Wave damping in a relativistic plasma

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We calculate the decrements of longitudinal and transverse wave damping by electron-ion collisions and by collective radiation damping of electrons in a relativistic plasma. It is shown that in many cases the radiative damping of the waves in the plasma exceeds the damping due to collisions.

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The dispersion and damping of Landau waves in a relativistic plasma have by now been sufficiently well investigated.[1–5] In this paper we consider the dissipation of the wave energy in a relativistic plasma as a result of electron-ion collisions and of collective radiative emission of electrons.

In analogy with the procedure in nonrelativistic theory,[6] the influence of the collisions can be taken into account by using a relativistic generalization of the Landau collision integral. This generalization was obtained by Belyaev and Budker.[7] The damping of plasma waves as a result of radiative scattering is taken into account by the Lorentz-Dirac deceleration force.[8–11] The covariant form of the kinetic equation for the electric component of the plasma is then

$$u^i\frac{\partial f}{\partial x^i}+\frac{1}{mc}\frac{\partial}{\partial u^n}\left[\left(\frac{e}{c}F^{ni}u_i+g^n\right)f\right]=-\frac{1}{m^2c^3}\frac{\partial J^i}{\partial u^i},\tag{1}$$

where u_i is the velocity 4-vector. The quantity J^i in the collision integral is given by (see Ref. 12, p. 250 of the original),

$$J^i=\frac{2\pi(ee')^2L}{c^2}\int d^4u'W^{ik}\left(f\frac{\partial f'}{\partial u'^k}-f'\frac{\partial f}{\partial u^k}\right),$$
$$W^{ik}=\frac{(u^lu_l')^2}{c[(u^lu_l')^2-1]^{3/2}}\{-[(u_lu'^l)^2-1]g^{ik}-u^iu^k-u'^iu'^k+u^lu_l'(u^iu'^k+u'^iu^k)\}\tag{2}$$

(L is the Coulomb logarithm). The force of deceleration by the radiation is of the form

$$g^n=\frac{2}{3}\frac{e^3}{mc^3}\frac{\partial F^{ni}}{\partial x^j}u_iu^j-\frac{2}{3}\frac{e^4}{m^2c^5}F^{ni}F_{ji}u^j+\frac{2}{3}\frac{e^4}{m^2c^5}(F_{ij}u^j)(F^{im}u_m)u^n.$$

The electromagnetic-field tensor is determined from the Maxwell equations

$$F^{ni}=\frac{\partial A^n}{\partial x_i}-\frac{\partial A^i}{\partial x_n},\quad \frac{\partial^2A^i}{\partial x_n\partial x^n}=\frac{4\pi}{c}\int eu^ifc^4d^4u+\frac{4\pi}{c}I_u^i.\tag{3}$$

The eight-dimensional distribution function f in (1)–(3) is connected with the seven-dimensional one by the relation[9]

$$f=f(x^i,u^1,u^2,u^3)2\theta(u_0)\delta(u_iu^i-1).\tag{4}$$

Linearizing the system (1)–(3) under the assumption that the ion mass greatly exceeds the electron mass, and assuming that all the perturbed quantities are proportional to $\exp(-ik_nx^n)$, we obtain from (1)–(3)

$$(-ik_nu^n)f+\frac{1}{mc}\frac{\partial}{\partial u^n}\left[\left(\frac{e}{c}F^{nm}u_m+g_L{}^n\right)f_0\right]=\frac{1}{m^2c^3}\frac{\partial}{\partial u^i}\left[nB_{ij}\frac{\partial f}{\partial u^j}\right],\tag{5}$$

$$F^{nm}=i(k^mA^n-k^nA^m),\quad A^n=-\frac{4\pi e}{k_ik^i}\int u^nfc^3d^4u,\tag{6}$$

where f_0 is the unperturbed distribution of the electrons, n is the electron density,

$$g_L{}^n=-i\frac{2e^3}{3mc^3}k_ju^jF^{nm}u_m,\tag{7}$$

$$B_{ij}=2\pi(ee')^2L\frac{u_0}{cu^3}(u^2\delta^{ij}-u^iu^j).\tag{8}$$

We solve Eq. (5) by successive approximations. To this end, we integrate it with respect to u^0 and neglect in first approximation the collision and radiative terms. We then obtain

$$f=\frac{e}{mc^2}\frac{1}{ik_ju^j}F^{nm}u_m\frac{\partial f_0}{\partial u^n}.$$

In expression (9) and elsewhere, $u_0=(1+u^2)^{1/2}$ while f_1 and f_0 are seven-dimensional distribution functions.

