# Critical phenomena in cooperative resonant fluorescence of a system of interacting atoms

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An exact stationary solution of the equation for the density matrix of a system of two-level atoms interacting with one another and with powerful resonant radiation is reported. The atoms are located in the volume much shorter than the wave length of the radiation. It is shown that in the limit of a large number of atoms, the system undergoes a nonequilibrium first-order phase transition when the parameters of the frequency detuning or of the exciting-radiation power are changed. The atomic correlation functions are obtained and discussed. It is shown that the quantum fluctuations are significant only in the phase-transition region. An analogy between the considered model and stimulated nonlinear resonant oscillations of an anharmonic oscillator is established.

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## 1. INTRODUCTION

It was shown recently in Refs. 1 and 2 and in the following papers[3-5] that a system of $N$ two-level atoms interacting with powerful resonant radiation and located in a volume much shorter than the radiation wavelength (the Dicke model) undergoes as $N\to\infty$ a nonequilibrium phase transition similar to a second-order phase transition. The transition manifests itself in the fact that up to a certain critical exciting-radiation power the system behaves in pure classical fashion, and quantum effects become substantial at higher values. In Ref. 6 it was found that in the case of a nonzero frequency detuning, there is no such critical phenomenon. On the other hand, if the atoms of the system in questions interact with one another, the coherence region with respect to the power of the exciting radiation obviously increases with increasing interaction. Therefore the competition between the dephasing effect of the frequency detuning and the synchronizing effect of the interaction of the atoms should lead to a qualitative change in the character of the motion in the considered model.

Indeed, in the present paper, in which the Dicke model is investigated with account taken of both the frequency detuning and the interaction between the atoms, it is shown that there exists a region of values of the parameters at which a nonequilibrium first-order phase transition can take place in the system.

The density matrix of the considered system of atoms satisfies the equation

$$\dot{\hat{\rho}}=-i[\hat{H}_{int},\hat{\rho}]-i[\hat{Q},\hat{\rho}]+(\gamma/2)([\hat{J}_-,\hat{\rho}\hat{J}_+]+[\hat{J}_-\hat{\rho},\hat{J}_+]),\qquad(1)$$
$$\hat{H}_{int}=2v\hat{J}_x+\varepsilon\hat{J}_z,$$

where $v$ is the matrix element of the interaction of the resonant radiation with the atoms, $\varepsilon=\omega_{ab}-\omega_0$ is the frequency detuning from resonance, $\hat{Q}$ is the interatomic interaction operator, $\gamma$ is the probability of the spontaneous transition $a\to b$, and

$$\hat{J}_\pm=\sum_{i=1}^{N}\hat{\sigma}_{\pm i}$$

are the collective operators of the transitions and satisfy the commutation relations $[\hat{J}_+,\hat{J}_-]=2\hat{J}_z$, $[\hat{J}_z,\hat{J}_\pm]=\pm\hat{J}_\pm$. Interatomic interactions of various types can be described with the aid of the operator[7,8] $\hat{Q}$:

$$\hat{Q}=\sum_i\varepsilon_i\hat{\sigma}_{zi}+\sum_{i,j}q_{ij}^z\hat{\sigma}_{zi}\hat{\sigma}_{zj}+\sum_{i,j}q_{ij}(\hat{\sigma}_{+i}\hat{\sigma}_{-j}+\hat{\sigma}_{-i}\hat{\sigma}_{+j}).\qquad(2)$$

Since the quantities $\varepsilon_i$, $q_{ij}^z$ and $q_{ij}$ are different for different pairs of atoms, the quantity

$$\langle\hat{J}^2\rangle=\langle\hat{J}_\pm\hat{J}_\mp\rangle+\langle\hat{J}_z^2\rangle\mp\langle\hat{J}_z\rangle$$

is not conserved in the general case according to (1). In the present article is considered the particular case when $\varepsilon_i=\varepsilon_q$, $q_{ij}^z=q^z$, $q_{ij}=q$, and consequently

$$\hat{Q}=\bar{\varepsilon}\hat{J}_z+\bar{q}\hat{J}_+\hat{J}_-,$$
$$\bar{\varepsilon}=\varepsilon_q+q^z-2q,\quad\bar{q}=q^z+2q.\qquad(3)$$

