Nonlinear theory of the "kinetic" excitation of waves

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A nonlinear theory of the kinetic instability (KI) that arises in a highly nonequilibrium system, and in the course of which the wave attentuation is negative in some region of **k** space because of the state of nonequilibrium of the system, is constructed. The amplitude of the secondary waves (SeW) increases in this region. The effect of the three- and four-wave scattering on the limitation of the KI is investigated in the approximation of a prescribed degree of nonequilibrium of the system. If the three-wave interaction processes involving the SeW are allowed, then the dominant mechanism underlying the limitation of the SeW amplitude is the growth of the total decrement of the SeW as the number of these waves increases, the negative contribution to the decrement being compensated for by this growth. If the three-wave processes are forbidden, then the limitation of the KI is due to the four-wave processes, and the amplitude of the SeW is then limited at a higher level. The shape of the SeW packet is determined.

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INTRODUCTION

In a recent work the present authors, Melkov, and Lavrinenko[1] theoretically predicted and experimentally detected the kinetic instability of the highly nonequilibrium state of parametrically excited spin waves (PSW) in a ferrite. This instability is caused by the decrease to zero of the decrement of the spin waves (SW) not connected with the parametric pump. A negative contribution to the decrement arises as a result of the four-magnon SW-scattering processes in which the highly nonequilibrium PSW participate. Beyond the kinetic-instability threshold, the occupation numbers of the "secondary" spin waves (SeSW) increase exponentially in time, and there arise the problems of finding 1) the nonlinear mechanisms that limit the development of the kinetic instability and 2) the steady state of the PSW system in a situation in which a particular limiting mechanism predominates.

It should be said that the decrement of the waves may turn out to be negative not only in the ferrites, but also in antiferromagnets located in a strong microwave field, in a two-stream plasma under conditions of beam instability, and in a number of other cases in which the medium is in a state far from thermodynamic equilibrium, Thus, the following general statement of the problem arises: To the kinetic equation describing the evolution of the wave occupation numbers $n_{\mathbf{k}}$ must be added the negative decrement, $\Gamma_{\mathbf{k}}$, arising as a result of the state of nonequilibrium of the medium:

$$dn_{\mathbf{k}}/2dt=\Gamma_{\mathbf{k}}n_{\mathbf{k}}+J_{col}\{n_{\mathbf{k}}\}. \qquad (1)$$

When the departure from equilibrium is appreciable, the quantity $\Gamma_{\mathbf{k}}$ can exceed the decrement $\gamma^0_{\mathbf{k}}$ of the waves at equilibrium for some **k**. In the instability region, where $\gamma^0_{\mathbf{k}}-\Gamma_{\mathbf{k}}<0$, the occupation numbers $n_{\mathbf{k}}$ increase rapidly, and it is necessary to find the new steady state for $t\to\infty$, the rate of relaxation to it, etc.

It should be said that the expression $\Gamma_{\mathbf{k}}n_{\mathbf{k}}$ for the source in Eq. (1) outwardly resembles the expression for the parametric pump: $hV_{\mathbf{k}}\sin(\varphi_{\mathbf{k}}+\varphi_{-\mathbf{k}}-\varphi_p)n_{\mathbf{k}}$, where the $\varphi_{\pm\mathbf{k}}$ are the phases of the waves with vectors $\pm\mathbf{k}$ and φ_p is the phase of the pump. In both cases the power of the source is proportional to the number $n_{\mathbf{k}}$ of available particles. But in the parametric excitation of waves the principal role is played by the phase relations in the $\pm\mathbf{k}$ pairs. First, they lead to a situation in which the parmetric waves are excited in a narrow, pump-frequency (ω_p) defined region in the vicinity of the resonance surface: $\omega_{\mathbf{k}}+\omega_{-\mathbf{k}}=\omega_p$. Secondly, they give rise to quite an effective "phase" mechanism for limiting the number of parametric waves, which mechamism is thoroughly analyzed in Ref. 2. In the "kinetic"-instability problem (1) considered here, the phase relations in the system of waves are absolutely unimportant, and we must study other, weaker mechanisms that lead to the limitation of the number of waves.

