Finite-size vortex lattice in a rotating superfluid liquid

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A vortex lattice consisting of a finite number of vortices is investigated. The displacements of the vortices from the sites of a regular triangular lattice and the shape of its boundary under equilibrium conditions are calculated within the framework of the continuum theory of elasticity.

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The theory of an infinite vortex lattice in rotating superfluid He^4 has been constructed by Tkachenko.[1-3] The existence of such a lattice has been confirmed both by the experimental observation of the waves by Tkachenko,[4-5] and also by direct photographing of the vortices in the rotating vessel.[6]

Recently, theoretical investigations of vortex lattices in bounded volumes—finite vortex crystals—have also appeared. These investigations are carried out both on the basis of the continuum theory of lattice vibrations (elasticity theory) with the addition of the corresponding boundary conditions,[7,8] and with the help of numerical calculations.[9,10] In particular, it has been established by the numerical calculations that distortions of the regular triangular lattice exist at equilibrium. They penetrate deep into the interior of the vortex crystal. According to Campbell and Ziff,[10] displacement of the lattice sites of the triangular lattice increase like ρ^5, where ρ is the distance to the axis of rotation. Close to the boundary of the crystal, the vortices align themselves into concentric rings, so that the boundary of the crystal takes on a regular cylindrical shape even without account of the walls of the vessel.

In the present work we attempt to obtain these results within the framework of the continuum theory. We note first that, because of the lack of correspondence between the symmetry of the cylinder, whose shape the vortex crystal tends to take on at equilibrium, and the hexagonal symmetry of the regular triangular lattice, the crystal shape deviates from cylindrical. Therefore, if we cut out a cylindrical region from the regular lattice, then not all its boundary vortices will be located on one circumference; some of them can be located at a distance from it of the order of the lattice constant. This leads to the appearance of the so-called destabilizing vortex velocity $\delta\mathbf{v}_s$, i.e., to a deviation from the velocity of rigid-body rotation $\mathbf{v}_{s0}=\mathbf{\Omega}\times\boldsymbol{\rho}$. According to the numerical calculations of Ref. 10,

$$\delta\mathbf{v}_s\sim\rho^5.$$

The destabilizing velocity makes the regular lattice unstable and deformed.

In addition, we shall consider a system of a large but finite number of vortices N in a rotating infinite superfluid liquid (the effect of the walls of the container is not taken into account).[1)] We first replace the discrete vortices by a continuous velocity field and neglect the rigidity of the lattice. We have

$$\int \operatorname{rot}\mathbf{v}_s d^2\rho=\varkappa N,\quad \varkappa=\frac{h}{m}.$$

After minimization of the energy,

$$E=\rho_s\int\left\{\frac{v_s^2}{2}-\Omega[\rho\mathbf{v}_s]\right\}d^2\rho,$$

(where ρ_s is the density, $\mathbf{v}_s$ is the velocity of the superfluid component, $\mathbf{\Omega}$ is the frequency of rotation of the vessel) we find in the rotating system of coordinates, that curl $\mathbf{v}_s=2\mathbf{\Omega}$ in a cylinder of radius $R_0=(\varkappa N/2\pi\Omega)^{1/2}$ that is coaxial with the axis of rotation, and curl $\mathbf{v}_s=0$ outside this cylinder.

In order to take into account the effect of the faceting in the given approximation, it is necessary to introduce on the boundary of the crystal a displacement of the order of the lattice constant b and directed normal to its boundary (tangential displacements do not affect the shape of the crystal),

$$\delta u_\rho^0=\delta u^0\cos n\vartheta,\quad \delta u^0\sim b, \tag{1}$$

where ϑ is the angle of rotation in the cylindrical system of coordinates. In the limit as $b\to 0$, we have the following for the destabilizing velocity field:

$$\operatorname{rot}\delta\mathbf{v}_s=2\Omega\delta(\rho-R_0)\delta u^0\cos n\vartheta. \tag{2}$$