Substituting (9) in (5) we obtain in the next order of approximation

$$f=\frac{e}{mc^2}\frac{1}{ik_ju^j}F^{nm}u_n\frac{\partial f_0}{\partial u^n}+\frac{u_0}{mc}\frac{1}{ik_ju^j}\frac{\partial}{\partial u^n}\left[g_L{}^n\frac{f_0}{u_0}\right]$$
$$-\frac{1}{ik_ju^j}\frac{u_0}{m^2c^3}\frac{\partial}{\partial u^i}\left[nB_{ij}\frac{\partial}{\partial u^j}\left(\frac{1}{ik_ru^r}\frac{e}{mc^2}F^{nm}u_m\frac{\partial f_0}{\partial u^n}\right)\right].\tag{10}$$

Substituting (10) in (6) and assuming that f_0 is a relativistic Maxwellian distribution,

$$f_0=n\exp\left(-\frac{mc^2}{\Theta}u_0\right)\Big/4\pi c^3\frac{\Theta}{mc^2}K_2\left(\frac{mc^2}{\Theta}\right),\tag{11}$$

we arrive at the dispersion equations. For longitudinal (Langmuir) waves we have

$$\varepsilon_1(k^0,k)+\delta\varepsilon_1{}^r(k^0,k)+\delta\varepsilon_1{}^s(k^0,k)=0,\tag{12}$$

where

$$\varepsilon_1(k^0,k)=1+\frac{m\omega_p{}^2}{\Theta k^2}\int_0^\infty\left(1+\frac{k_0u^0}{2ku}\ln\frac{k_0u^0-ku}{k_0u^0+ku}\right)f_0'd\Omega,\tag{13}$$

$$\delta\varepsilon_1{}^r(k^0,k)=i\frac{2}{3}r_0k^0\frac{\omega_p{}^2}{c^2k_0{}^2}\int_0^\infty\left[\frac{k_ik^i}{k^2}\left(1+\frac{k_0u^0}{2ku}\right.\right.$$
$$\left.\left.\times\ln\frac{k_0u^0-ku}{k_0u^0+ku}\right)+\frac{mc^2u^2}{3\Theta u_0}\right]f_0'd\Omega,\tag{14}$$

$$\delta\varepsilon_1{}^s(k^0,k)=i\frac{1}{3}r_0k_0L\frac{m\omega_p{}^4}{\Theta c^2k_0{}^2k^2}\int_0^\infty\frac{k_0{}^2u_0{}^2k^2u^2}{(k_0{}^2u_0{}^2-k^2u^2)^2}\frac{u_0}{u^3}f_0'd\Omega.\tag{15}$$

For transverse (electromagnetic) waves we obtain

$$\varepsilon_2(k^0,k)+\delta\varepsilon_2{}^r(k^0,k)+\delta\varepsilon_2{}^s(k^0,k)=0;\tag{16}$$

$$\varepsilon_2(k^0,k)=1-\frac{m\omega_p{}^2}{2\Theta k_ik^i}\int_0^\infty\left[\frac{k_0{}^2}{k^2}+\left(\frac{k_0{}^2}{k^2}-\frac{u^2}{u_0{}^2}\right)\frac{k_0u^0}{2ku}\ln\frac{k_0u^0-ku}{k_0u^0+ku}\right]f_0'd\Omega_r\tag{17}$$