Bearing in mind the experimental situation concerning the magnetic dielectrics, we shall in the present paper be interested in the limiting case in which the supercriticality is not very high, i.e., in which $|\Gamma_{\mathbf{k}}-\gamma^0_{\mathbf{k}}|\lesssim\gamma^0_{\mathbf{k}}$, and the deviations of the $n_{\mathbf{k}}$ from the thermodynamic-equilibrium values of $n^0_{\mathbf{k}}$ are large only in the vicinity of the instability region. Then, depending on the specific nature of the problem, one of the following different cases can arise: a) $\Gamma_{\mathbf{k}}-\gamma^0_{\mathbf{k}}$ first vanishes at a pair of points $\pm\mathbf{k}_0$; b) in the case of axial symmetry the vanishing occurs on the circle $k_z=0$, $|\mathbf{k}_\perp|=k_0$ or on the pair of circles: $k_z=\pm k^0_z$, $|k_\perp|=k_0$; c) in the case of spherical symmetry this occurs on the sphere $|\mathbf{k}|=k_0$. More complicated is the $k=0$ case, in which the secondary-wave distribution is not universal for a given symmetry of the problem. At very high supercriticalities (i.e., at $\Gamma_{\mathbf{k}}\gg\gamma^0_{\mathbf{k}}$), there arises an energy flux along the spectrum from the pump region to the region of effective attenuation, and the problem, as a rule, reduces to the problem of describing the scaling-invariant Kolmogorov spectrum of a weak wave turbulence.[3]

In §1 of the present paper we derive the kinetic equation for the waves, and compute the kinetic-instability threshold. A quite nontrivial answer (i.e., one for which $\Gamma_{\mathbf{k}_0}\neq\gamma^0_{\mathbf{k}_0}$ at the threshold) is obtained in the case in which the elastic scattering of the waves on the static inhomogeneities is important: see the formulas (1.21), (1.23), and (1.25).

In §2 we formulate for the "secondary"-wave (SeW) occupation numbers $N_{\mathbf{k}}=n_{\mathbf{k}}-n_{\mathbf{k}}^0$ nonlinear equations that take into account the fact that the $N_{\mathbf{k}}$ are different from zero only in the vicinity of the instability region, and presuppose that $\int N_{\mathbf{k}}d^3\mathbf{k}$ is small compared to $\int n_{\mathbf{k}}^0 d^3\mathbf{k}$ over the k-space region that makes a significant contribution to the wave attenuation.

In §3 we briefly consider the limitation of the kinetic instability as a result of the increase of the decrement of the secondary waves as their number increases:

$$\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\}>\gamma_{\mathbf{k}}\{0\}=\gamma_{\mathbf{k}}^0. \tag{2}$$

An important characteristic of the mechanism underlying the limitation of the nonlinear attenuation of the waves is the singularity of the SeW distribution function $N_{\mathbf{k}}$: if we neglect the thermal fluctuations, then $N_{\mathbf{k}}$ will be nonzero only at some points of k space or on some lines or surfaces (depending on the symmetry of the problem) whose positions are determined from the stability condition

$$\frac{\partial}{\partial\mathbf{k}}(-\Gamma_{\mathbf{k}}+\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\})=0. \tag{3}$$

The steady-state SeW distribution function $N_{\mathbf{k}}$ is then determined from the balance condition

$$(\Gamma_{\mathbf{k}}-\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\})N_{\mathbf{k}}=0. \tag{4}$$

A nontrivial situation arises in the theory when it becomes necessary to take the thermal fluctuations into account. Then there arise in the case of weak nonlinear attenuation corrections to the formula (4) that depend on the symmetry of the problem [see (3.15) and (3.19)]. If, on the other hand, the nonlinear attenuation is strong, then the SeW distribution undergoes a cardinal reconstruction, and the formula (4) is not even approximately valid. Instead of it, see (3.12), where $N\propto\sqrt{\Gamma}$.

