Reflection coefficient in the one-dimensional problem of self-action of a wave

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The problem is considered of normal incidence of a monochromatic wave on a half-space filled with a nonlinear medium whose dielectric constant is determined by the intensity of the wave field. The field on the boundary of the medium and the coefficient of reflection from the medium are investigated as functions of the incident-wave intensity for the simplest nonlinearity types typical of problems in nonlinear optics, ionospheric physics, and plasma physics. The analysis is based on the theory developed by Babkin and Klyatskin [Sov. Phys. JETP **52**, 416 (1980)].

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The problem of the incidence of a monochromatic wave on a layer $L_0 \leqslant x \leqslant L$ of a medium whose dielectric constant is determined by the intensity of the wave field is of great interest, and is the subject of a large number of investigations (for a bibliography see, e.g., Ref. 2). In the simplest one-dimensional case (normal incidence of the wave on an inhomogeneous layered medium), which is the one considered in the present paper, the wave field in the layer is described by the nonlinear Helmholtz equation

$$\frac{d^2U(x)}{dx^2}+k^2[1+\varepsilon(x,J(x))]U(x)=0, \quad J(x)=|U(x)|^2 \tag{1}$$

with conditions that the field and its derivative be continuous on the boundaries of the layer. If a wave $U_0(x) = v\exp\{ik(L-x)\}$ is incident on the layer, the solution of the problem can be represented in the form $U(x) = vu(x)$, where $u(x)$ satisfies the equation

$$\frac{d^2u(x)}{dx^2}+k^2[1+\varepsilon(x,wI(x))]u(x)=0, \quad w=|v|^2, \quad I(x)=|u(x)|^2 \tag{1'}$$

and the boundary conditions on the layer boundaries

$$u'(L)=-ik[2-u(L)], \quad u'(L_0)=-iku(L_0) \quad (u'(x)=du(x)/dx).$$

The coefficient of wave reflection from the layer is defined in this case as $\rho=u(L)-1$. In the general case ε is a complex quantity, i.e.,

$$\varepsilon(x,J(x))=\varepsilon_1(x,J(x))+i\gamma(x,J(x)),$$

where $\varepsilon_1=\varepsilon_1^*$ and the quantity γ describes the damping of the wave.

In the absence of damping (i.e., $\gamma=0$) and in the absence of inhomogeneities of the medium [i.e., $\varepsilon(x,J(x)) \equiv \varepsilon(J(x))$], there exist for the problem (1) two integrals[2,3]

$$U(x)\frac{dU^*(x)}{dx}-U^*(x)\frac{dU(x)}{dx}=\text{const},$$
$$\frac{dU(x)}{dx}\frac{dU^*(x)}{dx}+k^2\int_{J_0}^{J(x)}[1+\varepsilon(t)]dt=\text{const}, \tag{2}$$

which are customarily used to obtain the structure of the field inside the layer (by subsequent integration[2,3] and using phase-plane analysis[4]). The possible types of solutions are then matched on the boundary of the layer and the incident wave. In a number of cases ambiguities arise, since the same field inside the medium can correspond to several incident-wave intensities and hence to several reflection coefficients. This ambiguity can remain also in the presence of damping, when the integrals (2) are absent. It should also be noted that the solution of the problem (1) is determined as a rule by the critical (bifurcation) values of the parameters, assumption of which within the framework of the integrals (2) raises great difficulties.

An approach based on a different principle was developed for the solution of the problem in Ref. 1. Using the "imbedding" method, the original boundary-value problem reduces to an initial-value problem relative to the "imbedding" parameters, namely the position L of the right-hand layer boundary (on which the wave is incident) and the intensity w of the incident wave. The field at the boundary (or the reflection coefficient) is then described by a closed nonlinear equation, while the field inside the layer of the medium satisfies already a linear equation. This approach, as will be seen below, makes it possible to obtain, in the absence of damping, the dependence of the modulus of the reflection coefficient on the incident-wave intensity in the general case of arbitrary nonlinearity, and in the presence of damping, for concrete types of nonlinearity, the obtained equations are convenient for numerical analysis. No difficulties are raised when the problem parameters are equal to the birfurcation values.

It was shown in Ref. 1 that when account is taken of the dependence of the solution on L and w as parameters, the problem (1′) is described equivalently by the system of equations

$$\frac{\partial u(x;L,w)}{\partial L}=a(L,w)u(x;L,w)+b(L,w)w\frac{\partial u(x;L,w)}{\partial w}, \tag{3}$$
$$u(x;L,w)|_{L=x}=u_x(w);$$

$$\frac{\partial u_L(w)}{\partial L}=2ik[u_L(w)-1]+\frac{ik}{2}\varepsilon(L,wI_L)u_L^2+b(L,w)w\frac{\partial u_L(w)}{\partial w}, \quad u_{L_0}(w)=1, \tag{4}$$

where

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$$u_L(w)=u(L;L,w),\quad I_L(w)=|u_L(w)|^2,$$
$$a(L,w)=ik+{}^1/_2 ik\varepsilon(L,wI_L)u_L(w),$$
$$b(L,w)=a(L,w)+a^*(L,w).$$

The coefficient of reflection of the wave from the layer is determined by the equation $\rho_L(w)=u_L(w)-1$, with $|u_L(w)-1|^2\leq 1$.

