# One-dimensional spin chain in a random anisotropy field

V. M. Vinokur, M. B. Mineev, and M. V. Feĭgel'man

*Institute of Solid State Physics, Academy of Sciences, USSR, and L. D. Landau Institute of Theoretical Physics, Academy of Sciences, USSR*
(Submitted 2 June 1981)
Zh. Eksp. Teor. Fiz. **81**, 2142–2159 (December 1981)

The properties of the ground state of a classical one-dimensional system of $XY$ spins, with random anisotropy $V\cos(\varphi-\alpha)$ and regular anisotropy $\Lambda\cos m\varphi$ of different orders, are studied; the results apply also to charge-density waves in the presence of commensurability and random back-scattering effects. The dependence of the ground-state energy and of the order parameter on the values of $V$ and $\Lambda$ is found; the relative importance of the constant and random parts of the anisotropy is determined by the value of the parameter $\zeta=4\Lambda J^{1/3}/V^{4/3}$. When $\zeta\neq0$, the system has two correlation radii, a longitudinal $R_\parallel$ and a transverse $R_\perp$; and $R_\perp>R_\parallel$ always. The density of localized phonon mode, located below the edge of the continuous spectrum, is found. It is shown that introduction of a random perturbation in a regular system leads to the appearance of an exponential tail of the density of states below the edge of the continuous spectrum. It is shown also that in a system of the type of an incommensurable charge-density wave, the frequency dependence of the conductivity $\sigma(\omega)$ is given by the formula $\sigma(\omega)\sim\omega^2\ln^2(\omega_s/\omega)$, which has the same form as the formula of V. L. Berezinskiĭ [Sov. Phys. JETP **38**, 620 (1974)] for one-dimensional localization.

PACS numbers: 63.20.Pw

## 1. INTRODUCTION

In the present paper we continue the study of the low-temperature properties of classical one-dimensional periodic structures in a random field. For definiteness, we shall speak of a chain of classical, planar $XY$ spins with anisotropy, although the Hamiltonian that we shall consider describes other physical systems as well (for example, a charge-density wave (CDW), a Josephson transition, dislocations in crystals). In the preceding paper,[1] one of the authors developed a method that enables one to treat an incommensurable CDW in the field of randomly distributed impurities. It was shown that this problem is equivalent to the problem of a chain of classical planar spins with random second-order anisotropy. In the present paper, we consider a chain of $XY$ spins with random second-order anisotropy and constant anisotropy of arbitrary order (an external magnetic field may be regarded as first-order anisotropy).

In the language of CDW, allowance for a constant component of anisotropy corresponds to inclusion in the CDW Hamiltonian of commensurability energy. The Hamiltonian of a chain of $XY$ spins with random second-order anisotropy and constant anisotropy of arbitrary order has the form

$$H=\sum_n\left[{}^1/_2J(\varphi_n-\varphi_{n+1})^2-V\cos(\varphi_n-2\alpha_n)-\Lambda\cos m\varphi_n\right]. \qquad (1)$$

Where the spin of the $n$-th site is $S_n=(S\cos\theta_n, S\sin\theta_n)$, $\varphi_n=2\theta_n$, $\alpha_n$ is a random phase uniformly distributed over the circle,

$$J={}^1/_4J'S^2,$$

$J'$ is the exchange integral, $V=\frac{1}{2}S^2D$, $D$ is the random-anisotropy energy, $\Lambda=\frac{1}{2}D_0S^{2m}$, $D_0$ is the energy of the constant component of the anisotropy, and, finally, the order of the anisotropy is $2m$. An external field corresponds to $m=\frac{1}{2}$. This Hamiltonian is analogous to the Hamiltonian of a commensurable CDW, used in papers of Gor'kov[2] and of Lee, Rice, and Anderson.[3]

Our paper is devoted to investigation of the properties of a chain of $XY$ spins under conditions of weak anisotropy, $V\ll J$ (this corresponds to weak coupling of the CDW, $V\ll c\varepsilon_F$[1,4]), when formation of the ground state occurs at distances much larger than the lattice constant, and application of perturbation theory with respect to the weak anisotropy is impossible. In the presence of constant and random components of anisotropy, the behavior of the system is determined by competition between the energies $V$ and $\Lambda$, or in other words between characteristic distances: a correlation radius $R_s=a(J/V)^{2/3}$ due to the random phase and a correlation radius $d=a(J/\Lambda)^{1/2}$ of the regular system, where $a$ is the lattice constant (in the case when $\Lambda$ is the constant anisotropy energy, $d$ is simply the thickness of the domain wall). It then turns out that even an arbitrarily weak external field (or constant anisotropy) leads to the establishment in the system of a nonvanishing mean $\langle\cos m\varphi\rangle$, which vanishes only when $\Lambda=0$.

In this paper a general expression is found for the order parameter $\sigma=\langle\cos m\varphi\rangle$; in the limit for $R_s\ll d$, one gets $\sigma\sim(R_s/d)^2$. Also calculated are the energy of the ground state of the system, the correlation function, and the density of localized spin-wave states.

## 2. DISTRIBUTION FUNCTION W ($\beta$)

As was done before,[1] we obtain a recurrence relation connecting the free energies $\varepsilon(\varphi)$ of chains of $N$ and of $N+1$ spins, assuming that $\varphi$ and $\varepsilon(\varphi)$ change slowly with a shift of one step of the lattice:

$$|\varphi_{N+1}-\varphi_N|\ll1,\quad \left|\frac{\varepsilon_{N+1}(\varphi_{N+1})-\varepsilon_N(\varphi_N)}{\varepsilon_N(\varphi_N)}\right|\ll1. \qquad (2)$$

The conditions (2) are satisfied if the anisotropy is weak: $V\ll J$, $\Lambda\ll J$. The argument $\varphi$ of the function $\varepsilon_n$ is the phase at the last, $N$-th site of the chain.

The desired recurrence relation has the form

$$\varepsilon_{N+1}(\varphi)-\varepsilon_N(\varphi)=-\frac{1}{2J}\left(\frac{\partial\varepsilon_N}{\partial\varphi}\right)^2-V\cos(\varphi-\alpha_{N+1})-\Lambda\cos m\varphi. \qquad (3)$$


0038-5646/81/121138-09$02.40

(In the present paper, we restrict ourselves to the case of zero temperature, $T=0$.)

We now introduce $\gamma_N$, the value of the phase $\varphi_N$ that minimizes the energy $\varepsilon_N(\varphi_N)$, i.e.

$$\left.\frac{\partial\varepsilon_N(\varphi)}{\partial\varphi}\right|_{\varphi=\gamma_N}=0 \tag{4}$$

and the quantity

$$\beta_N=\left.\frac{\partial^2\varepsilon_N}{\partial\varphi^2}\right|_{\varphi=\gamma_N}. \tag{5}$$

Equation (3) determines a Markov process for $\varepsilon_N$ [a random walk is prescribed by the second term in the right side of (3)]. Accordingly, $\beta_N$ and $\gamma_N$ are also random variables. In order to calculate the basic physical characteristics of the system, it is necessary to find the distribution function $W(\beta,\gamma)$, to the calculation of which we now turn.