In this case $\langle\hat{J}^2\rangle=j(j+1)$, where $2j\equiv N$. We note that the system (1) and (3) describes also an extended system, averaged over space, of two-level atoms.[9] In the latter case $\bar{q}$ corresponds to the cooperative frequency shift considered in detail in Ref. 9. In addition, Eqs. (1) and (3) describe monochromatic excitation of a dissipative quantum system, whose energy spectrum $E_m$ is represented by a converging ($\bar{q}>0$) or diverging ($\bar{q}<0$) sequence of $N+1$ levels:

$$E_m=[\omega_{ab}+\bar{\varepsilon}+(N-1)\bar{q}]m-\bar{q}m^2,\quad m=0,1,\ldots,N.\qquad(4)$$

From (4) and from the commutation relations for the transition operators $\hat{J}_+$ and $\hat{J}_-$ it follows that at large $N$ this quantum system is analogous to an anharmonic oscillator with fundamental frequency $\bar{\omega}=\omega_{ab}+\bar{\varepsilon}+(N-1)\bar{q}$ and with anharmonic constant $\varkappa=\bar{q}/\bar{\omega}$, excited by quasiresonant radiation of frequency $\omega_0$.

The problem (1), (3), has an exact stationary solution. But before discussing it, we obtain an approximate solution that will be useful later on.

## 2. CLASSICAL SOLUTION

The equations of motion obtained from (1) and (3) for the mean values $\langle\hat{J}_\mu\rangle$ ($\mu=z,\pm$) are of the form

$$\langle\dot{\hat{J}}_+\rangle=-2iv\langle\hat{J}_z\rangle+i(\varepsilon+\bar{\varepsilon})\langle\hat{J}_+\rangle-(2i\bar{q}-\gamma)\langle\hat{J}_+\hat{J}_z\rangle,$$
$$\langle\dot{\hat{J}}_z\rangle=-iv(\langle\hat{J}_+\rangle-\langle\hat{J}_-\rangle)-\gamma\langle\hat{J}_+\hat{J}_-\rangle,\qquad\langle\hat{J}_-\rangle=\langle\hat{J}_+\rangle^*.\qquad(5)$$

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Assuming the quantum correlations to be insignificant, we can carry out in the right-hand side of (5) the factorization $\langle\hat{J}_\mu\hat{J}_\nu\rangle=\langle\hat{J}_\mu\rangle\langle\hat{J}_\nu\rangle$. Then the factorized equation can be conveniently represented in the form

$$\begin{gathered}dr_+/d\tau=-2i\beta z+i\Delta r_++2(1-i\bar{Q})zr_+,\\ dz/d\tau=-i\beta(r_+-r_-)-2r_+r_-;\\ \tau=\gamma t/2,\quad r_\pm=\langle\hat{J}_\pm\rangle/j,\quad z=\langle\hat{J}_z\rangle/j,\\ \beta=2v/\gamma j,\quad \Delta=2(\varepsilon+\bar{\varepsilon})/\gamma j,\quad \bar{Q}=2\bar{q}/\gamma.\end{gathered}\tag{6}$$

We note that for the factorized equation the "angular momentum" $r^2=z^2+r_+r_-$ remains a conserved quantity ($r^2=1$).