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$$\delta\varepsilon_2^r(k^0,k)=i\frac{1}{3}r_0k^0\frac{\omega_p^2}{c^2k_ik^i}\int\limits_0^\infty\left[\frac{2mc^2}{3\Theta}-\frac{k_ik^i}{k^2}+\frac{k_ik^i}{k_0^2}\right.$$

$$\left.\times\left(\frac{u^2}{u_0^2}-\frac{k_0^2}{k^2}\right)\frac{k_0u^0}{ku}\ln\frac{k_0u^0-ku}{k_0u^0+ku}\right]f_0'\,d\Omega, \tag{18}$$

$$\delta\varepsilon_2^s(k^0,k)=i\frac{1}{3}r_0k^0\frac{m\omega_p^4}{\Theta c^2k_0^2k_ik^i}\int\limits_0^\infty\frac{k_0^2}{k_0^2u_0^2-k^2u^2}\frac{u_0}{u}f_0'\,d\Omega. \tag{19}$$

In (11)–(19) we have $k^2=\mathbf{k}^2$, $u^2=\mathbf{u}^2$,

$$f_0'=f_0/n,\quad \omega_p^2=4\pi e^2n/m,\quad r_0=e^2/mc^2,\quad d\Omega=c^34\pi u^2du,$$

and Θ is the temperature.

Putting $ck_0=\omega-i\gamma$, where $|\gamma|\ll|\omega|$, and separating in (12) and (16) the real and imaginary parts, we obtain for the frequency ω and the damping decrements $\gamma_{1,2}$ of the longitudinal and transverse waves

$$\operatorname{Re}\varepsilon_{1,2}(\omega,k)=0,\quad \gamma_{1,2}=\gamma_{1,2}^r+\gamma_{1,2}^s+\gamma_{1,2}^L; \tag{20}$$

$$\gamma_{1,2}^{r,s}=-i\delta\varepsilon_{1,2}^{r,s}(\omega,k)/\operatorname{Re}\frac{\partial}{\partial\omega}\varepsilon_{1,2}(\omega,k), \tag{21}$$

$$\gamma_{1,2}^L=-\operatorname{Im}\varepsilon_{1,2}(\omega,k)/\operatorname{Re}\frac{\partial}{\partial\omega}\varepsilon_{1,2}(\omega,k). \tag{22}$$

The damping decrements $\gamma_{1,2}^r$ are determined by the radiative deceleration, while $\gamma_{1,2}^s$ are determined by the electron collisions. The decrements $\gamma_{1,2}^L$ (Landau damping) differ from zero only when $\omega/k<c$ and are due to resonant interaction of the electrons with the wave. This damping has been thoroughly studied for a relativistic plasma[2–5] and is not considered here.

We investigate now the decrements (21) in different asymptotic limits.

In the relativistic limit $\Theta\ll mc^2$ at $\omega^2/k^2\gg\Theta/m$ we can expand the integrands in (13)–(21) in powers of u/u_0 and uk/u_0k_0. We then obtain for the Langmuir waves

$$\omega^2=\omega_p^2,\quad \gamma_1^r=\frac{r_0\omega_p^2}{3c},\quad \gamma_1^s=\frac{r_0\omega_p^2}{3c}\frac{L}{\sqrt{2\pi}}\left(\frac{mc^2}{\Theta}\right)^{3/2}, \tag{23}$$

and for the electromagnetic waves

$$\omega^2=\omega_p^2+k^2c^2,\quad \gamma_2^r=\frac{r_0\omega_p^2}{3c},\quad \gamma_2^s=\frac{r_0\omega_p^2}{3c}\frac{L}{\sqrt{2\pi}}\left(\frac{mc^2}{\Theta}\right)^{3/2}\frac{\omega_p^2}{\omega_p^2+k^2c^2}. \tag{24}$$