Let the anisotropy constant first be of second order ($m=1$). Differentiating (3) the corresponding number of times with respect to $\varphi$ at the point $\varphi=\gamma_N$, we get, as before,[1] recurrence relations

$$\Delta\gamma=\gamma_N-\gamma_{N+1}=\frac{V}{\beta_N}\sin(\gamma_{N+1}-\alpha_{N+1})+\frac{\Lambda}{\beta_N}\sin\gamma_{N+1}+\frac{1}{2}\frac{\beta_N^{(3)}}{\beta_N}\frac{V^2}{\beta_N^2}\sin^2(\gamma_{N+1}-\alpha_{N+1}). \tag{6}$$

The last term in (6) is always small in comparison with the first with respect to the parameter $(V/J)^{1\ 3}$, but it must be retained (see below):

$$\begin{gathered}\Delta\beta=\beta_N-\beta_{N+1}=\beta_N^2/J-V\cos(\gamma_{N+1}-\alpha_{N+1})-\Lambda\cos\gamma_{N+1}+\beta_N^{(3)}\Delta\gamma-{}^1/{}_2\beta_N^{(4)}(\Delta\gamma)^2,\\ \Delta\beta^{(3)}=\beta_N^{(3)}-\beta_{N+1}^{(3)}=3\beta_N\beta_N^{(3)}/J+\beta_N^{(4)}\Delta\gamma-{}^1/{}_2\beta_N^{(5)}(\Delta\gamma)^2\\ +V\sin(\gamma_{N+1}-\alpha_{N+1})+\Lambda\sin\gamma_{N+1},\\ \Delta\beta^{(4)}=\beta_N^{(4)}-\beta_{N+1}^{(4)}=4\beta_N\beta_N^{(4)}/J+\beta_N^{(5)}\Delta\gamma-{}^1/{}_2\beta_N^{(6)}(\Delta\gamma)^2\\ +V\cos(\gamma_{N+1}-\alpha_{N+1})+\Lambda\cos\gamma_{N+1},\end{gathered} \tag{7}$$

etc.;

$$\beta_N^{(n)}=\left.\frac{\partial^n\varepsilon_N}{\partial\varphi^n}\right|_{\varphi=\gamma_N}.$$

In the preceding paper[1] it was asserted that the contribution of odd derivatives $\beta^{(2n+1)}$ can be neglected. This is in general incorrect: the mean values of these quantities vanish, but the equations for $\beta^{(2k)}$ contain [this is evident by substitution of (6) in (7)] products $\beta^{(3)}\beta^{(2k+1)}$, whose mean values are not small.

We shall determine the characteristic scale of the quantities $\beta$ by averaging (7) over the random phase $\alpha$ and the still unknown distribution $W(\beta,\gamma)$, transforming in the arguments of the cosines from $\gamma_{N+1}$ to $\gamma_N$ in accordance with (6) and taking into account that $\gamma_N$ is independent of $\alpha_{N+1}$. Taking into account that the characteristic scales of all the $\beta^{(2n)}$ coincide in order of magnitude, we have (the coefficient of the second term is given in order of magnitude)

$$-\frac{1}{J}\langle\beta^2\rangle+V^2\left\langle\frac{1}{\beta}\right\rangle+\Lambda\langle\cos\gamma\rangle=0. \tag{8}$$

Hence we have for the averages

$$\begin{aligned}&\langle\beta\rangle\sim\beta_s=(JV^2)^{1/3},\quad \langle\cos\gamma\rangle\ll1,\quad \text{if}\quad \zeta\ll1,\\ &\langle\beta\rangle\sim\beta_0=(J\Lambda)^{1/2},\quad \langle\cos\gamma\rangle\approx1,\quad \text{if}\quad \zeta\gg1,\end{aligned} \tag{9}$$

$$\zeta=4J^{1/3}\Lambda/V^{4/3}=4(\beta_0/\beta_s)^2. \tag{10}$$

The quantity $\zeta=(4R_s/d)^2$ introduced here characterizes the relative contribution of the constant and random components of the anisotropy.

Equations (6) and (7) are essentially the Langevin equations for the random variable $\xi=(\gamma,\beta,\beta^{(3)},\ldots)$, with an infinite number of components. The corresponding distribution function $W(\gamma,\beta,\beta^{(3)},\ldots)$ should satisfy an equation of the Fokker-Planck type. What interests us is the distribution function $W(\beta,\gamma)$ averaged over the remaining $\beta^{(n)}$. The equation for $W(\beta,\gamma)$ is most simply obtained by relating the distribution functions at neighboring sites by means of the formula for the total probability:

$$\begin{gathered}W(\beta_{N+1},\gamma_{N+1})\,d\beta_{N+1}d\gamma_{N+1}\\ =\langle W(\beta_N(\beta_{N+1},\gamma_{N+1}),\gamma_N(\beta_{N+1},\gamma_{N+1}))\,d\beta_N(\beta_{N+1},\gamma_{N+1})\,d\gamma_N(\beta_{N+1},\gamma_{N+1})\rangle.\end{gathered} \tag{11}$$

The values of $\beta_N$ and $\gamma_N$ on the right side of (11) are functions of $\beta_{N+1}$ and $\gamma_{N+1}$; the averaging is over the random phase $\alpha_{N+1}$. From the recurrence relations (6) and (7) it is evident that $\alpha$ and $\beta$ change little in a shift by one step of the lattice; this enables us to expand $W(\beta_N,\gamma_N)$ in the vicinity of $\beta_{N+1},\gamma_{N+1}$.

Writing further

$$d\beta_N d\gamma_N=d\beta_{N+1}d\gamma_{N+1}\frac{\partial(\beta_N,\gamma_N)}{\partial(\beta_{N+1},\gamma_{N+1})},$$

we get

$$\begin{gathered}\frac{\partial}{\partial\beta}\left(W\langle\Delta\beta\rangle+\frac{\partial W}{\partial\beta}\frac{\langle(\Delta\beta)^2\rangle}{2}+\frac{1}{2}\frac{\partial W}{\partial\gamma}\langle\Delta\beta\Delta\gamma\rangle+\frac{1}{2}W\left\langle\Delta\beta\frac{\partial\Delta\gamma}{\partial\gamma}\right\rangle\right.\\ \left.-\frac{1}{2}W\left\langle\frac{\partial\Delta\beta}{\partial\gamma}\Delta\gamma\right\rangle\right)+\frac{\partial}{\partial\gamma}\left(W\langle\Delta\gamma\rangle+\frac{1}{2}\frac{\partial W}{\partial\gamma}\langle(\Delta\gamma)^2\rangle\right.\\ \left.+\frac{1}{2}\frac{\partial W}{\partial\beta}\langle\Delta\beta\Delta\gamma\rangle+\frac{1}{2}W\left\langle\Delta\gamma\frac{\partial\Delta\beta}{\partial\beta}\right\rangle-\frac{W}{2}\left\langle\frac{\partial\Delta\gamma}{\partial\beta}\Delta\beta\right\rangle\right)=0.\end{gathered} \tag{12}$$

We shall solve Eq. (12) when

$$\Lambda\ll V. \tag{13}$$

This condition does not restrict the physical generality of the results, because it permits the existence of a region $\zeta\gg1$, i.e., such that

$$\Lambda^2\ll V^2\ll J^{1/2}\Lambda^{3/2}, \tag{14}$$

within which the properties of the system are determined principally by the constant component of the anisotropy.

On carrying out in (12) the averaging over the random phase $\alpha$, we get with the aid of (6) and (7) [and with use of (13)]

$$\begin{gathered}\frac{\partial}{\partial\beta}\left[-\frac{V^8}{2\beta}\left(1-\frac{1}{2}f(\beta)\right)W+\frac{\beta^2}{J}W-\Lambda W\cos\gamma+\frac{V^2}{4}\frac{\partial W}{\partial\beta}\right]\\ +\frac{\partial}{\partial\gamma}\left[\frac{\Lambda\sin\gamma}{\beta}W+\frac{V^2}{4\beta}\frac{\partial W}{\partial\gamma}\right]=0,\end{gathered} \tag{15}$$

where

$$f(\beta)=\frac{1}{\beta^2}\langle(\beta^{(3)})^2\rangle-\frac{1}{\beta}\langle\beta^{(4)}\rangle$$

(the averaging of $\beta^{(4)}$ and $\beta^{(3)}$ is done at prescribed $\beta$) is a certain function that is known with asymptotic accuracy for $\beta\ll\beta_s$ and for $\beta\gg\beta_s$ [we shall discuss the calculation of $f(\beta)$ in more detail below].

Equation (15) is easily solved if we note that it has the form of the divergence of a certain vector j in the parameter space in polar coordinates $(\beta,\gamma)$ ($\beta$ = radius, $\gamma$ = angle):

$$\frac{1}{\beta}\frac{\partial}{\partial\beta}(\beta j_\beta)+\frac{1}{\beta}\frac{\partial j_\gamma}{\partial\gamma}=0.$$

Here j is the diffusion current in the external field. Rewriting (15) in the form

$$\frac{\partial}{\partial\beta}\left\{\beta\left[-\frac{4}{V^2}\left(\frac{V^2}{2\beta}\left(1-\frac{1}{2}f(\beta)\right)-\frac{1}{J}\beta^2+\Lambda\cos\gamma\right)\frac{W}{\beta}+\frac{1}{\beta}\frac{\partial W}{\partial\beta}\right]\right\}+\frac{\partial}{\partial\gamma}\left[\frac{4\Lambda\sin\gamma}{V^2}\frac{W}{\beta}+\frac{1}{\beta}\left(\frac{1}{\beta}\frac{\partial W}{\partial\gamma}\right)\right]=0,$$

we get an equation formally equivalent to the equation of stationary diffusion in a force field **F** with components

$$F_\beta=\frac{4}{V^2}\left[\frac{V^2}{2\beta}\left(1-\frac{1}{2}f(\beta)\right)-\frac{\beta^2}{J}\right]+\frac{4\Lambda}{V^2}\cos\gamma,\qquad F_\gamma=-\frac{4\Lambda}{V^2}\sin\gamma, \tag{16}$$

the diffusion-coefficient tensor is isotropic, $D_{ij}=\beta^{-1}\delta_{ij}$.