The stationary solution of the Eqs. (6) is given by the relations

$$\beta^2=f(z_c)=(1-z_c^2)[(\Delta/2z_c-\bar{Q})^2+1],\tag{7}$$

or, equivalently

$$\Delta/2=[\bar{Q}\pm(\beta^2/(1-z_c^2)-1)^{1/2}]z_c,\tag{8}$$

$$y_c=\operatorname{Im}r_{+c}=(1-z_c^2)/\beta,\tag{9}$$

$$x_c=\operatorname{Re}r_{+c}=(\Delta/2z_c-\bar{Q})y_c.\tag{10}$$

The function $z_c(\beta)$ defined by (7) is shown in Fig. 1. According to this function, at certain relations (to be discussed later) between the parameters and $\Delta$ and $\bar{Q}$, there exist regions of $\beta$ in which each $\beta$ corresponds to three values of $z_c$ from the interval $[0,-1]$. Linearizing Eqs. (6) near the stationary points, it is easy to show that the points $a$ and $b$ (see Fig. 1) are respectively a node and a focus, while the point $c$ corresponds to an unstable state. The range of values of the parameters $\beta$ for which there are two stable states is located between the two points $\beta_c=f^{1/2}(\bar{z}c)$ and $\beta_B=f^{1/2}(\bar{z}_B)$ where $\bar{z}_C$ and $\bar{z}_B$ are the roots of the equation

$$\frac{df(z)}{dz}\sim\left\{(1+\bar{Q}^2)z^4-\frac{\Delta}{2}\bar{Q}z^3-\frac{\Delta}{2}\bar{Q}z+\left(\frac{\Delta}{2}\right)^2\right\}=0,\tag{11}$$

and are located in the interval $[0,-1]$. The functions $x_c(\beta)$ and $y_c(\beta)$ are not uniquely defined in the same interval.

In the case when $\bar{Q}^2\gg1$, $-1<\Delta/2\bar{Q}\leqslant0$, we can obtain from (11) approximate values of $\bar{z}_B$ and $\bar{z}_C$:

$$\bar{z}_B=\Delta/2\bar{Q},\quad \bar{z}_C=(\Delta/2\bar{Q})^{1/3}.$$

Accordingly, the boundaries of the ambiguity region (bifurcation region) are

$$\beta_B=[1-(\Delta/2\bar{Q})^2]^{1/2},\quad \beta_C=|\bar{Q}|[1-(\Delta/2\bar{Q})^{2/3}]^{3/2}.\tag{12}$$

It follows analogously from (8)–(10) in Fig. 2 that the dependences of $z_c$, $x_c$, and $y_c$ on $\Delta$ can likewise be ambiguous. At sufficiently small values of $\beta$ the value of $z_c$ differs little from $-1$, so that we can replace $\Delta/2z_c$ in (7) by $-\Delta/2$. As a result, the function $z_c(\Delta)$ is represented by the curve

$$z_c^2\approx1-\beta^2/[1+(\Delta/2+\bar{Q})^2]$$

with a maximum at point $\Delta/2=-\bar{Q}$ (see Fig. 2). With increasing $\beta$ the curve becomes deformed and its maximum shifts towards the point $\Delta=0$. At a certain value $\beta=\bar{\beta}$ an ambiguity sets in the function $z_c(\Delta)$. The limits of this region are determined by the condition

$$\frac{d\Delta(z)}{dz}=0,$$

where the function $\Delta(z)$ is defined by (8).

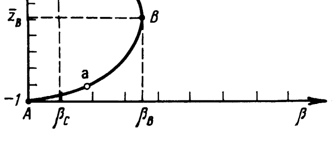

FIG. 1. Plot of $z_c(\beta)$ calculated on the basis of the classical solution at $\Delta=1$ and $\bar{Q}=-5$.

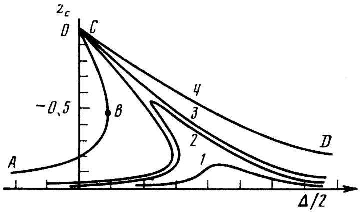

FIG. 2. Plot of $z_c(\Delta)$ calculated on the basis of the classical solution at $\bar{Q}=-5$, and $\beta=0.5$ (1); 0.9 (2); 1 (3); 3 (4).