The expressions for the decrements $\gamma_{1,2}^s$ coincide, apart from the notation, with the result given in the book by Ginzburg and Rukhadze.[6] It follows also from (23) and (24) that in the nonrelativistic limit we have $\gamma_1^s\gg\gamma_1^r$. For the electromagnetic waves, on the other hand, the radiative damping can exceed the damping due to collisions, provided the following inequality holds

$$\frac{\omega_p^2}{\omega_p^2+k^2c^2}\frac{L}{\sqrt{2\pi}}\left(\frac{mc^2}{\Theta}\right)^{3/2}<1. \tag{25}$$

We consider now Eqs. (20) and (21) in the ultrarelativistic limit $\Theta\gg mc^2$. In this case simple analytic expressions for the decrements (just as for the dispersion[1,5]) can be obtained in two asymptotic limits. First, at

$$\frac{m^2c^4}{\Theta^2}\Big/\left|1-\frac{k_0}{k}\right|\ll1,\quad \left|1-\frac{k_0}{k}\right|\ll1 \tag{26}$$

we can put $u_0\approx u$ in Eqs. (13)–(19). We then obtain for the Langmuir waves

$$\omega=kc\left[1+2\exp\left(-2-2\frac{k^2\Theta}{\omega_p^2m}\right)\right], \tag{27}$$

$$\gamma_1^r=\frac{8r_0\omega_p^2}{3c}\frac{k^2\Theta}{m\omega^2}\exp\left(-2-2\frac{k^2\Theta}{m\omega_p^2}\right), \tag{28}$$

$$\gamma_1^s=\frac{r_0\omega_p^2}{3c}\frac{L}{8}\frac{m^2c^4}{\Theta^2}\exp\left(2+2\frac{k^2\Theta}{m\omega_p^2}\right). \tag{29}$$

For the electromagnetic waves we get

$$\omega^2=k^2c^2+\omega_p^2\frac{mc^2}{2\Theta}, \tag{30}$$

$$\gamma_2^r=\frac{r_0\omega_p^2}{3c},\quad \gamma_2^s=\frac{r_0\omega_p^2}{3c}L\frac{mc^2}{\Theta}\frac{m\omega_p^2}{k^2\Theta}. \tag{31}$$

It is seen from (28) and (29) that in the region (26) the wave damping due to collisions can be larger as well as smaller than the radiative damping. In particular, at

$$\frac{k^2\Theta}{m\omega_p^2}\gg\frac{Lm^2c^4}{8\Theta^2}\exp\left(4+4\frac{k^2\Theta}{m\omega_p^2}\right) \tag{32}$$

the radiative damping turns out to be larger than γ_1^s. As for the electromagnetic waves, in the region (26) their radiative damping turns out to exceed the damping due to the electron-ion collisions. It follows also from (26)–(31) that the values of the damping decrements lie in the range

$$\frac{r_0\omega_p^2}{3c}\frac{m^2c^4}{\Theta^2}\ll\gamma_1^r\ll\frac{r_0\omega_p^2}{3c}, \tag{33}$$

$$\frac{r_0\omega_p^2}{3c}\frac{L}{8}\left(\frac{mc^2}{\Theta}\right)^2\ll\gamma_1^s\ll\frac{r_0\omega_p^2}{3c}L\frac{mc^2}{8\Theta}, \tag{34}$$

$$\frac{r_0\omega_p^2}{3c}L\left(\frac{mc^2}{\Theta}\right)^3\ll\gamma_2^s\ll\frac{r_0\omega_p^2}{3c}L\frac{mc^2}{\Theta}. \tag{35}$$

We see therefore that in the considered limit (26), the radiative damping of the Langmuir waves and the damping $\gamma_{1,2}^s$ due to the collisions are smaller in absolute magnitude than the corresponding nonrelativistic values (23) and (24). The radiative damping of electromagnetic waves are of the same order in the considered approximation.