It is easily shown that

$$(\mathrm{rot}\,F)_z=\frac{\partial F_\beta}{\partial\gamma}-\frac{\partial(\beta F_\gamma)}{\partial\beta}=0,$$

that is, the external field is irrotational, and the solution of the stationary Fokker-Planck equation is a Boltzmann distribution with potential energy $\Phi$, obtained by integration of the components of force (16):

$$\Phi=-\frac{4\beta\Lambda}{V^2}\cos\gamma+\Phi_0(\beta),\qquad W=A\exp(-\Phi(\beta))=A\exp\left(\frac{4\beta\Lambda}{V^2}\cos\gamma\right)\exp(-\Phi_0(\beta)). \tag{17}$$

Here $\Phi_0$ represents the external field in the absence of the constant anisotropy, and the constant $A$ is determined by the normalization condition. We see from the structure of the solution that the angular part, due to the action of the constant field, separates out; the value of $\Phi_0$ is independent of $\Lambda$ and can be determined in the ranges $\beta\gg\beta_s$ and $V\ll\beta\ll\beta_s$.

The corresponding calculations have already been made[1]; here we shall improve them somewhat. It is easy to show (see Appendix A) that in the range $\beta\gg\beta_s$

$$\langle(\beta^{(3)})^2\rangle/\beta^2\sim O(\beta^{-3}),\quad \langle\beta^{(4)}\rangle/\beta=-{}^2/_5,$$

and therefore $f(\beta)=2/5$. This enables us to determine the preexponential power factor in the asymptotic behavior of $W(\beta)$ for $\beta\gg\beta_s$. In the range $V\ll\beta\ll\beta_s$, it is convenient to use an expansion of the function $\varepsilon(\varphi)$ as a Fourier series (the condition $V\ll\beta$ here insures the smallness of the changes of all quantities on shift by one step of the lattice). As a result (see Appendix B)[1)]

$$W_0(\beta)=\exp(-\Phi_0(\beta))\sim\frac{\beta}{\beta_s^2}\ln\left(\frac{\beta_s}{\beta}\right).$$

This expression would be obtained from (16) and (17) for

$$f(\beta)=1+[\ln(\beta_s/\beta)]^{-1}.$$

Thus

$$f(\beta)=1+1/\ln(\beta_s/\beta),\quad V\ll\beta\ll\beta_s,$$
$$f(\beta)={}^2/_5,\quad \beta\gg\beta_s.$$

For the potential energy $\Phi_0$, we have

$$\Phi_0(\beta)=-\ln(\beta/\beta_s)-\ln\ln(\beta_s/\beta)\quad V\ll\beta\ll\beta_s,$$
$$\Phi_0(\beta)={}^4/_3(\beta/\beta_s)^3-{}^8/_5\ln(\beta/\beta_s),\quad \beta\gg\beta_s.$$

Thus the distribution function $W(\beta,\gamma)$ has the form

$$W(\beta,\gamma)=\frac{A}{2\pi}W_0(\beta)\exp\left(\frac{4\Lambda\beta}{V^2}\cos\gamma\right); \tag{18}$$

$$W_0(\beta)=\exp(-\Phi_0(\beta))=\begin{cases}\left(\dfrac{\beta}{\beta_s}\right)^{8/5}\exp\left[-\dfrac{4}{3}\left(\dfrac{\beta}{\beta_s}\right)^3\right], & \beta\gg\beta_s,\\[2ex] \dfrac{\beta}{\beta_s}\ln\dfrac{\beta_s}{\beta}, & V\ll\beta\ll\beta_s.\end{cases} \tag{19}$$

The distribution function $W(\beta)$ is obtained by integration of (18) with respect to $\gamma$ and has the form

$$W(\beta)=AW_0(\beta)I_0\left(\frac{4\beta\Lambda}{V^2}\right),$$

where $I_0(x)$ is a zero-order modified Bessel function. The constant $A$ is determined from the normalization condition

$$1=\int_0^\infty W(\beta)\,d\beta=A\beta_s\int_0^\infty \tilde{W}_0(x)I_0(\zeta x)\,dx;\qquad \tilde{W}_0(x)=\begin{cases}x\ln x^{-1}, & x\ll1,\\ x^{8/5}\exp(-{}^4/_3x^3), & x\gg1.\end{cases} \tag{20}$$

For a concrete calculation it is necessary to use some interpolation formula joining the asymptotes, for example

$$\tilde{W}_0(x)=[x\ln(1+x^{-1})+x^{8/5}]\exp(-{}^4/_3x^3).$$

The final results are not very sensitive to the choice of this interpolation.

From (20) we have

$$A=a\beta_s^{-1},\quad \zeta\ll1\quad(a\sim1),\qquad A=2^{21/10}\beta_s^{-1}\zeta^{1/5}\exp(-{}^1/_3\zeta^{3/2}),\quad \zeta\gg1. \tag{21}$$

When $\zeta\ll1$, the distribution function increases almost linearly over the interval $(V,\beta_s)$ and drops exponentially for $\beta\gg\beta_s$. In the case $\zeta\gg1$, $W(\beta)$ has a well expressed maximum at the point $\beta=\beta_0$. We recall[1] that the range $\beta\ll V$ corresponds to a Poisson tail of the distribution function:

$$W(\beta)\sim\exp(-\mathrm{const}/\beta).$$

The physical characteristics of the system will be calculated in the following sections; nevertheless we should like to mention immediately one important consequence of our result for $W(\beta,\gamma)$. It is clearly evident from formula (18) that the distribution of the angle $\gamma$ around the circle is no longer uniform.

In the presence of a constant component $\Lambda$ of the anisotropy, the most probably values of $\gamma$ are concentrated about the value $\gamma=0$, but the randomness blurs the distribution. As a result, for arbitrary $\Lambda\neq0$ there is a nonvanishing mean $\langle\cos\gamma\rangle$:

$$\langle\cos\gamma\rangle=A\int d\beta\,W_0(\beta)I_1\left(\frac{4\beta\Lambda}{V^2}\right)\sim\begin{cases}\zeta, & \zeta\ll1,\\ 1-O(\zeta^{-3/2}), & \zeta\gg1.\end{cases}$$

We note that $\langle\cos\gamma\rangle$ is not equal to the characteristic order parameter of the system, $\sigma=\langle\cos\varphi\rangle$, because by

$\gamma$ we always mean the phase $\varphi$ at the last point of the chain. But as will be shown below, $\sigma$ behaves similarly to $\langle\cos\gamma\rangle$ and is also nonzero when $\Lambda\neq 0$.

We turn to consideration of the general case $m\neq 1$. Equation (15) takes the form

$$\frac{\partial}{\partial x}\left[\frac{\partial W}{\partial x}-\left(\frac{2-f(x)}{x}+m^2\zeta\cos m\gamma\right)W\right]+\frac{\partial}{\partial\gamma}\left(\frac{1}{x^2}\frac{\partial W}{\partial\gamma}+\frac{m\zeta}{x}\sin m\gamma\right)=0. \tag{22}$$

(Here we have introduced the new variable $x=\beta/\beta_s$.) In the general case $m\neq 1$, the external field is no longer irrotational in polar coordinates, and solution of (22) is very complicated. We shall restrict ourselves to analysis of the situations $\zeta\gg 1$ and $\zeta\ll 1$.