At $w=0$, Eqs. (3) and (4) reduce to the corresponding relations of the linear problem, derived in Ref. 5 (see also Ref. 6). It is natural to assume that their solutions coincide with the limiting values of the solutions of Eqs. (3) and (4) as $w\to 0$.

In the case $\varepsilon(x,J)\equiv\varepsilon(J)$ one can go to the limit as $L_0\to-\infty$, corresponding to incidence of a wave on a half-space $x<L$. In this case Eqs. (3) and (4) take the form

$$\frac{\partial u(\xi,w)}{\partial\xi}=a(w)u(\xi,w)+b(w)w\frac{\partial u(\xi,w)}{\partial w},\quad u(0,w)=u(w);\tag{5}$$

$$b(w)w\frac{du(w)}{dw}=2i[1-u(w)]-\frac{i}{2}\varepsilon(wI(w))u^2(w),\tag{6}$$

where now

$$\xi=k(L-x)\geqslant 0\quad(k=k^*),$$
$$a(w)=i+\frac{i}{2}\varepsilon(wI)u(w),\quad b(w)=a(w)+a^*(w),\quad I(w)=|u(w)|^2.$$

At $w=0$, Eqs. (5) and (6) determine the initial values

$$u(0)=\frac{2}{1+\alpha},\quad \alpha=[1+\varepsilon(0)]^{1/2},\quad \operatorname{Im}\alpha>0,\quad \operatorname{Re}\alpha>0,$$
$$u(\xi,0)=\frac{2}{1+\alpha}\exp\{i\alpha\xi\},\tag{7}$$

which correspond to the solution of the linear problem. All the equations written out above were derived in Ref. 1. We consider below in greater detail their structure and their corollaries.

From (5) we obtain for the intensity $J(\xi,w)=w|u(\xi,w)|^2$ of the wave field inside the medium the equation

$$\frac{\partial}{\partial\xi}J(\xi,w)=b(w)w\frac{\partial}{\partial w}J(\xi,w)\tag{8}$$

with the initial condition $J(0,w)=wI(w)$, where $I(w)$ is obtained from the solution of Eq. (6).

A parametric representation of the solution of Eq. (8) can be easily obtained by the method of characteristics:

$$\xi=-\int_{\bar w}^{w}\frac{dw}{wb(w)},\quad J(\xi,w)=\bar wI(\bar w).\tag{9}$$

Eliminating the parameter $\bar w$, we ultimately obtain in explicit form the function $J=J(\xi,w)$.

Thus, the solution of the problem reduces to finding the field $u(w)$ on the boundary or finding the reflection coefficient $\rho(w)=u(w)-1$. We assume $\varepsilon(J)=\varepsilon_1(J)+i\gamma(J)$, where $\varepsilon_1=\varepsilon_1^*$, and the quantity $\gamma(J)$ describes the damping of the wave. Putting $u(w)=R(w)+iS(w)$ and separating in (6) the real and imaginary parts, we obtain

$$wb(w)dR/dw=2S+\varepsilon_1(wI)RS+{}^1/_2\gamma(wI)(R^2-S^2),$$
$$wb(w)dS/dw=2(1-R)-{}^1/_2\varepsilon_1(wI)(R^2-S^2)+\gamma(wI)RS,\tag{10}$$

where $I(w)=R^2+S^2$ and $b(w)=-[\gamma(wI)R+\varepsilon_1(wI)S]$. We note that these equations lead to the equalities

$$b(w)dwR/dw=2S-{}^1/_2\gamma(wI)I,\quad b(w)dwI/dw=4S.\tag{11}$$

From the condition $|\rho|^2\leq 1$ follow the restrictions $0\leq R\leq 2$ and $-1\leq S\leq 1$, the equal signs being applicable only at $\gamma\equiv 0$. In this latter case, when there is no damping, the system (10) takes the simpler form

$$w\varepsilon_1(wI)SdR/dw=-S[2+\varepsilon_1(wI)R],$$
$$w\varepsilon_1(wI)SdS/dw=2(R-1)+{}^1/_2\varepsilon_1(wI)(R^2-S^2),\tag{12}$$

and relations (11) become

$$\varepsilon_1(wI)S\frac{dwR}{dw}=-2S,\quad \varepsilon_1(wI)S\frac{dwI}{dw}=-4S.\tag{13}$$

Considering Eq. (12) to be a system of ordinary differential equations (without allowance for the initial conditions), we see that all the solutions of this system break up into solutions of two types.