Another asymptotic limit for an ultrarelativistic plasma corresponds to

$$\left(\frac{m^2c^4}{\Theta^2}\right)\Big/\left|1-\frac{k_0}{k}\right|\gg1. \tag{36}$$

In this limit the integrands in (13)–(19) can be expanded in powers of the parameter $u_0^{-2}/|1-k_0/k|$. In first-order approximation in this parameter, the dispersion of the Langmuir wave is $\omega=kc$, with $k^2c^2\sim mc^2\omega_p/\omega_p^2/\Theta$, and the damping decrements take the form

$$\gamma_1^r=\frac{r_0\omega_p^2}{3c}\frac{k^2\Theta}{6m\omega_p^2}\frac{m^2c^4}{\Theta^2}\sim\frac{r_0\omega_p^2}{3c}\frac{m^2c^4}{\Theta^2},\quad \gamma_1^s=\frac{r_0\omega_p^2}{3c}L. \tag{37}$$

In the same approximation we have for the electromagnetic waves

$$\gamma_2^r=\frac{r_0\omega_p^2}{3c},\quad \gamma_2^s=\frac{r_0\omega_p^2}{3c}L\frac{\omega_p^2m}{k^2\Theta}\ll\frac{r_0\omega_p^2}{3c}L\frac{m^2c^4}{\Theta^2}. \tag{38}$$

We see that in the region (36) the radiative damping of the electromagnetic waves greatly exceeds the damping due to the electron-ion collisions. For Langmuir waves, the opposite inequality holds: $\gamma_1^s\gg\gamma_1^r$.

Finally, Eqs. (12)–(21) for arbitrary Θ become much simpler in the long-wave limit

$$(k^2/k_0^2)\,\overline{(u^2/u_0^2)}\ll1, \tag{39}$$

where the superior bar denotes averaging. In this case it follows from (19)–(21), the first order in the parameter $ku/k_0\mu_0$, that the dispersion and the damping decrements coincide for both longitudinal and transverse waves, and are equal to

$$\omega^2=\omega_p^2\frac{mc^2}{\Theta}K_2\left(\frac{mc^2}{\Theta}\right)\int\limits_{mc^2/\Theta}^{\infty}d\alpha\frac{K_2(\alpha)}{\alpha^2}, \tag{40}$$

$$\gamma^r=\frac{1}{3}\frac{r_0\omega_p^2}{c}\left[1-\frac{mc^2}{\Theta}\int\limits_{mc^2/\Theta}^{\infty}d\alpha\frac{K_2(\alpha)}{\alpha^2}\bigg/K_2\left(\frac{mc^2}{\Theta}\right)\right], \tag{41}$$

$$\gamma^s=\frac{1}{3}\frac{r_0\omega_p^2}{c}\frac{L}{2}\exp\left(-\frac{mc^2}{\Theta}\right)\bigg/\left(\frac{mc^2}{\Theta}\right)\int\limits_{mc^2/\Theta}^{\infty}d\alpha\frac{K_2(\alpha)}{\alpha^2}, \tag{42}$$

where $K_2(\alpha)$ is a MacDonald function. In the nonrelativistic limit Eqs. (40)–(42) coincide under condition (41) with expressions (23) and (24). In the ultrarelativistic limit, Eq. (40)–(42) leads to the result

$$\omega^2=\omega_p^2\frac{mc^2}{3\Theta},\quad \gamma^r=\frac{r_0\omega_p^2}{3c}\frac{2}{3},$$
$$\gamma^s=\frac{r_0\omega_p^2}{3c}3L\frac{m^2c^4}{\Theta^2}. \tag{43}$$

It is seen therefore that in an ultrarelativistic plasma the longitudinal and transverse long waves (39) are attenuated more strongly by scattering than by energy transfer to the ions.

The decrements γ^s and γ^r, represented analytically by Eqs. (41) and (42), is plotted in Fig. 1. It is seen that the radiative damping depends little on the temperature and, as follows from estimates, decreases with the increasing temperature by a factor 1.5. Using also the asymptotic expansions for the MacDonald function, it is easy to verify that in the nonrelativistic region the damping decrement due to the electron-ion collisions decreases with increasing electron temperature in accord with the power law $\sim\Theta^{-3/2}$. In the ultrarelativistic region this dependence is different: $\gamma\sim\Theta^{-2}$ [see also Eqs. (23), (24), (29), and (31)].