When $\zeta\gg 1$, the principal contribution comes from the region near $\gamma\approx 0$, and therefore we can expand the trigonometric functions in (22). Solving the resulting equation approximately, we find

$$W(\beta)\sim\tilde{W}_0(x)\exp[m^2\zeta x(1-{}^1/_2\gamma^2)]. \tag{23}$$

In the case $\zeta\ll 1$, we may regard the regular potential as a perturbation and seek a solution of (22) in the form

$$W(x)=\tilde{W}_0(x)+\zeta W_1(x)\cos m\gamma.$$

Then it is not difficult to show that

$$W(x)\approx\tilde{W}_0(x)[1+\zeta\varphi(x)\cos m\gamma]\quad(\zeta\ll 1); \tag{24}$$

where

$$\begin{gathered}\varphi(x)\sim x^m\quad\text{for}\quad m<1,\quad x\ll 1,\\ \varphi(x)\sim x/\ln x^{-1}\quad\text{for}\quad m>1,\quad x\ll 1,\\ \varphi(x)\approx m^2x\quad\text{for}\quad x\gg 1.\end{gathered} \tag{25}$$

Comparison of (23)–(25) and (18) shows that the general character of the function $W(\beta,\gamma)$ when $m\neq 1$ is the same as when $m=1$.

## 3. PHYSICAL CHARACTERISTICS OF THE SYSTEM

### 1. Energy of the ground state and order parameter

The total energy of the ground state of a chain of $N$ spins is by definition $\varepsilon_N(\gamma_N)$.[1] The mean energy $E$ per lattice site is determined by the expression

$$E=\langle\varepsilon_{N+1}(\gamma_{N+1})-\varepsilon_N(\gamma_N)\rangle, \tag{26}$$

where the averaging is over the random phase $\alpha$ and the distribution function $W(\beta,\gamma)$ (in this section we shall, for brevity, restrict ourselves to the case $m=1$; the results for arbitrary $m$ are analogous). Using equation (3) and the relations (5) and (7), we easily get

$$\varepsilon_{N+1}(\gamma_{N+1})-\varepsilon_N(\gamma_N)=-V\cos(\gamma_{N+1}-\alpha_{N+1})-\Lambda\cos\gamma_{N+1}-\frac{V^2}{2\beta_N}\sin^2(\gamma_{N+1}-\alpha_{N+1}). \tag{27}$$

The averaging gives

$$E=-\int(V^2/4\beta+\Lambda\cos\gamma)W(\beta,\gamma)d\beta d\gamma,$$

and after integration over $\gamma$ we get

$$E=-A\int\left[\frac{V^2}{4\beta}I_0\left(\frac{4\beta\Lambda}{V^2}\right)+\Lambda I_1\left(\frac{4\beta\Lambda}{V^2}\right)\right]W_0(\beta)d\beta$$
$$=\frac{\beta_s^2}{4J}\left(\int\tilde{W}_0(x)I_0(\zeta x)dx\right)^{-1}\int\left[\frac{I_0(\zeta x)}{x}+\zeta I_1(\zeta x)\right]\tilde{W}_0(x)dx, \tag{28}$$

where we have substituted $A$ from (21).

When $\zeta\ll 1$, this expression reduces to $E\sim J(V/J)^{4/3}$ (in the CDW case this corresponds to the expression obtained earlier,[1] $E\sim\varepsilon_F(cV^2/\varepsilon_F^2)^{2/3}$); and when $\zeta\gg 1$, $E\sim\Lambda$.

By use of the relation (28), we can calculate the order parameter

$$\sigma=\langle\cos\varphi\rangle=-\partial E/\partial\Lambda. \tag{29}$$

When $\zeta\ll 1$, we find with the aid of equations (21) and (22)

$$E=-\frac{\beta_s^2}{4J}\left(\left\langle\frac{1}{x}\right\rangle_0+\frac{3}{4}\zeta^2\langle x\rangle_0-\frac{\zeta^2}{4}\left\langle\frac{1}{x}\right\rangle_0\langle x^2\rangle_0\right), \tag{30}$$

where the brackets $\langle\ldots\rangle_0$ denote an average with the function $\tilde{W}_0(x)$, and

$$\sigma=-\frac{\partial E}{\partial\Lambda}=\frac{\beta_s^2}{4J}\left(\frac{3\zeta^2}{2\Lambda}\langle x\rangle_0-\frac{\zeta^2}{2\Lambda}\langle x^2\rangle_0\left\langle\frac{1}{x}\right\rangle_0\right)=B\zeta,\quad B\sim 1. \tag{31}$$

Comparing the last expression with the mean

$$\langle\cos\gamma\rangle=\frac{2\Lambda}{V^2}\langle\beta\rangle_0\approx\frac{1}{2}B\zeta, \tag{32}$$

we see that for $\zeta\ll 1$, $\langle\cos\gamma\rangle$ is in fact proportional to the order parameter.

When $\zeta\gg 1$, we integrate in (28) by the method of steepest descents and, with the aid of (29), get

$$\sigma=1-\zeta^{-2/3}. \tag{33}$$

2. In this section we shall calculate (restricting ourselves for simplicity to the case $m=1$) the correlation functions

$$K_\parallel=\langle\cos\gamma_N\cos\gamma_0\rangle,\quad K_\perp=\langle\sin\gamma_N\sin\gamma_0\rangle.$$

Here it should be mentioned that the calculation of the correlation radius given earlier[1] is in error; the present paper presents a systematic calculation, which shows in particular that $R_s^{-1}$ is well defined within the Gaussian-fluctuation range and contains no large logarithm $\ln(\beta_s/V)$.

The correlation function $K_\parallel(N)$ is determined by the expression

$$K_\parallel(N,0)=\int\mathcal{P}(\gamma_N\beta_N;\gamma_0\beta_0)\cos\gamma_N\cos\gamma_0 d\gamma_0 d\gamma_N d\beta_0 d\beta_N\times\left[\int\mathcal{P}(\gamma_N,\beta_N;\gamma_0,\beta_0)d\gamma_0 d\gamma_N d\beta_0 d\beta_N\right]^{-1}, \tag{34}$$

and similarly for

$$K_\perp(N)=\langle\sin\gamma_N,\ \sin\gamma_0\rangle. \tag{35}$$

Since the random changes of the values of $\beta_N$ and $\gamma_N$ from site to site represent a Markov process, the distribution function $\mathcal{P}$ is

$$\mathcal{P}=P(\gamma_N,\beta_N,N|\gamma_0,\beta_0)W(\gamma_0,\beta_0), \tag{36}$$

where $P$ is the conditional probability distribution of the values of $\beta$ and $\gamma$ at site $N$ for given $\beta_0$ and $\gamma_0$ at $N=0$; it is determined by the nonstationary Fokker-Planck equation with the corresponding initial conditions:

$$\frac{\partial P}{\partial t}-\hat{L}P=\delta(\gamma-\gamma_0)\delta(\beta-\beta_0)\delta(t); \tag{37}$$

$$\hat{L}=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}-\left(\frac{2}{x}-\frac{f(x)}{x}-4x^2+\zeta\cos\gamma\right)\right]+\frac{1}{x^2}\frac{\partial}{\partial\gamma}\left[\frac{\zeta\sin\gamma}{x}+\frac{\partial}{\partial\gamma}\right],$$
$$x=\beta/\beta_s,\quad t=V^2N/4\beta_s^2=Na/4R_s. \tag{38}$$

Equations (37) and (38) are an obvious generalization of (15) to include the nonstationary problem.

Thus $P$ is the Green's function of the operator $\partial/\partial t - \tilde{L}$. It is convenient to transform from the operator $\tilde{L}$ to an operator of the Schrödinger type by the transformation

$$L=e^{\Phi/2}\tilde{L}e^{-\Phi/2}, \tag{39}$$

where $\Phi$ is determined by formulas (17)–(19). Then the Green's function $P$ is expressed in terms of the eigenfunctions $\Psi_\nu$ of the operator $L$:

$$P(x,\gamma,t|x_0,\gamma_0)=\exp[-{}^1/_2\Phi(x,\gamma)-{}^1/_2\Phi(x_0,\gamma_0)] \times\sum_\nu \Psi_\nu^*(x_0,\gamma_0)\Psi_\nu(x,\gamma)\exp(-\lambda_\nu t), \tag{40}$$

$$L\Psi_\nu=-\lambda_\nu\Psi_\nu, \tag{41}$$

where $\nu$ is a set of two numbers $(n, l)$ corresponding to "radial" and "angular" quantum numbers.