I. $S(w)\equiv 0$. The first equation of (12) is then identically satisfied, and the second leads to a transcendental equation for $R(w)$:

$$4(1-R)=R^2\varepsilon_1(wR^2).\tag{14}$$

In this case $b(w)\equiv 0$, and the solution of (8) is

$$J(\xi,w)=wR^2(w),\tag{15}$$

which corresponds to a plane wave propagating in a nonlinear medium. The wave field is determined from (5):

$$u(\xi,w)=R(w)\exp\left\{i\xi\frac{2-R(w)}{R(w)}\right\}.$$

This type of solution was analyzed in Ref. 2.

II. $S(w)\neq 0$. Assume that at $w=w_0$ we have $R(w_0)=R_0$ and $S(w_0)=S_0$. Then, dividing both sides of (12) and (13) by S, we obtain the system of equations

$$w\varepsilon_1(wI)dR/dw=-[2+\varepsilon_1(wI)R],$$
$$w\varepsilon_1(wI)SdS/dw=2(R-1)+{}^1/_2\varepsilon_1(wI)(R^2-S^2)$$

and the equalities

$$\varepsilon_1(wI)\frac{dwR}{dw}=-2,\quad \varepsilon_1(wI)\frac{dwI}{dw}=-4.$$

Integrating these last relations, we obtain the equations

$$\int_{w_0I_0}^{wI(w)}\varepsilon_1(t)dt=-4(w-w_0),\quad I_0=R_0^2+S_0^2,$$
$$wI(w)-2wR(w)=w_0(I_0-2R_0),\tag{16}$$

which define $I(w)$ and $R(w)$ as functions of w. The function $S(w)$ is determined from the obvious relation

$$S(w)=\pm[I(w)-R^2(w)]^{1/2},\tag{17}$$

where the sign of the square root should be chosen to satisfy the initial condition, or, if $S(w_0)=0$, the requirement that the field be bounded at $\xi>0$.

From (16) we obtain

$$|\rho(w)|^2=(R-1)^2+S^2=1-\frac{w_0}{w}(2R_0-I_0), \tag{18}$$

i.e., we always have $2R_0 \geqslant I_0$, the modulus of the reflection coefficient increases with increasing incident-wave intensity. Note that solutions of this type can exist only if the radicand in (17) is positive; at the points where $I(w_1)=R^2(w_1)$, an interchange of the solution regimes can take place.

For the solutions of the type considered, $b(w)\neq 0$ so that the equations that follow from (9) for the wave intensity inside the medium are valid and, naturally, coincide with the earlier results (see, e.g., Refs. 2 and 3) by directly integrating Eq. (1) at $\gamma\equiv 0$ and $\varepsilon\equiv\varepsilon(wI)$ with the aid of two integrals.[2] In our case the corresponding equations yield an explicit dependence of all the quantities on the intensity of the incident wave [e.g., Eq. (18), which cannot be separated from the integrals in (2) in simple manner].

As indicated above, the approach developed in Ref. 1 is based on the following principle. At low incident-wave intensities we have a linear problem. Further evolution of the field with increasing w is described by the nonlinear system (10) with specified initial conditions, and the intensity of the incident wave, naturally, changes adiabatically. It is natural to expect this evolution to single out from among the possible solutions those which can in fact be realized. We should then obtain automatically the type of solution (I or II) which corresponds to the initial data, and the possible transition from one type to another. In the presence of damping it is impossible to solve Eqs. (10) analytically. An analysis of the system (10) in the absence of damping makes it possible to establish the solution singular points in whose vicinities a numerical calculation can be used at $\gamma\neq 0$.

We examine specifically several simplest types of nonlinearity in the absence of damping:

a) let $\varepsilon_1(t)=\beta t$ and $\beta>0$. From (7) it follows that $R(0)=1$ and $S(0)=0$. Since the parameter β enters only in the combination $\beta w>0$, we can set it equal to unity without loss of generality (we proceed similarly in all the examples that follow). The system (12) take the form

$$(R^2+S^2)w^2S\,dR/dw=-S[2+wR(R^2+S^2)],$$
$$(R^2+S^2)w^2S\,dS/dw=2(R-1)+{}^1/_2w(R^4-S^4).$$