It should be noted here that at small pump intensities (at $x_c^2+y_c^2=1-z_c^2\ll1$) the equation of the bifurcation curve (7) or (8) coincides with the analogous equation for stimulated nonlinear resonant oscillations of an anharmonic damped oscillator (see e.g., Ref. 10). Indeed, putting $1-z_c^2=b^2\ll1$ in (7) we obtain

$$|2v|^2=b^2[(\bar{\omega}-\omega_0+\varkappa\bar{\omega}jb^2)^2+(\gamma j)^2],$$

which coincides, when the parameters are redesignated, with Eq. (29.4) of the book by Landau and Lifshitz.[10]

Returning to the exact equation (1) and (3), we easily see that its stationary solution is uniquely determined by specifying the parameters $\beta$, $\Delta$, and $\bar{Q}$ [this follows from the fact that the determinant of the right-side of Eq. (1) differs from zero when account is taken of the normalization $\operatorname{Tr}\hat{\rho}=1$]. The difference between the classical (i.e., obtained on the basis of direct factorization of the equations of motion) solution and the solution based on the complete equation is explained by the influence of the quantum fluctuations which are not taken into account by the classical description. The next section of the article is devoted in fact to the infleuence of the quantum fluctuations on the stationary solution.

## 3. EXACT SOLUTION

The exact stationary solution of Eq. (1), (3) can be obtained on the following manner. Following Refs. 1, 2, and 6 we assume that the stationary density matrix $\hat{\rho}_c$ can be represented in the form

$$\hat{\rho}_c=A^{-1}R(\hat{J}_-)R^+(\hat{J}_+),\quad A=\operatorname{Sp}R(\hat{J}_-)R^+(\hat{J}_+),\tag{13}$$

where $R(\hat{J}_-)$ and $R^+(\hat{J}_+)$ are functions of the operators $\hat{J}_-$ and $\hat{J}_+$, i.e., are operators defined by the relation[11]

$$R(\hat{J}_-)=\sum_{n=0}c_n\hat{J}_-^n,\tag{14}$$

where $c_n$ are the coefficients of the Taylor expansion of the $c$-number function $R(x)$. Then, taking into account the easily proved relations

$$\hat{J}_z R(\hat{J}_-) = -R(\hat{J}_-)\hat{J}_z - [\textstyle\int R(\hat{J}_-)\,d\hat{J}_-, \hat{J}_+], \tag{15a}$$

$$[\hat{J}_+\hat{J}_-, R(\hat{J}_-)G(\hat{J}_+)] = [\hat{J}_+, \hat{J}_- R(\hat{J}_-)]G(\hat{J}_+) - R(\hat{J}_-)[\hat{J}_+ G(\hat{J}_+), \hat{J}_-], \tag{15b}$$

where, according to (14)

$$\int R(\hat{J}_-)\,d\hat{J}_- = \sum_n \frac{c_n}{n+1}\hat{J}_-^{n+1},$$

we obtain from (1) and (3) at $d\hat{\rho}_c/dt = 0$

$$[\hat{J}_+, \{R(\hat{J}_-)(\hat{J}_- + i\bar{v}\hat{1}) + i\bar{Q}\hat{J}_- R(\hat{J}_-) + i\nu\int R(\hat{J}_-)\,d\hat{J}_-\}]\,R^+(\hat{J}_+) + \mathrm{H.c.} = 0,$$
$$\bar{v} = 2v/\gamma, \quad \bar{Q} = 2\bar{q}/\gamma, \quad \nu = 2(\varepsilon + \bar{\varepsilon})/\gamma. \tag{16}$$

If we denote the operator in the curly brackets of (16) by $F(\hat{J}_-)$, relation (16) states that the operator $i[\hat{J}_+, F(\hat{J}_-)]R^+(\hat{J}_+)$ is Hermitian. The latter can be satisfied either if $[\hat{J}_+, F] = 0$ or if $i[\hat{J}_+, F(\hat{J}_-)] = R(\hat{J}_-)$ apart from a real factor. The latter cannot take place by virtue of the commutation relations between the $\hat{J}_\nu$. Therefore the operator $F(\hat{J}_-)$ must commute with $\hat{J}_+$, i. e., we have accurate to a real factor chosen from the normalization condition

$$R(\hat{J}_-)[(1 + i\bar{Q})\hat{J}_- + i\bar{v}\hat{1}] + i\nu\int R(\hat{J}_-)\,d\hat{J}_- = 1, \tag{17}$$

where $\hat{1}$ is the unit operator. Since (17) contains only the operators $\hat{J}_-$, it is possible to operate with all the quantities in (17) as with $c$-numbers.