As the supercriticalities rise, the SeW numbers $N_{\mathbf{k}}$ increase, so that not only the interaction of the SeW with the thermal waves, but also their collision with each other, becomes important. The four-wave processes,

$$\omega_{\mathbf{k}_1}+\omega_{\mathbf{k}_2}=\omega_{\mathbf{k}_3}+\omega_{\mathbf{k}_4},\quad \mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4, \tag{5}$$

in which only the SeW participate do not remove energy from the SeW system, and therefore cannot, one would think, limit the kinetic instability. But the SeW-scattering processes (5) lead to the broadening of the SeW-distribution function $N_{\mathbf{k}}$ beyond the limits of the instability region, and as the supercriticality increases, the width of the distribution function $N_{\mathbf{k}}$ increases in such a way that the net energy balance is satisfied:

$$\int(\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\}-\Gamma_{\mathbf{k}})N_{\mathbf{k}}\,d^3\mathbf{k}=0. \tag{6}$$

This condition determines the dependence of the width of the function $N_{\mathbf{k}}$ on the supercriticality: the total number should be such that the processes (5) ensure the required width of $N_{\mathbf{k}}$.

In §4 we present a quantitative theory of the above-described "collision" mechanism of kinetic-instability limitation. The simplest case is the one in which the problem has spherical symmetry, and $\gamma_{\mathbf{k}}^0-\Gamma_{\mathbf{k}}$ first vanishes on the sphere of radius k_0. In this case

$$4\pi k^2N_k=\frac{N/\pi\varkappa_0}{\operatorname{ch}[\pi(k-k_0)/2\varkappa_0]}. \tag{7}$$

Here N is the total number of SeW and $\varkappa_0$ is the characteristic width of the k-modulus distribution:

$$N=\int N_{\mathbf{k}}\,d^3\mathbf{k}=\frac{1}{|T|}\left(\frac{\pi k_0 v\Gamma_{eff}}{2}\right)^{1/2}, \tag{8}$$

$$\varkappa_0=(2\Gamma_{eff}/\gamma_{kk}'')^{1/2}, \tag{9}$$

where T is the mean value of the matrix element of the four-wave interaction describing the processes (5) of SeW scattering on the sphere of radius k_0,

$$\gamma_{kk}''=(\partial^2\gamma_k/\partial k^2)_{k=k_0},\quad v=(\partial\omega_{\mathbf{k}}/\partial\mathbf{k})_{k=k_0} \tag{10}$$

is the group velocity of the SeW, and

$$\Gamma_{eff}=(\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\}-\Gamma_{\mathbf{k}})_{k=k_0}. \tag{11}$$

We are not able to solve exactly the obtained integral equations (4.18) and (4.23) for $N_{\mathbf{k}}$ in the lower-symmetry cases in which $\gamma_{\mathbf{k}}^0-\Gamma_{\mathbf{k}}$ vanishes on a line (axial symmetry) or at a pair of points $\pm\mathbf{k}_0$. Nevertheless, we determine the asymptotic shape of the SeW packet, and derive interpolation formulas, (4.20) and (4.24), that approximately describe the SeW distribution in the entire $\mathbf{k}$ space.

If the kinetic-instability threshold is attained at the point $k_0=0$, or if the value of k_0 is so small that the width $\varkappa_0$ of the packet at some supercriticality is comparable to, or exceeds, k_0, then the SeW distribution becomes nonsymmetric with respect to $k-k_0$. In this case the shape of the SeW packet is not universal: it depends only on the symmetry of the problem, and to determine it we must know the dispersion law for the waves and the dependence of $\gamma_{\mathbf{k}}-\Gamma_{\mathbf{k}}$ on the wave vector in the region of small k. In Subsec. 4.4 we derive an approximate expression for the distribution of SeW with the dispersion law

$$\omega_k=\omega_0+\alpha k^2 \tag{12}$$

under conditions when γ_k is a linear function of k:

$$\gamma_k=\gamma_0+\gamma_k'k \tag{13}$$

(the isotropic-ferromagnetic model). A characteristic feature of this distribution is the exponential decrease of $N_{\mathbf{k}}$ with increasing frequency:

$$N_k=\frac{N}{2(\pi^{1/2}\varkappa_1)^3}\exp\left[\frac{-(\omega_k-\omega_0)}{\alpha\varkappa_1^2}\right],\qquad \varkappa_1=\frac{\pi^{1/2}\Gamma_{eff}}{\gamma_k'}. \tag{14}$$