It is easy to show that in a certain vicinity of the origin we have $S\equiv 0$ by virtue of the initial conditions, i.e., we have solutions of type I. The function $R(w)$ is obtained from the algebraic equation $4(1-R)=wR^4$, which follows from (12) and always has two real roots of opposite sign. The branch that satisfies the condition $R(0)=1$ is determined from the Ferrari formulas

$$R(w)=[2/w(2y)^{1/2}-y/2]^{1/2}-(2y)^{1/2}/2,$$
$$y=\frac{4}{(3w)^{1/2}}\,\mathrm{sh}\,\frac{\varphi}{3},\quad \mathrm{sh}\,\varphi=\frac{9}{8(3w)^{1/2}},$$

from which we obtain asymptotically

$$R(w)\sim 1-w/4,\quad \rho(w)\sim -w/4\quad (w\to 0),$$
$$R(w)\sim 2^{1/2}w^{-1/4},\quad \rho(w)\sim -1+2^{1/2}w^{-1/4}\quad (w\to\infty).$$

The function $R(w)$ decreases monotonically to zero with increasing w, and the reflection coefficient tends to -1. This solution corresponds to a plane wave in a nonlinear medium, and the intensity of the wave field inside the layer is described by Eq. (15).

b) Let now $\varepsilon(t)=-\beta t$ with $\beta>0$. In this case the solution of the corresponding system of equations ($\beta=1$)

$$w^2S(R^2+S^2)\,dR/dw=S[2+wR(R^2+S^2)],$$
$$w^2S(R^2+S^2)\,dS/dw=2(1-R)-{}^1/_2w(R^4-S^4)$$

belongs in a certain vicinity of the origin to type I, and now the function $R(w)$ is determined from the algebraic equation $4(R-1)=wR^4$, which has two real roots at $0<w\leqslant w_0=(\frac{3}{4})^3$. The branch that satisfies the condition $R(0)=1$ is confined to the region $1\leqslant R\leqslant R_\rho=\frac{4}{3}$ and is again obtained from the Ferrari formula

$$R(w)=(2y)^{1/2}/2-[2/w(2y)^{1/2}-y/2]^{1/2},$$

where now

$$y=\frac{4}{(3w)^{1/2}}\,\mathrm{ch}\,\frac{\varphi}{3},\quad \mathrm{ch}\,\varphi=\frac{9}{8(3w)^{1/2}}=\left(\frac{w_0}{w}\right)^{1/2}.$$

Asymptotically we have

$$R(w)\sim 1+w/4,\quad \rho(w)\sim w/4\quad (w\to 0),$$
$$R(w)\sim {}^4/_3-({}^2/_3)(1-w/w_0)^{1/2}\quad (w\to w_0-0),$$

whence, in particular, it follows that $dR/dw\to\infty$ as $w\to w_0-0$. Solutions of this type likewise correspond to a plane wave with wave-field intensity (15) inside the layer. At the critical point we have $J(\xi,w_0)=\frac{3}{4}$, and here $1+\varepsilon(J)=\frac{1}{4}$.

At $w>w_0$, the algebraic equation for R has no real roots, but this means that we must go over to solutions of type II, for which $S(w)\neq 0$. Assuming continuity of the transition from the solutions of one type to another, i.e., assuming $R(w_0)=\frac{4}{3}$ and $S(w_0)=0$, we obtain from (16) and (17) at $w>w_0$

$$I(w)=\frac{1}{4w}Q(w),\quad R(w)=\frac{1}{8w}\left[Q(w)+\frac{3}{2}\right],$$
$$S(w)=\frac{1}{16\sqrt{2}\,w}[Q(w)-3][Q(w)-2]^{1/2}; \tag{19}$$
$$Q(w)=[128w-45]^{1/2}.$$

For the reflection coefficient we have $|\rho(w)|^2=1-3/8w$ and $|\rho(w_0)|^2=\frac{1}{9}$. By quadratures, relations (9) can be rewritten in the form

$$\xi=\sqrt{2}\ln\left(\frac{t(w)-1}{t(w)+1}\frac{t(\bar{w})+1}{t(\bar{w})-1}\right),\quad t=[Q(w)-2]^{1/2},\quad J=\frac{1}{4}Q(\bar{w}),$$

whence, eliminating $\bar{w}$, we ultimately obtain the intensity of the field inside the medium

$$J(\xi,w)=\frac{1}{2}\left\{1+\frac{1}{2}\left[\frac{1+q(w)\exp\{\xi/\sqrt{2}\}}{q(w)\exp\{\xi/\sqrt{2}\}-1}\right]^2\right\}, \tag{20}$$

where

$$q(w)=[t(w)+1]/[t(w)-1].$$

Since $\varepsilon(J)=-J$ in our problem, Eq. (20) describes simultaneously the dielectric-constant distribution, produced by the incident wave, as a function of w and ξ. We see that the change of the field from the plane-wave conditions to the more complicated conditions (20), comes into play before the quantity $\tilde{\varepsilon}(J)=1+\varepsilon(J)$ vanishes. At $w_0<w<w_1=61/128$ the value of $\tilde{\varepsilon}(J)$ differs from zero; at $w \geq w_1$ there always exists a point