For the correlation functions $K_\parallel$ and $K_\perp$ we have

$$K_\parallel=\sum_\nu |f_\nu|^2\exp(-\lambda_\nu t), \tag{42}$$

$$K_\perp=\sum_\nu |g_\nu|^2\exp(-\lambda_\nu t); \tag{43}$$

$$f_\nu=\int dxd\gamma\exp(-{}^1/_2\Phi(x,\gamma))\Psi_\nu(x,\gamma)\cos\gamma,$$
$$g_\nu=\int dxd\gamma\exp(-{}^1/_2\Phi(x,\gamma))\Psi_\nu(x,\gamma)\sin\gamma. \tag{44}$$

From (38) and (39) we easily find

$$L=L_0-\zeta U_1(x)\cos\gamma-\zeta^2/4, \tag{45}$$

$$L_0=\frac{\partial^2}{\partial x^2}+\frac{1}{x^2}\frac{\partial^2}{\partial\gamma^2}-U_0(x), \tag{46}$$

$$U_0(x)=-\frac{1}{2}\left[\frac{2}{x^2}+\left(\frac{f}{x}\right)'+8x\right]+\frac{1}{4}\left[\frac{2}{x}-\frac{f}{x}-4x^2\right]^2, \tag{47}$$

$$U_1(x)=-[2x^2-(1-f)/2x]. \tag{48}$$

It follows from (42) and (43) that the behavior of the correlation functions at large distances is determined by the quantities $\lambda_\nu$. Thus our problem reduces to a search for the eigenvalues $\lambda_\nu$ of the operator $L$. It is easy to see that the unperturbed (for $L=L_0$) values

$$f_\nu^{(0)}=f_{nl}^{(0)},\quad g_\nu^{(0)}=g_{nl}^{(0)}$$

are nonzero only in the case $l=1$. In fact, as is clear from (47), the eigenfunctions of the operator $L_0$ have the form

$$\Psi_{nl}^{(0)}=\psi_{n0}^{(0)}(x),\quad l=0,$$
$$\Psi_{nl}^{(0)}=\psi_{nl}^{(0)}(x)\begin{Bmatrix}\cos l\gamma\\ \sin l\gamma\end{Bmatrix},\ l\neq0. \tag{49}$$

and the integral over $\gamma$ in (44) and (45) vanishes for $l\neq1$.

In view of the fact that we are interested in the behavior of the correlation functions at large distances, all that is important is the smallest $(n=0)$ eigenvalue of the set $\lambda_{n1}^{(0)}$. Thus

$$K_\parallel^{(0)}=|f_1^{(0)}|^2\exp(-\lambda_{01}^{(0)}t),\quad K_\perp^{(0)}=|g_1^{(0)}|^2\exp(-\lambda_{01}^{(0)}t). \tag{50}$$

We note that the operator $L$ of (46) has $\lambda_0=0$ as its lowest eigenvalue, and the corresponding eigenfunction corresponds to the stationary solution of equation (16) found above:

$$\Psi_{00}=\Psi_0=e^{-\Phi/2}. \tag{51}$$

The equation for $\psi_{01}^{(0)}$ when $\zeta=0$ has the form

$$\frac{d^2\psi_{01}^{(0)}(x)}{dx^2}+\left(\lambda_{01}^{(0)}-U_0(x)-\frac{1}{x^2}\right)\psi_{01}^{(0)}(x)=0. \tag{52}$$

The value of $\lambda_{01}^{(0)}$ is determined by numerical solution of (52) for various choices of the interpolation formula for the function $f(x)$ that occurs in $U_0(x)$. Calculation shows that the eigenvalues are practically independent of the choice of $f$. We shall give the results obtained for

$$f=\frac{1+{}^2/_5x^3+1/\ln x^{-1}}{1+x^3}.$$

The unperturbed level for $\zeta=0$ is

$$\lambda_{01}^{(0)}\approx3.22. \tag{53}$$

When $\zeta\neq0$, there appears in $K_\parallel$ a contribution independent of the distance $t$:

$$K_\parallel=\left(\int dx\,d\gamma\exp[-\Phi(x,\gamma)]\cos\gamma\right)^2+|f_1|^2\exp(-\lambda_{01}t)$$
$$=\langle\cos\gamma\rangle^2+|f_1|^2\exp(-\lambda_{01}t).$$

The term $\zeta U_1(x)$ in the operator $L$ splits the doubly degenerate level $\lambda_{01}^{(0)}$; this leads to the occurrence of two different correlation radii, $R_\parallel$ and $R_\perp$. When $\zeta\ll1$, $R_\parallel(\zeta)$ and $R_\perp(\zeta)$ can be found by perturbation theory, which it is convenient to construct as follows. We represent the wave function corresponding to the level $\lambda_{01}$ as an expansion in powers of $\zeta$:

$$\Psi_{01}=\psi_{01}\cos\gamma+\zeta a_0(x)+\zeta a_2(x)\cos2\gamma+\zeta^2a_1(x)\cos\gamma. \tag{54}$$

On substituting this expression in (41) and equating coefficients of identical harmonics, we get equations for the functions $a_0(x)$ and $a_2(x)$:

$$a_0''+a_0[\lambda-U_0(x)]-{}^1/_2\psi_{01}(x)U_1(x)=0,$$
$$a_2''-4/x^2+a_2[\lambda-U_0(x)]-{}^1/_2\psi_{01}(x)U_1(x)=0. \tag{55}$$

The corrections $\delta\lambda_{01}^\parallel$ and $\delta\lambda_{01}^\perp$ to the eigenvalue $\lambda_{01}$ are expressed in terms of these functions:

$$\delta\lambda_{01}^\parallel=\zeta^2\left({}^1/_4+C\int U_1(x)(a_0(x)+{}^1/_2a_2(x)\psi_{01}(x))dx\right),$$
$$\delta\lambda_{01}^\perp=\zeta^2\left({}^1/_4+{}^1/_2C\int U_1(x)a_2(x)\psi_{01}(x)dx\right), \tag{56}$$
$$C^{-1}=\int|\psi_{01}(x)|^2dx.$$

Numerical integration of (55) and (56) gives

$$\lambda_{01}^\parallel=\lambda_{01}^{(0)}+\delta\lambda_{01}^\parallel=3.22+0.12\zeta^2,$$
$$\lambda_{01}^\perp=\lambda_{01}^{(0)}+\delta\lambda_{01}^\perp=3.22+0.03\zeta^2. \tag{57}$$

Thus for $\zeta\ll1$ we get from (57)

$$R_\parallel\approx(1.24-0.05\zeta^2)R_*,\qquad R_\perp\approx(1.24-0.01\zeta^2)R_*. \tag{58}$$

The relation between the longitudinal correlation radius $R_\parallel$ and the transverse $R_\perp$ is easily determined in the limit $\zeta\gg1$, because in this case the characteristic $\gamma$ are $\ll1$ and the distribution functions with respect to $\gamma$ are Gaussian. We have

$$K_\perp=\langle\sin\gamma_N\sin\gamma_0\rangle\approx\langle\gamma_0\gamma_N\rangle\sim\exp(-x/R_\perp),$$
$$K_\parallel=\langle\langle\cos\gamma_N\cos\gamma_0\rangle\rangle\approx{}^1/_4\langle\langle\gamma_N{}^2\gamma_0{}^2\rangle\rangle$$
$$={}^1/_2\langle\gamma_N\gamma_0\rangle^2\sim{}^1/_2\exp(-2x/R_\perp)\sim\exp(-x/R_\parallel).$$

Therefore for $\zeta\gg1$ we get

$$R_\perp=2R_\parallel.$$

It is easy to show further that in the limit $\zeta \gg 1$

$$R_{\perp}=d=a(J/\Lambda)^{1/2}.$$

We note that, as is evident from (58), the transition from the regime $\zeta \ll 1$ to the regime $\zeta \gg 1$ actually occurs at $\zeta \approx 5$.