Differentiating (17) with respect to $\hat{J}_-$ in solving the resultant equation we obtain

$$R(\hat{J}_-) = (\hat{J}_- + i\tilde{v}\hat{1})^{-1-i\tilde{\varepsilon}}; \quad \tilde{v} = \bar{v}/(1 + i\bar{Q}), \quad \tilde{\varepsilon} = \nu/(1 + i\bar{Q}). \tag{18}$$

The final form of the sought stationary density matrix is

$$\hat{\rho}_c = A^{-1}(\hat{J}_- + i\tilde{v}\hat{1})^{-1-i\tilde{\varepsilon}}(\hat{J}_+ - i\tilde{v}^*\hat{1})^{-1+i\tilde{\varepsilon}^*}$$
$$= A^{-1}\sum_{m,n=0}^{2j}(-i\tilde{v})^{-m}(i\tilde{v}^*)^{-n}\frac{(1+i\tilde{\varepsilon})_m(1-i\tilde{\varepsilon}^*)_n}{m!n!}\hat{J}_-^m\hat{J}_+^n, \tag{19}$$

where

$$(\lambda)_n = \lambda(\lambda+1)\ldots(\lambda+n-1) \equiv \Gamma(\lambda+n)/\Gamma(\lambda),$$
$$A = \sum_{n=0}^{2j} a_n, \quad a_n = |\tilde{v}|^{-2n}|(1+i\tilde{\varepsilon})_n|^2 C_{2j+n+1}^{2n+1}. \tag{20}$$

The coefficients $a_n$ determine the distribution of the quantum system over superposition states with density operators proportional to $\hat{J}_-^n\hat{J}_+^n$.

From (19) and from an earlier[1] analysis of Eqs. (1) at $\nu = \bar{Q} = 0$ it follows that at zero frequency detuning allowance for the interatomic interaction leads to an increase of the coherence region with respect to the power of the exciting radiation—time radiation scattered by the atoms is in the limit $j \gg 1$ coherent at powers up to values $\sim(|\bar{v}|j)^2(1 + \bar{Q}^2)$, and above them quantum effects become significant. In addition, in contrast to the case $\bar{Q} = 0$, the interaction between the atoms leads to a nonzero mean value $\langle\hat{J}_x\rangle$. In the other limiting case, when $\bar{Q} = 0$ but $\nu \neq 0$, we can find from (19) (see also Ref. 6) that no such critical value of the power exists.

We dwell in detail in the present paper on the combined influence exerted on the atomic correlation functions by the frequency detuning and by the interaction between the atoms. These functions are defined by the expressions

$$G^{(m,n)} = \langle\hat{J}_+^m\hat{J}_-^n\rangle, \tag{21}$$

and are proportional to the correlation functions $\langle\hat{E}_-^m\hat{E}_+^n\rangle$ of the scattered radiation.[12] Using (19), we obtain

$$G^{(m,n)} = A^{-1}\sum_{r=\max(m,n)}^{2j}(-i\tilde{v})^{-r+n}(i\tilde{v}^*)^{-r+m}\frac{(1+i\tilde{\varepsilon})_{r-n}(1-i\tilde{\varepsilon}^*)_{r-m}(2j+r+1)!(r!)^2}{(r-n)!(r-m)!(2j-r)!(2r+1)!}. \tag{22}$$

From among the functions (21), most information is provided by $G^{(1,0)}$, $G^{(1,1)}$, and $G^{(2,2)}$. The function $G^{(1,0)}$ is proportional to the average amplitude of the scattered radiation, $G^{(1,1)}$ is proportional to its intensity, and $G^{(2,2)}$ is proportional to the intensity correlation function. The correlation properties of the scattered radiation are usually determined by the normalized functions of first and second order:

$$g^{(1)} = G^{(1,1)}/|G^{(1,0)}|^2, \quad g^{(2)} = G^{(2,2)}/(G^{(1,1)})^2. \tag{23}$$