$$\xi_0(w)=\sqrt{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\frac{[Q(w)-2]^{1/2}-1}{[Q(w)-2]^{1/2}+1}$$

at which $\tilde{\varepsilon}(J)=0$. In the region $0\leq\xi\leq\xi_0$ we then have $\tilde{\varepsilon}(J)\leq 0$, and at $w=w_1$ we have $\xi_0=0$, and at $w\gg w_1$ we obtain $\xi_0\approx\sqrt{2}\ln[\sqrt{2}+1)/(\sqrt{2}-1)]\approx 2.5$. In the remaining part of space, $\xi>\xi_0$, we have $\tilde{\varepsilon}(J)>0$, and it is precisely this which causes the field to penetrate deep into the medium with increasing intensity of the incident wave [at $\xi\gg 1$ we have $J(\xi,w)\sim\frac{3}{4}$] at critical point w_0.

We have studied above the case when the solution is continuous but the derivatives of all the considered quantities become discontinuous. Whether this is indeed the case can be determined by constructing the solution in the presence of finite (albeit arbitrarily small) damping. To this end, Eq. (10) at constant $\gamma>0$ was integrated numerically. The solutions obtained exhibited as $\gamma\to 0$ a distinct tendency to converge towards the described piecewise-analytic solution, continuous in w but with a discontinuity of the first derivative at the point[1)] w_0.

c) We investigate now the more complicated case $\varepsilon_1(t)=-\delta e^{-\beta t}$, $\beta>0$, $\delta>1$, which describes the penetration of an electromagnetic field into an electron plasma and is considered in Ref. 3. From (7) we obtain $R(0)=2/\delta$, $S(0)=-2(\delta-1)^{1/2}\delta$. We note that $\tilde{\varepsilon}(0)=1+\varepsilon_1(0)<0$, and that the corresponding linear problem ($w=0$) describes total reflection of the wave from the layer ($|\rho(0)|=1$). The wave attenuates exponentially into the interior of the layer, and the depth of the skin layer is $\sim 1/(\delta-1)^{1/2}$. With increasing incident-wave intensity, $\tilde{\varepsilon}(J)$ also increases, and the depth of the skin layer increases until the medium begins to transmit.

Since $S(0)\neq 0$, the solution that is continuous in the vicinity of the origin is of type II. From (16) and (17) we obtain in this case

$$I(w)=\frac{1}{w}\ln\frac{1}{1-4w/\delta},\quad R(w)=\frac{1}{2}I(w),$$
$$|\rho(w)|^2=1,\quad S(w)=-\{R(w)[2-R(w)]\}^{1/2}. \tag{21}$$

Such a solution exists so long as $R(w)\leq 2$. Thus, a singularity appears at the point w_0 determined by the transcendental equation

$$4w_0/\delta=1-e^{-4w_0}. \tag{22}$$

It follows from the integral (9) that the amplitude $E(\xi,w)=[J(\xi,w)]^{1/2}$ of the field inside the medium is described at $0\leq w\leq w_0$ by the relation

$$\delta^{1/2}\xi=\int\limits_{E(\xi,w)}^{E(w)}\frac{dt}{[1-e^{-t^2}-t^2/\delta]^{1/2}}, \tag{23}$$

where $E(w)=E(0$ and $w)=[wI(w)]^{1/2}$. Expression (23) was obtained in Ref. 3 with the aid of the integrals (2) and was tabulated for the critical value w_0, when $E(w_0)=2w_0^{1/2}$ is the zero of the radicand in the right-hand side of (23).

The behavior of the solution at above-critical values $w>w_0$ and at zero damping cannot be determined, since it becomes eventually discontinuous and there are no dynamic conditions that make it possible to determine the sizes of the discontinuities. In fact, as seen from (1′), $|\rho(w)|\to 0$ as $w\to\infty$, and consequently we should have asymptotically a solution of the first type for (14)

$$S(w)=0,\quad 4(R-1)=\delta R^2e^{-wR^2}, \tag{24}$$

corresponding to the plane-wave regime. The theory does not tell us when and how the transition from the type II solution to the solution (24) takes place, at $w=w_0$ or via a sequence of jumps. It is clear, however, that the solutions will be subject to hysteresis. If we move from very large values towards a decrease of the parameter w, the solution (24) will exist all the way to the value $w_1=\frac{1}{4}\ln\delta<w_0$ at which $R(w_1)=2$. With further decrease of w, a jump (or a number of jumps) to the solution (21) should take place.