In Eqs. (1) and (2), n is determined by the symmetry of the regular lattice, $n=6, 12, 18$ and so on. Further, only the fundamental frequency is taken into account. According to (2), curl $\delta\mathbf{v}_s$ is equal to zero everywhere except at the boundaries, which corresponds to the condition of incompressibility of the vortex crystal (see below). It follows from the condition of incompressibility of the liquid itself that div $\delta\mathbf{v}_s$ also vanishes. The following destabilizing velocity field satisfies these conditions in the cylindrical system of coordinates:

$$\delta v_{s\rho}=-\Omega\delta u^0\sin n\vartheta\left\{\theta(R_0-\rho)\left(\frac{\rho}{R_0}\right)^{n-1}+\theta(\rho-R_0)\left(\frac{R_0}{\rho}\right)^{n+1}\right\},$$

$$\delta v_{s\vartheta}=-\Omega\delta u^0\cos n\vartheta\left\{\theta(R_0-\rho)\left(\frac{\rho}{R_0}\right)^{n-1}-\theta(\rho-R_0)\left(\frac{R_0}{\rho}\right)^{n+1}\right\}, \tag{3}$$

$$\theta(x)=\begin{cases}1, & x>0\\ 0, & x<0\end{cases}.$$

Such a field increases the energy of the liquid by an amount

$$\delta E(n,\ \delta u^0)=\pi\rho_s\Omega^2R_0^2(\delta u^0)^2/n. \tag{4}$$

Attention must be paid to the fact that the energy of the boundary distortion is proportional in the two-dimensional case not to its "area" (i.e., R_0), as in an ordinary crystal, but to the "volume" R_0^2. This is due to the long-range interaction of the vortices with one another.

The destabilizing velocity $\delta\mathbf{v}_s$, which determines the deviation from rigid-body rotation, makes the state

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unstable. Absence of motion in the rotating system of coordinates always corresponds to minimum energy. Obviously, the transition to the state with minimum energy will be accompanied by a decrease in the energy of distortion of the cylindrical shape of the boundary due to displacements of the vortices in the volume, so that the elastic energy of the crystal is increased.

The hexagonal symmetry of the triangular lattice allows us to describe the vortex crystal in terms of an isotropic solid.[13] Moreover, it must be taken into account that only transverse deformations can exist in a vortex crystal in the stationary state.[3] Longitudinal deformations always lead to motion with a finite frequency.[7,8] Therefore, the theory of the elasticity of a vortex crystal is obtained from the general theory after transition to infinite modulus of uniform compression.[13] The elastic energy is determined by the single transverse shear modulus G:

$$E_{\text{elast}}=G\int u_{ik}^2 d^2\rho,\quad G=\rho_s c_T^2=\frac{\rho_s \varkappa\Omega}{8\pi},\tag{5}$$

where c_T is the velocity of the Tkachenko waves and u_{ik} is the transverse deformation tensor connected with the displacement $\mathbf{u}=\operatorname{curl}\boldsymbol{\psi}$ in the usual fashion.

The potential $\boldsymbol{\psi}=\psi\mathbf{e}_z$ ($\mathbf{e}_z$ is a unit vector along the axis of rotation) should the biharmonic equation $\Delta\Delta\psi=0$. Its solution, which gives the displacements the boundary that depend also on ϑ, just as the unrenormalized displacements (1), is

$$\begin{aligned} u_\rho&=(An\rho^{n-1}+Bn\rho^{n+1})\cos n\vartheta,\\ u_\vartheta&=-(An\rho^{n-1}+B(n+2)\rho^{n+1})\sin n\vartheta.\end{aligned}\tag{6}$$

The constants A and B are determined from the condition that the sum of the energy δ_E of distortion of the shape of the vortex crystal and the energy of elastic deformations be a minimum. Here we must recognize that the resulting displacements on the boundary are composed of the unrenormalized displacement $\delta^0_{u_\rho}$ and the elastic displacements $u_0(R_0,\theta)$. Therefore, we must replace δ^0_u in formulas (3) and (4) by the renormalized amplitude of the displacements

$$\delta\tilde{u}=\delta u^0+AnR_0^{n-1}+BnR_0^{n+1}.$$

The minimum of the expression $\delta E(n,\delta\tilde{u})+E_{\text{elast}}$ gives the values of the constants:

$$A=-\frac{\delta u^0(n+1)}{2(1+\alpha)R_0^{n-1}n},\quad B=\frac{\delta u^0(n-1)}{2(1+\alpha)R_0^{n+1}n},\quad \alpha=\frac{\varkappa(n^2-1)}{8\pi\Omega R_0^2}.\tag{7}$$

It can be verified that the vortices in the ground state of the obtained finite vortex crystal are immobile in a rotating system of coordinates. For purely transverse displacements of an infinite lattice, the velocity of the vortices in the rotating system of coordinates is determined by the expression[7,8]

$$\frac{d\mathbf{u}}{dt}=\frac{\varkappa}{16\pi}[\Delta\mathbf{e}_z\times\mathbf{u}].$$