## 4. DENSITY OF STATES

The equation of small oscillations of an $XY$ spin chain can be written in the form

$$a^2\frac{J}{u^2}\omega^2\theta=-a^2J\theta''+\theta(V\cos\varphi(x)+\Lambda\cos m\varphi(x)), \tag{59}$$

where $u$ is the velocity of spin waves in the pure system; for applicability of quasiclassical methods, it is necessary that $u/aJ \ll 1$. Arguments similar to those given earlier[1] show that the density of localized spin-wave states is determined by the distribution function $\overline{W}(\beta)$ of the quantity $\beta$ within a chain [in contrast to the distribution function $W(\beta)$ at the last site of the chain]. More exactly, for the density of states normalized on the mean length of a localized state we shall have

$$\tilde{\rho}(\omega)\,d\omega=\overline{W}(\beta)\,d\beta,\quad \omega=\beta u/aJ. \tag{60}$$

In the limit $\zeta \ll 1$, the form of $\rho(\omega)$ differs little from the case $\zeta=0$; therefore, proceeding as in Ref. 1 but using the correct distribution function

$$W(\beta)\sim\beta\ln(\beta_s/\beta),$$

we get

$$\rho(\omega)\sim\omega^3\ln^2(\omega_s/\omega)$$

[$\omega_s$ is defined in (65)]; the previously treated[1] frequency dependence of the conductivity of a charge density wave has in fact the form $\sigma(\omega)\sim\omega^2\ln^2(\omega_s/\omega)$.

In this section we shall further determine the values of $\overline{W}(\beta)$ and $\rho(\omega)$ in the opposite limit $\zeta \gg 1$; therefore we are interested in the range $\beta\sim\beta_0\gg\beta_s$. We fix the phase $\varphi_n$ at an arbitrary point $N$ within a chain, and we introduce the energies, dependent on the phase $\varphi_N$, $\varepsilon_N^{<}(\varphi_N)$ and $\varepsilon_N^{>}(\varphi_N)$ of the right and left semi-infinite chains.[1] We are interested in the range $\beta_s \ll \beta \ll \beta_0$, in which the density of states rises exponentially with increase of $\beta$. In this range, we can write an expansion

$$\varepsilon_N^{>}(\varphi)=\varepsilon_N^{>}(\gamma_N^{>})+\sum_{n=2}^{\infty}\frac{\beta_N^{(n)>}}{n!}(\varphi-\gamma_N^{>})^n,\quad \beta_N^{(2)>}=\beta_N^{>},$$

where $\gamma_N^{>}$ is the phase that minimizes the free energy $\varepsilon_N^{>}$ of the right semi-infinite chain (we recall that in the limit $\beta\gg\beta_s$ we have $\beta^{(2k+1)}\ll\beta^{(2n)}$). A similar expansion can be written for $\varepsilon_N^{<}$.

For the energy of the whole chain, $\varepsilon(\varphi)=\varepsilon^{<}+\varepsilon^{>}$, we have (we retain the first two terms in the expansion)

$$\begin{aligned}\varepsilon(\varphi)&={}^1/_2\beta^{<}(\varphi-\gamma^{<})^2+{}^1/_{24}\beta^{(4)<}(\varphi-\gamma^{<})^4\\&+{}^1/_2\beta^{>}(\varphi-\gamma^{>})^2+{}^1/_{24}\beta^{(4)}(\varphi-\gamma^{>})^4.\end{aligned} \tag{61}$$

We now introduce the quantity $\gamma$, the phase that minimizes $\varepsilon(\varphi)$, and $\beta=(\partial^2\varepsilon/\partial\varphi^2)_\gamma$. Expressions for $\gamma$ and $\beta$ are obtained, as above, by differentiation of the relation (61) with respect to $\varphi$. By taking into account that for $\zeta\gg1$

$$\gamma^{<},\ \gamma^{>}\ll1,\quad |\gamma^{<}-\gamma^{>}|\ll1,$$

one can easily eliminate $\beta^{(4)>}$ and $\beta^{(4)<}$ and obtain expressions for $\gamma$ and $\beta$ as functions of $\gamma^{<}$, $\gamma^{>}$, $\beta^{<}$, and $\beta^{>}$:

$$\begin{aligned}\gamma&=\frac{\beta^{<}\gamma^{<}+\beta^{>}\gamma^{>}}{\beta^{<}+\beta^{>}},\\ \beta&=\beta^{<}+\beta^{>}-\frac{1}{5}\frac{(\gamma^{<}-\gamma^{>})^2}{\beta^{>}+\beta^{<}}\beta^{<}\beta^{>}.\end{aligned} \tag{62}$$

The distribution function $\overline{W}$ is now found with the aid of the well known formula of the theory of probability:

$$\begin{aligned}\overline{W}(\beta)=\int d\gamma\int d\beta^{<}d\beta^{>}\int d\gamma^{<}d\gamma^{>}W(\beta^{<},\gamma^{<})\,W(\beta^{>},\gamma^{>})\\ \times\,\delta(\beta-\beta(\beta^{<},\beta^{>},\gamma^{<},\gamma^{>}))\,\delta(\gamma-\gamma(\beta^{<},\beta^{>},\gamma^{<},\gamma^{>})).\end{aligned} \tag{63}$$

Using (18), (19), and (62), we get, after simple calculations by the method of steepest descents [it is easily shown that for calculation with asymptotic accuracy, it is sufficient to replace (62) by $\beta=\beta^{>}+\beta^{<}$],

$$\overline{W}(\beta)\approx\frac{C}{\beta_s}\zeta^{-3/5}\exp\left(-\frac{2}{3}\zeta^{3/2}\right)\left(\frac{\beta}{\beta_s}\right)^{17/10}\exp\left[\zeta\left(\frac{\beta}{\beta_s}\right)-\frac{1}{3}\left(\frac{\beta}{\beta_s}\right)^3\right],\ C\sim1. \tag{64}$$

The expression (64) was obtained for the case when the order of the anisotropy is $m=1$. This formula remains correct also when $m\neq1$, if in it we make the substitution $\zeta\to m^2\zeta$. [This is evident from a comparison of expressions (23) and (18) in the limit $\zeta\gg1$.]

Finally, for the density of states at a lattice site,

$$\rho(\omega)=u(\Lambda/J)^{1/2}\tilde{\rho}(\omega)$$

[the characteristic dimension of a state is of order $d=a(J/\Lambda)^{1/2}$], we get with the aid of (60) and (64)

$$\begin{aligned}\rho(\omega)=\frac{C}{\omega_s}\left(\frac{\Lambda}{J}\right)^{1/2}\zeta^{-3/5}\exp\left(-\frac{2}{3}m^3\zeta^{3/2}\right)\left(\frac{\omega}{\omega_s}\right)^{17/10}\\ \times\exp\left[-m^2\zeta\left(\frac{\omega}{\omega_s}\right)-\frac{1}{3}\left(\frac{\omega}{\omega_s}\right)^3\right],\quad \omega_s=\beta_s\frac{u}{Ja}.\end{aligned} \tag{65}$$

The expression (65) has a sharp peak at

$$\omega=\omega_s m\zeta^{1/2}=\omega_0.$$

In an ordered system, with $V=0$, the edge of the spectrum would be located at $\omega=\omega_0$. The random potential leads to the appearance at $\omega<\omega_0$ of localized equalizing fluctuations with the density (65). At $\omega\gtrsim\omega_0$, the chief contribution to the density of states comes from the continuous spectrum; therefore the expression (65) is literally applicable at such frequencies $\omega<\omega_{01}$ that $\rho(\omega)\ll\rho(\omega_0)$.

## CONCLUSIONS

We have investigated the properties of a uniform chain of classical $XY$ spins in the field of a weak random anisotropy $V$ and constant anistropy $\Lambda$. It was shown that the relative importance of the random and constant parts of the anisotropy is determined by the ratio of the quantities $(JV^2)^{1/3}$ and $(\Lambda J)^{1/2}$; therefore the system behaves as an "almost regular" one as soon as $\Lambda\gg V(V/J)^{1/3}$, even though $\Lambda$ may be $\ll V$ (since $V\ll J$). If, in particular, $\Lambda$ corresponds to an external magnetic field ($m=\frac{1}{2}$), then the order parameter reaches saturation at fields much smaller than the random anisotropy field. A constant part of the anisotropy occurs also when the distribution of random axes is "not completely random": $\langle\cos\alpha\rangle\neq 0$ (in this case $m=1$). Here $\Lambda\sim V\langle\cos\alpha\rangle$; therefore as soon as $(V/J)^{1/3}\ll\langle\cos\alpha\rangle\ll 1$, the system behaves as an almost regular one. The general expression for the order parameter $\sigma=\langle\cos m\varphi\rangle$ is given by formulas (28) and (29). The order parameter $\sigma$ is nonzero for all $\Lambda\neq 0$; Fukuyama's assertion[5] that in such a system there is a phase transition, with appearance of an order parameter at $\Lambda=\Lambda_c>0$, is in error.