At a constant positive damping coefficient γ, the problem is described by the complex equation (6) or by the system (10). As $w\to\infty$, the solution of Eq. (6) takes the stationary form

$$u_\infty=\frac{2}{1+(1+i\gamma)^{1/2}}. \tag{25}$$

At sufficiently large w, we put $u(w)=u_\infty+\tilde{u}(w)$ and linearize (6) with respect to $u(w)$, neglecting the quantity $\delta\exp(-wI)$. As a result we obtain the equation

$$-\frac{\gamma}{2}w(u_\infty+u_\infty^*)\frac{d\tilde{u}}{dw}=-2i\tilde{u}+\gamma u_\infty\tilde{u},$$

whose solution at $\gamma\ll 1$ has a rapidly oscillating structure

$$\tilde{u}(w)=\frac{C}{w}\exp\left\{i\frac{2}{\gamma}\ln w\right\}, \tag{26}$$

meaning apparently that it is unstable with respect to a change of w in the opposite direction.

Equation (6) [or the system (10)] can have singularities at the points at which the coefficient $b(w)$ vanishes. To establish the character of these singularities, we change over to new variables with the aid of the relations

$$u(w)=\frac{2}{1+\Phi(J)},\quad J=wI(w). \tag{27}$$

Since $b(w)d/dw=2i(u^*-u)d/dJ$ by virtue of (11), Eq. (6) is rewritten in the form

$$(\Phi-\Phi^*)J\frac{d\Phi}{dJ}=1-\Phi^2+\varepsilon(J).$$

Putting $\Phi=U+iV$ and separating the real and imaginary parts in the last equation, we obtain

$$2VJ\frac{dU}{dJ}=\gamma(J)-2UV,\quad 2VJ\frac{dV}{dJ}=U^2-V^2-1-\varepsilon_1(J), \tag{28}$$

and according to (27) we get

$$b(w)=\frac{J}{2w}[\varepsilon(J)V-\gamma(1+U)], \quad |\rho(w)|^2=\frac{(U-1)^2+V^2}{(U+1)^2+V^2}. \qquad (29)$$

We note that the systems (10) and (28) are equivalent if a one-to-one correspondence exists between w and J.

The system (28) has singularities at those points at which V vanishes. Taking into account, however, the first equation of (29), we see that at $\gamma \neq 0$ there are no points at which the functions $b(w)$ and $V(J)$ vanish simultaneously ($U \geqslant 0$ always). On this basis we can propose the following calculation procedure. We solve first the system (10). On approaching the singular point, where $b(w)$ vanishes, we calculated from the known quantities w, R, and S the values

$$J=w[R^2+S^2], \quad U=\frac{2R}{R^2+S^2}-1, \quad V=-\frac{2S}{R^2+S^2}$$

and change over to the second system of (28), which is integrated before passing through the value $b=0$. Next, from the given values of J, U, and V we calculate

$$w=\frac{1}{4}J[(1+U)^2+V^2], \quad R=\frac{2(1+U)}{(1+U)^2+V^2}, \quad S=-\frac{2V}{(1+U)^2+V^2}$$

and again solve the system (10) prior to the approach to the new singular point, if it exists, etc.

The described algorithm was realized on the basis of a numerical Runge-Kutta-Felberg scheme of fifth-order accuracy with the aid of the subprogram RKF45 given in Ref. 7. The program was tested, using as the example the exact solutions of type II, obtained at $\gamma=0$ and discussed above. Control integrations of the system (10) at $\gamma \neq 0$, carried out with w increased to a certain $w^* > 0$ followed by a return to the initial point of the final value (round trip) has demonstrated good computational stability of the scheme.

The right-hand sides of the system (10) in the normal form at the point $w=0$ are indeterminacies of the 0/0 type. To overcome the corresponding difficulty we used at the start of the calculation asymptotic representations of the solution in the vicinity of the origin, in the form of power-law expansions up to terms w^3 inclusive, with the aid of which we calculated new initial data at a certain point $w > 0$.

The change from the system (10) to the system (28) and back was effected in the following manner. If, e.g., in the course of the numerical integration of (10) the coefficient $b(w)$ assumed at the point $\bar{w}$ an absolute value smaller than a certain specified small number, a transition to the system (28) took place, and the latter solved then in the same direction of the change of the new variable J, which was fixed at the point $\bar{w}$. The transition from (28) to (10) was made similarly when the coefficient V became small enough. Of course, if $b(w)$ did not reverse sign, only the system (10) worked. This method constructed numerically continuous functions $R(w)$ and $S(w)$, but these could become non-single-valued at sufficiently small γ.

It is of interest to note that at all realizations of the described algorithm, the argument (phase) of the complex reflection coefficient varied monotonically on moving along the curve in the (R, S) plane.