The experimentally studied situation—the kinetic excitation of SW under the action of PSW—differs from the above-discussed cases in two important characteristics: first, the SW spectrum is anisotropic, and therefore the SeSW can interact with each other inside a narrow cone[1]; secondly, the elastic scattering of the SeSW by the PSW is not weak. In §5 we take these facts into account within the framework of the following crude model: the surface $\omega_{\mathbf{k}}$=const for the secondary waves is the two portions cut out of a sphere by a cone of angle of taper θ_m ($\theta_m\ll 1$) and the elastic scattering is of the greatest possible intensity. As a result, the SeSW turn out to be isotropically distributed inside the cone $\theta \leq \theta_m$. The distribution of the SeSW over the absolute k values coincides with the expression (7), which was obtained in the fully isotropic case, but the number of SeSW turns out to be higher than (8) by a factor of θ_m^{-1}.

§1. THE KINETIC INSTABILITY

1.1 The choice of the collision term. As indicated in the Introduction, we shall, in constructing the theory, proceed from the kinetic equation (1), in which the collision term J_{col} consists, generally speaking, of terms describing the two-, three-, and four-wave processes:

$$J_{col}\{n_{\mathbf{k}}\}=J_{col}^{(2)}+J_{col}^{(3)}+J_{col}^{(4)}. \tag{1.1}$$

The two-wave processes are the processes of elastic scattering of the waves by the random static inhomogeneities. In the case of point defects having a concentration c (Ref. 4)

$$J_{col}^{(2)}=\pi c\int|g_{\mathbf{kk}'}|^2(n_{\mathbf{k}'}-n_{\mathbf{k}})\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})d^3\mathbf{k}'. \tag{1.2}$$

If the kinetic instability of the waves is due to their interaction with the parametric waves having the same frequency, then the elastic scattering of the SeW arises as a result of the four-wave processes, when "before" and "after" were with respect to one parametric wave. The corresponding expression for $J_{col}^{(2)}$ will be derived below in Subsec. 1.4.

The inelastic three-wave processes, if they are allowed by the energy and momentum conservation laws, make, as a rule, the main contribution to the thermodynamic-equilibrium establishment process and, consequently, to the logarithmic decrement $\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\}$ of the waves:

$$J_{col}^{(3)}=-\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\}N_{\mathbf{k}}+\pi\Phi_{\mathbf{k}}. \tag{1.3}$$

Below a significantly different role will be played in the theory by the three-wave processes of SeW decay:

$$\omega_{\mathbf{k}}=\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'} \tag{1.4}$$

and combination of a SeW with another SeW or a thermal wave:

$$\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}=\omega_{\mathbf{k}+\mathbf{k}'}. \tag{1.5}$$

The expressions for the departure and arrival terms, $\gamma_{\mathbf{k}}n_{\mathbf{k}}$ and $\pi\Phi_{\mathbf{k}}$ respectively, of the kinetic equation have the standard form, and we give them primarily in order to introduce the notation. In the processes (1.4)

$$\gamma_{3d}=\pi\int|V_{\mathbf{k},12}|^2(n_1+n_2+1)\delta(\omega_{\mathbf{k}}-\omega_1-\omega_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)d\mathbf{k}_1d\mathbf{k}_2. \tag{1.6}$$

In the processes (1.5)

$$\gamma_{3c}=2\pi\int|V_{1,\mathbf{k}2}|^2(n_2-n_1)\delta(\omega_1-\omega_{\mathbf{k}}-\omega_2)\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)d\mathbf{k}_1d\mathbf{k}_2, \tag{1.7}$$

so that

$$\gamma_3(\mathbf{k})=\gamma_{3d}(\mathbf{k})+\gamma_{3c}(\mathbf{k}), \tag{1.8}$$

$$\Phi_3(\mathbf{k})=\Phi_{3d}(\mathbf{k})+\Phi_{3c}(\mathbf{k}), \tag{1.9}$$

$$\Phi_{3d}=\int|V_{\mathbf{k},12}|^2n_1n_2\delta(\omega_{\mathbf{k}}-\omega_1-\omega_2)\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)d\mathbf{k}_1d\mathbf{k}_2, \tag{1.10}$$

$$\Phi_{3c}=\int|V_{1,\mathbf{k}2}|^2n_1n_2\delta(\omega_1-\omega_{\mathbf{k}}-\omega_2)\delta(\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2)d\mathbf{k}_1d\mathbf{k}_2. \tag{1.11}$$

