).

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Translated by Julian B. Barbour