At $g^{(1)} = 1$, the scattered radiation is fully coherent; in other words, it is identical with the classical radiation and a deviation of $g^{(1)}$ from unity is evidence of loss of coherence. The function $g^{(2)}$ determines the rate of the counting of the photon coincidences by the two counters. The values $g^{(2)} > 1$ are evidence of the presence of bunching of the scattered photons, and $g^{(2)} < 1$ evidence of antibunching; if $g^{(2)} = 2$, then the properties of the scattered light are equivalent to the properties of the equilibrium thermal radiation. In addition, interest attaches to the mean value $\langle\hat{J}_z\rangle$, which is proportional to the power absorbed by the atomic system. According to (19)

$$\langle\hat{J}_z\rangle = -(2A)^{-1}\sum_{n=0}^{2j} n a_n = \frac{1}{4}|\tilde{v}|\frac{\partial \ln A}{\partial|\tilde{v}|}. \tag{24}$$

Figures 3 and 4 shows the dependences, calculated on the basis of (22)–(24), of $z = \langle\hat{J}_z\rangle/j$, $g^{(1)}$, and $g^{(2)}$ on the parameter $\beta = 2|v|/\gamma j$, that characterizes the amplitude of the exciting radiation, and on the normalized frequency detuning $\Delta = 2(\varepsilon + \bar{\varepsilon})/\gamma j$ at various particle numbers $N = 2j$. It follows from the curves of Fig. 3 that for those values of the parameters $\bar{Q}$ and $\Delta$ for which Eq. (11) has two roots in the region $[0, -1]$ there exists a region of values of $\beta$ where the function $z(\beta)$ undergoes an abrupt jump, and the functions $g^{(1)}(\beta)$, $g^{(2)}(\beta)$ have a sharp maximum. With increasing number of particles in the system (with increasing $j$) these singularities become more strongly pronounced. In the limit as $j \to \infty$ the region of the transition with respect to $\beta$ converges to the point $\beta = \beta_{\mathrm{cr}}$, and the mean value of $z(\beta)$ tends to the values $z_c(\beta)$ obtained on the basis of the factorization method. At $\beta < \beta_{\mathrm{cr}}$ these values correspond to one stable branch of $z_c$, and at $\beta > \beta_{\mathrm{cr}}$ to the other. On the other hand, the functions $g^{(1)}(\beta)$ and $g^{(2)}(\beta)$ tend as $j \to \infty$ to unity (i. e., the system behaves classically) at $\beta = \beta_{\mathrm{cr}}$ and take on values larger than

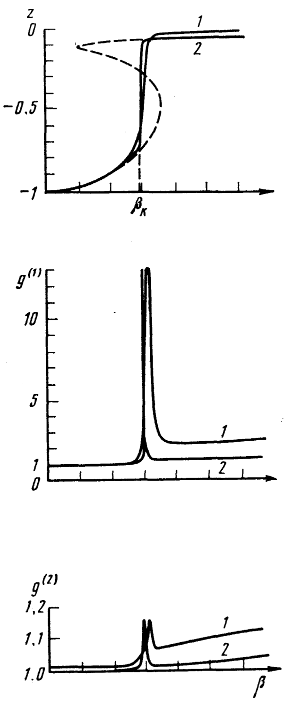

FIG. 3. The functions $z(\beta)$, $g^{(1)}(\beta)$, $g^{(2)}(\beta)$, calculated on the basis of the exact solution (19) at $\bar{Q}=-5$, $\Delta=1$, $j=15$ (1), $j=50$ (2). The dashed curves represent the corresponding classical quantities.

unity at the point $\beta=\beta_{\rm cr}$.

It can thus be assumed that in the "thermodynamic" limit a nonequilibrium phase transition, similar to an equilibrium first-order phase transition, takes place at the point $\beta=\beta_{\rm cr}$. A similar critical transition takes place also when the frequency detuning is varied at a fixed value of the exciting-radiation power, as is confirmed by Fig. 4.