Calculation of the variation of the correlation radii with $\zeta=4\Lambda J^{1/3}V^{-4/3}$ shows [see (58)] that both radii, $R_\parallel$ and $R_\perp$, decrease with increase of $\zeta$, and that $R_\parallel$ is always smaller than $R_\perp$. Because of the small numerical coefficients of $\zeta^2$ in (58), the dependence of $R_\parallel$ and $R_\perp$ on $\zeta$ becomes significant when $\zeta=3$ to 5.

The fluctuational tail of the density of states is given (for $\zeta\gg 1$) by formula (65), which is applicable under the conditions

$$\omega<\omega_0,\quad \rho(\omega)\ll\rho(\omega_0),$$

these are necessary in order that it may be possible to neglect mixing of the fluctuational levels with the states of the continuous spectrum, beginning with a pure ($V=0$) system with $\omega=\omega_0$. The result obtained means that introduction of randomly located impurities in a regular system leads to the appearance of an exponential tail of the density of states below the edge of the continuous spectrum.

For $\zeta\ll 1$, we have for the density of the fluctuational levels

$$\sigma(\omega)\sim\omega^2\ln^2(\omega_s/\omega). \tag{66}$$

As has already been mentioned, the problem under consideration is formally equivalent to the problem of the catching of a charge density wave on impurities, in which calculation of the frequency dependence of the conductivity $\sigma(\omega)$ is of interest. It is interesting to note that the behavior of $\sigma(\omega)$ found by us coincides with that obtained by Berezinskiĭ[6] for the case of noninteracting particles in a random potential:

$$\rho(\omega)\sim\omega^3\ln^2(\omega_s/\omega). \tag{67}$$

We are grateful to S. A. Brazovskiĭ, V. L. Pokrovskiĭ, and D. E. Khmel'nitskiĭ for useful discussions of the work. We are also very grateful to L. N. Shchur, who patiently taught us the use of the computer.

## APPENDIX A

We shall find the distribution function $F(\beta,\beta^{(4)})$ for $\beta\gg\beta_s$. As will be shown at the end of this Appendix, in the limit under consideration all $\beta^{(2k+1)}\ll\beta^{(2n)}$; therefore in equations of the type (7) we may at once drop terms containing $\beta^{(2k+1)}$. The Fokker-Planck equation for $F$ is derived from the recurrence relations (7) with $\Lambda=0$ by use of a total-probability formula of the type (11) [we recall that in the absence of a regular potential, the distribution function $F(\beta\ldots)$ is independent of $\gamma$]. On averaging over the disorder and integrating over $\beta^{(6)}\ldots$, we get

$$\frac{\partial}{\partial x}\left[\frac{1}{4}\frac{\partial F}{\partial x}+\left(x^2-\frac{1}{2x}+\frac{y}{4x^2}\right)F\right]+\frac{\partial}{\partial y}\left[\frac{1}{4}\frac{\partial F}{\partial y}+\frac{1}{2}\frac{\partial F}{\partial x}+\left(4xy-\frac{1}{2x}-\frac{\langle z\rangle}{4x^2}\right)F\right]=0, \tag{A.1}$$

$$x=\beta/\beta_s,\quad y=-\beta^{(4)}/\beta_s,\quad z=\beta^{(6)}/\beta_s.$$

We shall seek the function $F$ in the form

$$F(x,y)=W(x)\delta(y-ax). \tag{A.2}$$

On substituting (A.2) in (A.1) and then integrating the equation over $y$, we determine the form of the function

$$W(x)=x^{2-a}e^{-4/3x^3}. \tag{A.3}$$

When $\beta\gg\beta_s(x\gg 1)$, equation (A.2) can be rewritten in the form

$$\frac{\partial}{\partial x}\left[\frac{1}{4}\frac{\partial F}{\partial x}+x^2F\right]+\frac{\partial}{\partial y}\left[\frac{1}{4}\frac{\partial F}{\partial y}+\frac{1}{2}\frac{\partial F}{\partial x}+4xyF\right]=0. \tag{A.4}$$

On substituting (A.3) and (A.4) in (A.1) and retaining the terms that are important when $x\to\infty$, we get

$$W(x)\left[(4xy+x^2(a-2))\delta'(y-ax)+4x\delta(y-ax)\right]=0. \tag{A.5}$$

It is easily seen that when $a=2/5$, the left side of (A.5) reduces to

$$4xW(x)\frac{\partial}{\partial y}\left[(y-{}^2/_5x)\delta(y-{}^2/_5x)\right],$$

which is equal to zero in consequence of the equality $x\delta(x)=0$. Thus the substitution (A.2) with $a=2/5$ in fact gives the solution of (A.1) in the limit $x\to\infty$; the distribution function $W(x)$ has the form

$$W(x)=x^{8/5}e^{-4/3x^3},\quad x\gg 1. \tag{A.6}$$

To avoid misunderstanding, we note that the $\delta$ function in (A.2) corresponds simply to the fact that the characteristic spread of the difference $(y-\frac{2}{5}x)$ is much smaller than $x$ for $x\to\infty$; and in fact it can be shown that

$$x^{-1}\langle(y-{}^2/_5x)^2\rangle)^{1/2}\sim x^{-3}.$$

We shall now show that when $\beta\gg\beta_s$, the relation $\langle(\beta^{(3)})^2\rangle\ll\beta^2$ is satisfied. For this purpose, we consider the Fokker-Planck equation for the distribution function $Q(x,z)(x=\beta/\beta_s,z=\beta^{(3)}/\beta_s)$, integrated over all the other $\beta^{(n)}$. With the aid of (7), retaining the terms important when $x\gg 1$, we get

$$\frac{\partial}{\partial x}\left[\frac{1}{4}\frac{\partial Q}{\partial x}+x^2Q\right]+\frac{\partial}{\partial z}\left[\frac{1}{4}\frac{\partial Q}{\partial z}+3xzQ\right]=0. \tag{A.7}$$

The important difference of this equation from (A.1) consists in the absence of a term containing the mixed second derivative $Q''_{xz}$; this leads to a substantially different asymptotic solution:

$$Q(x,z)\sim W(x)e^{-5z^2x}. \tag{A.8}$$

As is evident from (A.8),

$$\langle z^2\rangle|_x\sim x^{-1}\ll x^2.$$

It is clear that this is true also for all the other odd derivatives $\beta^{(2k+1)}$.

## APPENDIX B

To determine the asymptotic behavior of $W_0(\beta)$ when $\beta \ll \beta_s = (JV^2)^{1/3}$, it is convenient to use an expansion of the function $\varepsilon(\varphi)$ as a Fourier series:

$$\varepsilon(\varphi) = -\sum_{m=1}^{\infty} a_m \cos((\varphi-\delta_m)m). \tag{B.1}$$

It is natural to suppose that when $\beta \sim |\varepsilon(\varphi)| \ll \beta_s$, the coupling of different harmonics produced by the nonlinear term $J^{-1}(\partial\varepsilon/\partial\varphi)^2$ in (3) will be small. If we generally discard all harmonics with $m \geq 2$, then, as was shown earlier,[1] the equation for the distribution function $W_1(a_1)$ will have the form

$$\frac{\partial}{\partial a_1}\left(\frac{\partial W_1}{\partial a_1} - \frac{W_1}{a_1}\right) = 0. \tag{B.2}$$

This equation has two solutions:

$$W_1 \sim a_1; \quad W_1 \sim a_1 \ln(1/a_1).$$

In order to explain which of these solutions is realized, we must allow for the second harmonic of the function $\varepsilon(\varphi)$. Substituting in (3)

$$\varepsilon_N(\varphi) = -a_{1N}\cos(\varphi-\delta_{1,N}) + a_{2N}\cos 2(\varphi-\delta_{2,N}),$$

we find the amplitudes $a_1$ and $a_2$ and the phases $\delta_1$ and $\delta_2$ in the next step:

$$\begin{aligned}
a_{1,N+1} &= a_{1,N}\cos(\delta_{1,N}-\delta_{1,N+1}) + V\cos(\delta_{1N+1}-\alpha_{N+1}) \\
&\quad - \frac{1}{J} a_{1,N}a_{2,N}\cos(\delta_{1,N}+\delta_{1,N+1}-2\delta_{2,N}), \\
a_{2,N+1} &= a_{2,N}\cos 2(\delta_{2,N}-\delta_{2,N+1}) + \frac{1}{4J} a_{1,N}^2 \cos 2(\delta_{1,N}-\delta_{2,N}), \\
\delta_{1,N+1} &= \delta_{1,N} - \frac{V}{a_{1,n}}\sin(\delta_{1,N}-\alpha_{N+1}) + \frac{1}{J} a_{2,N}\sin 2(\delta_{1,N}-\delta_{2,N}), \\
\delta_{2,N+1} &= \delta_{2,N} + \frac{1}{J}\frac{a_{1,N}}{8a_{2,N}}\sin 2(\delta_{1,N}-\delta_{2,N}).
\end{aligned} \tag{B.3}$$