This velocity cancels exactly the destabilizing velocity $\delta\mathbf{v}_s(n,\delta\tilde{u})$. As a result, we find that inside the finite vortex crystal the elastic displacements of the sites of the regular lattice increase like $(\rho/R_0)^5$, in complete agreement with the results of the numerical experiment of Ref. 10. Such a dependence is valid not too close to the boundary of the crystal, since close to it we would need to take into account the terms $(\rho/R_0)^7$ and subsequent harmonics with $n=12, 18$ and so on. The experimental results[10] provide the possibility of estimating δ^0_u/b. For a vortex region consisting of 217 vortices this ratio is equal to 0.35 ± 0.15.

Thus, the assumption that unrenormalized displacements of the order of the lattice constant arise on the boundary of the vortex crystal is verified. For the harmonics, that can be considered within the framework of the continuum theory ($n\ll R_0/b$), the total normal displacement on the crystal boundary

$$\delta\tilde{u}\cos n\vartheta=\frac{\alpha}{1+\alpha}\delta u^0\cos n\vartheta$$

tends to zero, since $\alpha\to0$ for large R_0. Such a behavior indicates a tendency of the boundary of the finite vortex crystal to take the shape of a cylinder coaxial with the rotation axis. This agrees with the alignment of vortices on the boundary of the crystal along the circumference at equal distances from one another, as observed in numerical experiments. However, the exact determination of the location of the vortices on the boundary requires a calculation of the displacement harmonics with very high $n\sim R_0/b$ which cannot be done within the framework of the continuum theory.

1) There is an irrotational region in the rotating vessel near the walls[11] and reduces their effect. In addition, metastable states can develop in any change in the rotational velocity of the vessel, when the number of vortices formed is less than equilibrium.[12] In this case, they always congregate around the axis of the vessel so that their density corresponds to its rotational velocity. The size of the irrotational region increase in this case and the effect of the walls decreases.

1V. K. Tkachenko, Zh. Eksp. Teor. Fiz. **49**, 1875 (1965); **50**, 1574 (1966); **56**, 1763 (1969) [Sov. Phys. JETP **22**, 1282 (1966); **23**, 1049 (1966); **29**, 974 (1969)].

2V. K. Tkachenko, Pis'ma Zh. Eksp. Teor. Fiz. **17**, 617 (1973) [JETP Lett. **17**, 434 (1973)].

3V. K. Tkachenko, Zh. Eksp. Teor. Fiz. **67**, 1984 (1974) [Sov. Phys. JETP **40**, 985 (1975)].

4S. D. Takadze, Fiz. Nizk., Temp. **4**, 148 (1980) [Sov. J. Low Temp. Phys. **4**, 72 (1980)]; J. S. Tsakadze, S. J. Tsakadze and E. B. Sonin, Phys. Rev. **B21**, 3028 (1980).

5C. D. Andereck, J. Chalupa and W. I. Glauberson, Phys. Rev. Lett. **44**, 33 (1980).

6M. J. Gordon, C. A. Williams, and R. E. Packard, J. de Phys. **39**, C6-172; E. J. Yarmchuk, M. J. V. Gordon and R. E. Packard, Phys. Rev. Lett. **43**, 214 (1979).

7E. B. Sonin, Zh. Eksp. Teor. Fiz. **70**, 1970 (1976) [Sov. Phys. JETP **43**, 1027 (1976)].

8M. R. Williams and A. L. Fetter, Phys. Rev. **B16**, 4846 (1977).

9L. V. Kiknadze and Yu. G. Mamaladze, Fiz. Nizk. Temp. **12**, 1501 (1976) [Sov. J. Low Temp. **12**, 731 (1976)].

10L. J. Campbell and R. M. Ziff, Phys. Rev. **B20**, 5, 1886 (1979).

11I. M. Khalatnikov, Teoriya sverkhtekuchesti (Theory of Superfluidity) Moscow, Nauka, 1971. Sec. 31.

12Zh. S. Tsakadze and S. Zh. Tsakadze., Usp. Fiz. Nauk **115**, 503 (1975) [Sov. Phys. Uspekhi **18**, 242 (1975)].

13L. D. Landau and E. M. Lifshitz, Teoriya uprugosti (Theory of Elasticity) Moscow, Nauka, 1965, Chap. 4. (English translation, Pergamon Press, Oxford 1970).

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