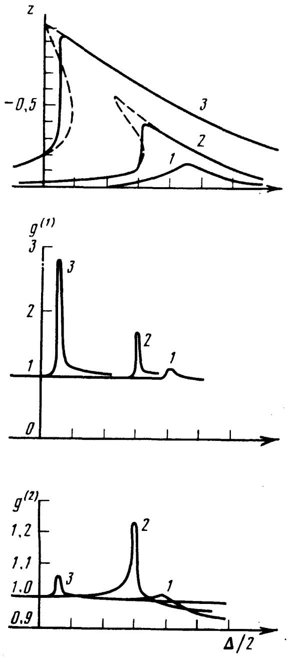

FIG. 4. The same quantities as in Fig. 3, as functions of $\Delta$, calculated at $j=50$ and $\beta=0.5$ (1), 0.9 (2), and 3 (3).

It follows from the results that the quantum fluctuations taken into account in the complete description influence greatly the character of the behavior of the investigated systems at large $j$ only in the vicinity of the region of $\beta_{\rm cr}$ (or $\Delta_{\rm cr}$), and that in the thermodynamic limit ($j\to\infty$, with $\beta$ and $\Delta$ finite) they are effective only at the point $\beta_{\rm cr}$ (or $\Delta_{\rm cr}$) itself.

## 4. QUASICLASSICAL SOLUTION

To find the phase-transition points $\beta_{\rm cr}$ and $\Delta_{\rm cr}$ one could seek asymptotic expressions for the exact formulas (19)–(24) as $j\to\infty$. There is, however, a simpler method of taking into account the influence of the quantum fluctuations in the thermodynamic limit. We note for this purpose that we can write for the coefficients $a_n$ in (19) and (20) the simple recurrence relation

$$a_n=\frac{1+(\Delta j/n+\bar{Q})^2}{(\beta j)^2}(2j-n+1)(2j+n+1)\frac{n}{2(2n+1)}a_{n-1},\quad a_0=1. \qquad (25)$$

Replacing at $j\gg1$ the discrete variable $n$ by the continuous variable

$$w=n/2j, \qquad (26)$$

we obtain from (25) a differential equation for the function $a(w)$:

$$\frac{da}{dw}=-2j\left(1-\frac{f(w)}{\beta^2}\right)a, \qquad (27)$$

where $f(w)$ coincides, apart from the substitution $w\to-w$, with the function introduced by relation (7). The solution of (27) is

$$a(w)=c\exp(-2j\Phi(w)); \qquad (28)$$

$$\Phi(w)=\int\left[1-\frac{f(w)}{\beta^2}\right]dw \qquad (29)$$

$$=w-\beta^{-2}[(1+\bar{Q}^2)(w-w^3/3)-(\Delta/2)^2(w+w^{-1})+\bar{Q}\Delta(\ln w-w^2/2)].$$

The constant in (28) is chosen from the normalization condition

$$\int_0^1 a(w)dw=1. \qquad (30)$$

Since $2j\gg1$ in (28), the function $a(w)$ takes the shape of a set of steep peaks whose positions are determined by the minima of the function $\Phi(w)$, i.e., by the equations

$$\Phi'(w)=1-f(w)/\beta^2=0, \qquad (31a)$$

$$\Phi''(w)=-f'(w)/\beta^2>0. \qquad (31b)$$

Equation (31a) coincides, following the substitution $w\to-w$, with Eq. (7) that determines the mean value $z_c=\langle\hat{J}_z\rangle/j$ in the factorized description. Consequently, the positions of the maxima of $a(w)$ coincides with the value $-z_c$, and their dependences on $\beta$ and $\Delta$ are shown in Figs. 1 and 2. The condition (31b) determines the stable branches $AB$ and $CD$ on these figures. Thus, expanding $\Phi(w)$ in a power series near the minimum points and retaining only the quadratic terms, we obtain an asymptotic representation of the function $a(w)$:

$$a(w)=\sum_{k=1,2}\alpha_k\exp[-j\Phi''(-z_{ck})(w+z_{ck})^2], \qquad (32)$$

where $-z_{c1}$ and $-z_{c2}$ are stable roots of Eq. (31a), and the weights $\alpha_{1,2}$ are, according to (30)

$$\alpha_i = \frac{[\Phi''(-z_{ci})]^{-1/2}\exp[-2j\Phi(-z_{ci})]}{\sum\limits_{k=1,2}[\Phi''(-z_{ck})]^{-1/2}\exp[-2j\Phi(-z_{ck})]}. \tag{33}$$