From (B.3) it is easy to obtain the corresponding Fokker-Planck equation, which in the dimensionless variables

$$x = a_1/\beta_s, \quad \varepsilon = 2(\delta_1-\delta_2), \quad u = a_2/\beta_s$$

has the form

$$\frac{\partial}{\partial x}\left[\frac{1}{4}\frac{\partial W_3}{\partial x} - \left(\frac{1}{4x} - xu\cos\varepsilon\right)W\right] + \frac{\partial}{\partial u}\left[\left(-\frac{x^2}{4}\cos\varepsilon\right)W_3\right] + \frac{\partial}{\partial\varepsilon}\left[\frac{1}{x^2}\frac{\partial W_3}{\partial\varepsilon} + \left(\frac{x^2}{4u} - 2u\right)\sin\varepsilon W_3\right] = 0. \tag{B.4}$$

We shall seek a solution of (B.4) for $x \ll 1$ in the self-similar form

$$W_3 = xR(t,\varepsilon)x^{-4} \quad (t = u/x^4).$$

For $R(t,\varepsilon)$ we have

$$4t^2\frac{\partial^2 R}{\partial t^2} + 12t\frac{\partial R}{\partial t} + 4R + \frac{\partial^2 R}{\partial\varepsilon^2} + \frac{1}{4t}\frac{\partial}{\partial\varepsilon}(R\sin\varepsilon) - \frac{1}{4}\frac{\partial R}{\partial t}\cos\varepsilon = 0. \tag{B.5}$$

When $t \gg 1$, the principal term of the asymptotic $R(t,\varepsilon)$ is determined by the first three terms in (B.5) and has the form $1/t$. When $t \ll 1$, the most important terms in (B.5) are the last two, which give $R(t,\varepsilon) \sim t$. It is clear in advance that discarding of all the higher harmonics of $\varepsilon(\varphi)$ is not justified if the amplitude of any of the remaining ones is of the order of or larger than unity (in dimensionless variables). Therefore the behavior $R(t,\varepsilon) \sim 1/t$ occurs when $u \lesssim 1$, i.e., when $t \lesssim x^{-4}$; when $t \gg x^{-4}$, the function $R(t,\varepsilon)$ decreases exponentially. Therefore the distribution function is

$$W_1(x) = \int_0^\infty W_3(x,u,\varepsilon)\,du\,d\varepsilon \approx x\int_0^\infty R(t,\varepsilon)\,dt\,d\varepsilon \sim x\ln x^{-1}, \quad x \ll 1. \tag{B.6}$$

We note here an important fact: integration of equation (B.4) over $u$ and $\varepsilon$ leads to the previous equation (B.2) for $W_1(x)$, since the additional term

$$x\int u\cos\varepsilon W_3(x,u,\varepsilon)\,du\,d\varepsilon$$

is much smaller than the others for $x \ll 1$. Therefore consideration of the second harmonic was necessary only for choice of one of the two solutions of equation (B.2). We note that these same considerations permit immediate discarding of the solution $t^{-1}\ln t^{-1}$ for $R(t,\varepsilon)$ when $t \gg 1$ [which is also allowed by the terms in (B.5) that are important when $t \gg 1$], since then integration in (B.6) would give

$$W_1(x) \sim x\ln^2(1/x),$$

which is not a solution of equation (B.2) and therefore is incorrect.

We shall now, by using the already known properties of $W_3(x,u,\varepsilon)$, find the distribution function $W_0(\beta)$ [previously, only the terms with $m = 1, 2$ were retained in (B.1)]. The value of $\beta$ as a function $\beta(x,u,\varepsilon)$ is given implicitly by the equations

$$\left.\frac{\partial\varepsilon(\varphi)}{\partial\varphi}\right|_{\varphi=\gamma} = \beta_s[x\sin\gamma - 2u\sin(2\gamma+\varepsilon)] = 0, \tag{B.7}$$

$$\beta = \left.\frac{\partial^2\varepsilon(\varphi)}{\partial\varphi^{(2)}}\right|_{\varphi=\gamma} = \beta_s(x\cos\gamma - 4u\cos(2\gamma+\varepsilon)); \tag{B.8}$$

the distribution function $W_0(\beta)$ is given by the expression

$$W_0(\beta) = \iint_0^\infty dx\,du\int_0^{2\pi} d\varepsilon W_3(x,u,\varepsilon)\,\delta(\beta-\beta(x,u,\varepsilon)). \tag{B.9}$$

Elimination of $\gamma$ from (B.7) and (B.8) leads to cumbersome expressions for $\beta(x,u,\varepsilon)$; therefore it is more convenient to introduce under the integral in (B.9) the $\delta$ function from equation (B.7) and an additional integration over $d\gamma$:

$$W_0(\beta) = \iint_0^\infty dx\,du \iint_0^{2\pi} d\varepsilon\,d\gamma W_3(x,u,\varepsilon)\,\delta[\beta - (x\cos\gamma - 4u\cos(2\gamma+\varepsilon))]\cdot\delta(x\sin\gamma - 2u\sin(2\gamma+\varepsilon))\beta\beta_s^{-1}. \tag{B.10}$$

The factor

$$\beta\beta_s^{-1} = x\cos\gamma - 4u\cos(2\gamma+\varepsilon)$$

is the Jacobian corresponding to integration over $d\gamma$. The principal contribution to the integral (B.10) comes from the range $x \ll 1$, $x^4 \ll u \ll 1$, in which

$$W_3(x,u,\varepsilon) \sim x/u. \tag{B.11}$$

Substituting (B.11) in (B.10), we have

$$W_0(\beta) \sim \frac{\beta}{\beta_s^2}\iint d\varepsilon\,d\gamma \times \frac{\theta(\sin\gamma\sin(2\gamma+\varepsilon))\,\theta(\cos\gamma - 2\sin\gamma\,\mathrm{ctg}(2\gamma+\varepsilon))}{|\sin\gamma|\,|\cos\gamma - 2\sin\gamma\,\mathrm{ctg}(2\gamma+\varepsilon)|}. \tag{B.12}$$

The integral (B.12) contains a logarithmic divergence;

actually, the divergence is cut off because of the boundedness of the $x, u$ region within which $W_3(x, u, \varepsilon)$ has the form (B.11). We have finally

$$W_0(\beta) \sim \frac{\beta}{\beta_*^2} \ln \frac{\beta_s}{\beta}. \tag{B.13}$$

The function $W_0(\beta)$ has the same form as does the distribution function of the amplitude of the first harmonic, $W_1(x)$; it is easily shown that allowance for higher ($m \geqslant 3$) harmonics also does not change the result (B.13).

[1)]The result obtained earlier,[1] $W(\beta) \sim \beta^{\eta}$, $\eta = 1 - 100a^2/3$, where $a$ is the ratio of the amplitudes of the second and first harmonics of the potential energy, is in error. The properties of systems with a random potential containing several harmonics will be treated separately.


[1]M. V. Feĭgel'man, Zh. Eksp. Teor. Fiz. **79**, 1095 (1980) [Sov. Phys. JETP **52**, 555 (1980)].

[2]L. P. Gor'kov, Pis'ma Zh. Eksp. Teor. Fiz. **25**, 384 (1977) [JETP Lett. **25**, 358 (1977)].

[3]P. A. Lee, T. M. Rice, and P. W. Anderson, Solid State Commun. **14**, 703 (1974).

[4]H. Fukuyama and P. A. Lee, Phys. Rev. **B17**, 535 (1978).

[5]H. Fukuyama, J. Phys. Soc. Jpn. **45**, 1474 (1978).

[6]V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. **65**, 1251 (1973) [Sov. Phys. JETP **38**, 620 (1974)].


Translated by W. F. Brown, Jr.