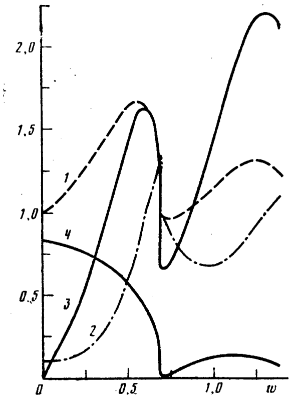

FIG. 1. Behavior of the solution of the problem at $\delta=2$ and $\gamma=0.18$. Curve 1 corresponds to $R(w)$, curve 2 to $1+S(w)$, curve 3 to $J(W)$, and curve 4 to $|\rho(w)|^2$.

We proceed now to discuss the results of the numerical experiments, most of which were performed for $\varepsilon_1(t)=-\delta e^{-\delta t}$ at $\delta=1.25$, 2, and 5 with corresponding critical points $w_0=0.12$, 0.4, and 1.25 [see Eq. (22)]. The intermediate case $\delta=2$ was analyzed in detail.

At a sufficiently large value of the parameter γ ($\gamma > 0.18$) all the functions have a smooth dependence on w. For example, the function R reaches a maximum and then, oscillating in accord with (26), reaches its asymptotic value determined from (25). With decreasing γ, the maximum of $R(w)$ shifts towards the critical value $w_0=0.4$, and the curve itself begins to drop steeply after passing through the maxima. Finally, at $\gamma=0.18$ all the curves, as functions of w, have a practically vertical section at $\tilde{w}_0=0.69$, which is in essence a discontinuity. This section of the curves was obtained by integrating the system (28), and a one-to-one correspondence between J and w took place in this case. The behavior of the solution of the problem for this case is shown in Fig. 1. After passing through the discontinuity, all the curves reach their asymptotic values via oscillations (26). For the critical value w_0, the reflection coefficient vanishes with high accuracy, i.e., the wave is almost completely absorbed by the nonlinear medium. With further increase of w, the modulus of the reflection coefficient increases.

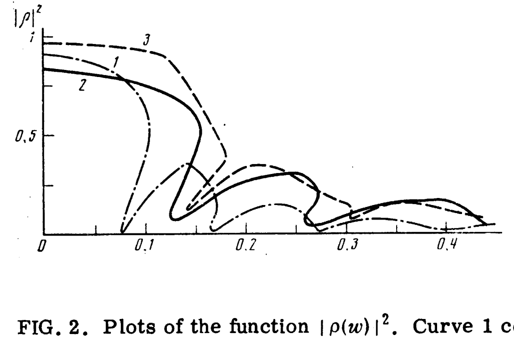

FIG. 2. Plots of the function $|\rho(w)|^2$. Curve 1 corresponds to the parameters $\delta=2$ and $\gamma=0.1$ ($\tilde{w}=w/5$), curve 2 corresponds to $\delta=1.25$, $\gamma=0.06$ ($\tilde{w}=w$), curve 3 to $\delta=5$, $\gamma=0.15$ ($\tilde{w}=w/10$). The ordinates are the values of w.

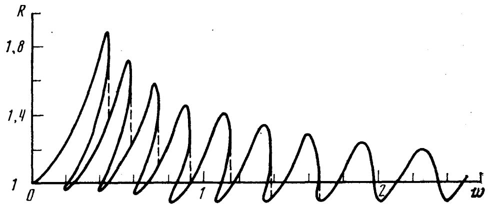

FIG. 3. Plot of the function $R(w)$ at $\delta = 2$ and $\gamma = 0.05$.

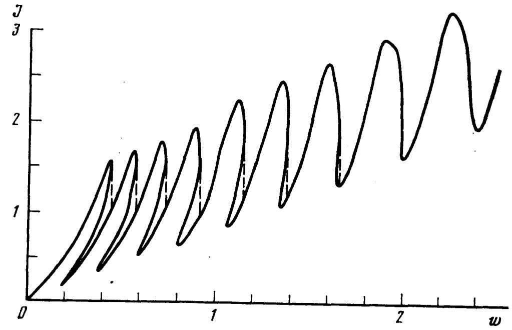

FIG. 5. Plot of the function $J(w) = w[R^2 + S^2]$ at $\delta = 2$ and $\gamma = 0.05$.

The formation of discontinuity at a finite and rather large value of the parameter γ means that it is physically incorrect to introduce a constant damping in the nonlinear problem considered. The damping should be a nonlinear function of the field intensity in the medium, just as the real part of the dielectric constant. Recognizing, however, that we are primarily interested in the tendency of the behavior of the solution of the problem as $\gamma \to 0$, we can formally consider also discontinuous functions, in analogy with the procedure used, e.g., in the problem of formation of discontinuities in a Riemann wave (shock wave).