The widths of the Gaussian curves in (32) are proportional to $[2j\Phi''(-z_{ci})]^{-1/2}$. With increasing $j$, they tend to zero and in the limit as $j\to\infty$ the corresponding functions in (32) can be replaced by $\delta$-functions. According to (32) and (33), in the regions of the parameters $\beta$ and $\Delta$, where the function (7) or (31a) has a single root, the distribution function $a(w)$ is represented by a one maximum. In the region where there are two roots, the function $a(w)$ has two minima. The ratio of the weights of these maxima is

$$\frac{\alpha_1}{\alpha_2} = \left[\frac{\Phi''(-z_{c2})}{\Phi''(-z_{c1})}\right]^{1/2}\exp\{-2j[\Phi(-z_{c1})-\Phi(-z_{c2})]\}. \tag{34}$$

Since $2j \gg 1$, the predominant maximum will be the one for which the function $\Phi(w)$ calculated at the corresponding point of the bifurcation curve is smaller. If the functions $\Phi_1(\mu)=\Phi(-z_{c1}(\mu))$ and $\Phi_2(\mu)=\Phi(-z_{c2}(\mu))$, where $\mu$ is equal to $\beta$ or $\Delta$, have an intersection point, then the abscissa of this point is in fact the sought phase-transition point in the thermodynamic limit. If, however, there is no intersection point, the phase-transition point is one of the limits of the bifurcation region. The plots of $\Phi_{1,2}(\Delta)$ shown in Fig. 5 and a comparison of these plots with Fig. 4 confirm this assumption. Figures 5b and 5c represent the case when the phase-transition point $\Delta_{cr}$ coincides with the boundary of the bifurcation region, while in Fig. 5d the phase-transition point is where $\Phi_1(\Delta)$ and $\Phi_2(\Delta)$ intersect.

It should be noted in conclusion that in a number of papers (see, e.g., Refs. 3 and 4) the presence of critical phenomena such as first-order phase transitions and hysteresis in cooperative resonant fluorescence was attributed to nonconservation of the square of the angular momentum $\hat{J}^2$. The model considered in the present article shows that this is not so. In our case $\hat{J}^2$ is conserved both in the complete and in the factorized (classical) description. However, as seen from the results, hysteresis phenomena appear in the factorized model, and a nonequilibrium first-order phase transition appears in the exact solution in the thermodynamic limit.

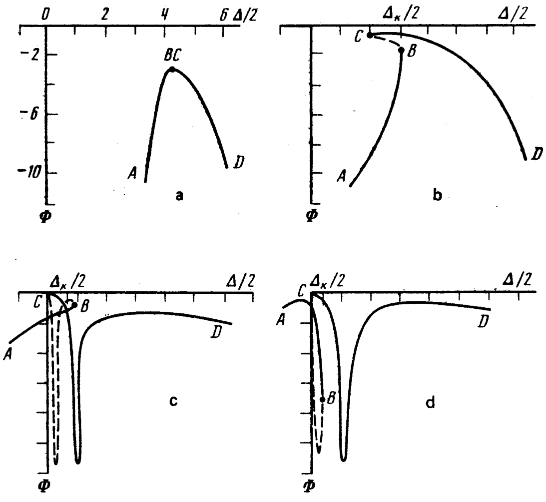

FIG. 5. The functions $\Phi_{1,2}(\Delta)=\Phi[-z_{c1,2}(\Delta)]$, are calculated at $\beta = 0.5$ (a); 0.9 (b); 3 (c); 5 (d). $\Phi_1$ and $\Phi_2$ correspond to the branches *AB* and *CD* respectively.

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