The indicated dependence of γ^s in the region (39) on the temperature can be qualitatively estimated on the basis of the proportionality of the damping decrement to the effective collision frequency $\nu_{\rm eff}$ (Ref. 6). Indeed, it follows from (8) that

$$\nu_{eff}\sim u_0/u^3.$$

Therefore in the nonrelativistic limit $\nu_{\rm eff}\sim\Theta^{-3/2}$, and in the ultrarelativistic limit $\nu_{\rm eff}\sim\Theta^{-2}$. Outside the region (39) the temperature dependence of the decrement is determined essentially by the dispersion of the waves, which, e.g., in the region (26), contains an exponential temperature dependence.

Thus, it is seen from the equations obtained for the decrements of the wave damping in a relativistic plasma on account of electron-ion collisions, and from the damping decrements due to deceleration by the radiation, that in many of the cases indicated above the radiative damping predominates. This result confirms the qualitative conclusion drawn by Lifshitz and Pitaevskii (Ref. 12, p. 254 of the original), that radiative scattering of the waves in a relativistic plasma is important. We note also that the magnitude of the radiative damping of the Langmuir and electromagnetic waves in all the cases considered above, despite the conclusions of Ref. 13, do not exceed in absolute magnitude the damping decrement of the waves in a nonrelativistic plasma.

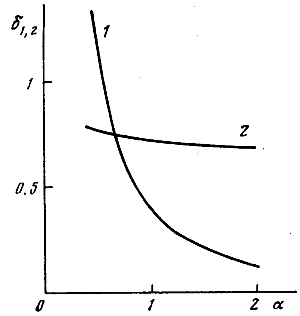

FIG. 1. Curves 1 and 2 are plots of the dimensionless decrements of plasma-wave damping by collisions $\delta_1=\gamma^s/(r_0\omega_p^2L/3c)$ and by radiative deceleration of the electrons $\delta_2=\gamma^r/(r_0\omega_p^2/3c)$ as functions of the dimensionless temperature $\alpha=\Theta/mc^2$.

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[1]V. P. Silin, Zh. Eksp. Teor. Fiz. **38**, 1537 (1960) [Sov. Phys. JETP **11**, 1136 (1960)].
[2]B. Buti, Phys. Fluids **5**, 1 (1962).
[3]D. G. Lominadze, G. Z. Machabeli, A. B. Mikhaĭlovskiĭ, Fiz. Plazmy **5**, 1337 (1979) [Sov. J. Plasma Phys. **5**, 748 (1979)].
[4]D. G. Lominadze and A. B. Mikhailovskiĭ, Zh. Eksp. Teor. Fiz. **67**, 959 (1979) [Sov. Phys. JETP **49**, 483 (1979)].
[5]A. B. Mikhailovskii, Plasma Phys. **22**, 133 (1980).
[6]V. L. Ginzburg and A. A. Rukhadze, Volny v magnitoaktivnoĭ plazme (Waves in Magnetoactive Plasma), Nauka, 1970, pp. 5–207.
[7]S. T. Belyaev and G. I. Budker, Dokl. Akad. Nauk SsSR **107**, 807 (1956) [Sov. Phys. Dokl. **1**, 218 (1956)].
[8]N. B. Rubin, Zh. Tekh. Fiz. **34**, 676 (1964) [Sov. Phys. Tech. Phys. **9**, 521 (1964)].
[9]R. J. Hakin, J. Math. Phys. **8**, 1379 (1967).
[10]L. S. Kuz'menkov, Dokl. Akad. Nauk SSSR **241**, 322 (1978) [Sov. Phys. Dokl. **23**, 469 (1978)].
[11]Kwok-Kee Tam and D. Tang, Progr. Theoret. Phys. **62**, 1245 (1979).
[12]E. M. Lifshitz and L. I. Pitaevskiĭ, Fizikcheskaya kinetika (Physical Kinetics), Nauka, 1979.
[13]Kwok-Kee Tam, and J. O'Hanlon, Nuovo Cimento **63B**, 241 (1969).

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