With further decrease of the parameter γ, the one-to-one correspondence between J and w is already lost on finite sections of the w axis. It is formally possible to continue the integration of the systems (10) and (28) in accord with the algorithm indicated above, treating the appearance of such an ambiguity as the analog of the breaking of a nonlinear wave. Then each loop in the solution will yield the value of the discontinuity for monotonic change of the parameter w. Thus, Fig. 2 shows a plot (curve 1) of the square of the modulus of the reflection coefficient at $\gamma = 0.1$. This plot has already two discontinuities, and the reflection coefficient at the point of the first discontinuity no longer vanishes. The set of figures 3–6 shows the behavior of the solution of the problem at $\gamma = 0.05$, with already seven discontinuities. The tendency of the behavior of the solution as $\gamma \to 0$ is clear from a comparison of Fig. 1 with Figs. 3–6. First, the region where oscillatory solutions of the type (26) exists shifts towards the critical point w_0. Secondly, each oscillation compresses in the direction towards the point w_0, and this compression depends on the value of the function itself, and it is this which leads to formation of the discontinuity ("breaking of the wave") in full analogy with the dispersion effect of nonlinear waves. Finally and thirdly, since the period of the oscillations (26) increases with increasing w, the number of discontinuities is always finite for any arbitrarily small value of γ. It appears that as $\gamma \to 0$ the system of discontinuities will be grouped at sufficiently large w relative to the solution (24) corresponding to the plane-wave regime.

The picture is similar for other values of the parameter δ, as seen from Fig. 2, which shows also plots of $|\rho|^2$ for $\delta = 1.25$ and $\delta = 5$, with two discontinuities each.

We note in conclusion that the curves shown in Figs. 2–6 show hysteresis, for when the discontinuities move in the opposite direction along w they should pass on the other sides of the loops. This type of hysteresis, however, is not physical, since a constant γ is not physically tenable. When adequate account is taken of the nonlinear damping the observed hysteresis will apparently give way to fronts of the discontinuities, but in this case another type of hysteresis will remain, connected with the accurate motion along the w axis in the region of large w, where the system has already reached its stationary value.

[1)]We note that the results obtained above do not agree with those given in Ref. 4.

[1]G. I. Babkin and V. I. Klyatskin, Zh. Eksp. Teor. Fiz. **79**, 817 (1980) [Sov. Phys. JETP **52**, 416 (1980)].
[2]F. G. Bass and Yu. G. Gurevich, Goryachie élektrony i sil'nye élektromagnitnye volny v plazme poluprovodnikov i gazovogo razryada (Hot Electrons and Strong Electromagnetic

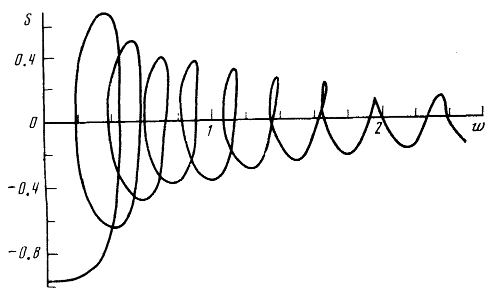

FIG. 4. Plot of the function $S(w)$ at $\delta = 2$ and $\gamma = 0.05$.

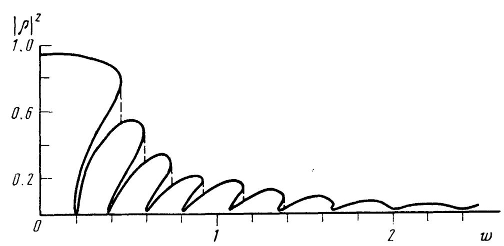

FIG. 6. Plot of the function $|\rho(w)|^2$ at $\delta = 2$, $\gamma = 0.05$.

Waves in the Plasma of Superconductors and of Gas Discharge), Nauka, 1975.
[3]V. P. Silin, Zh. Eksp. Teor. Fiz. **53**, 1663 (1967) [Sov. Phys. JETP **26**, 955 (1968)].
[4]O. V. Bagdasaryan and V. A. Permyakov, Izv. vyssh. ucheb. zaved. Radiofizika **21**, 1352 (1978).
[5]G. I. Babkin and V. I. Klyatskin, *ibid.* **34**, 1185 (1980).
[6]V. I. Klyatskin, Stokhasticheskie uravneniya i volny v sluchaǐno-neodnorodnykh sredakh (Stochastic Equations and Waves in Randomly Inhomogeneous Media), Nauka, 1980.
[7]G. E. Forsythe and C. Moler, Computer Methods for Mathematical Computations, Prentice-Hall, 1977.

Translated by J. G. Adashko