The four-wave processes are also important in our problem. This is because, first, the three-wave processes may be forbidden, or their matrix elements may turn out to be anomalously small, as, for example, in the case of spin waves in ferro- and antiferromagnets. Secondly, and this is the main thing, it is precisely the four-wave scattering processes (5) in which only the SeW participate that lead to the collision mechanism, of interest to us here, of kinetic-instability limitation. As we shall see in §4, the intensity of these processes will be sharply increased because of the narrowness of the SeW packet. As is well known, in the scattering processes (5)

$$\gamma_{4s}=2\pi\int|T_{\mathbf{k}1,23}|^2[n_1(n_2+n_3)-n_2n_3]\delta(\omega_{\mathbf{k}}+\omega_1-\omega_2-\omega_3)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3, \tag{1.12}$$

$$\Phi_{4s}=2\int|T_{\mathbf{k}1,23}|^2n_1n_2n_3\delta(\omega_{\mathbf{k}}+\omega_1-\omega_2-\omega_3)\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3. \tag{1.13}$$

1.2. The kinetic-instability (KI) threshold. If elastic scattering of the SeW does not occur, and $J_{col}^{(2)}=0$, then the kinetic-instability threshold is determined from the condition for the vanishing of the total decrement, namely, from the equation

$$\Gamma_{\mathbf{k}}=\gamma_{\mathbf{k}}^0,\quad \gamma_{\mathbf{k}}^0=\gamma_3^0+\gamma_4^0, \tag{1.14}$$

where the superscript 0 attached to γ_3 and γ_4 indicates that these quantities are computed in the thermodynamic-equilibrium spectrum.

$$n_{\mathbf{k}}=n_{\mathbf{k}}^0=[\exp(\hbar\omega_{\mathbf{k}}/\Theta)-1]^{-1} \tag{1.15}$$

(Θ is the temperature). If, on the other hand, $J_{col}^{(2)}\neq 0$, then the situation becomes complicated, and the equation for the determination of the KI threshold turns out to be an integral equation:

$$dn_{\mathbf{k}}/dt=(\Gamma_{\mathbf{k}}-\gamma_{\mathbf{k}}^0)n_{\mathbf{k}}-\pi c\int|g_{\mathbf{kk}'}|^2(n_{\mathbf{k}}-n_{\mathbf{k}'})\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})d^3\mathbf{k}'. \tag{1.16}$$

The solution to this equation is singular: $N_{\mathbf{k}}$ is nonzero on a constant-frequency surface:

$$N_{\mathbf{k}}=v_\Omega k_\Omega^{-2}N_\Omega\delta(\omega_{\mathbf{k}}-\omega_{k_0}). \tag{1.17}$$

Here k_Ω is the radius of this surface at the point with the angular coordinate Ω and v_Ω is the group velocity. In the simplest case, in which $g_{\mathbf{kk}'}$ can be assumed to be constant over the surface (1.17), it follows from (1.16) that

$$N_\Omega=\langle N_\Omega\rangle\gamma_{def}(\Omega)/(\gamma_\Omega^0+\gamma_{def}-\Gamma_\Omega). \tag{1.18}$$

Here $\langle\,\rangle$ denotes averaging over the solid angle:

$$\langle f_\Omega\rangle=\frac{1}{4\pi}\int f_\Omega d\Omega, \tag{1.19}$$

$\gamma_{def}(\Omega)$ is the static-inhomogeneity-induced-damping constant of the waves:

$$\gamma_{def}(\Omega)=4\pi^2c|g|^2k_\Omega^2/v_\Omega,\quad \gamma_{def}=\langle\gamma_{def}(\Omega)\rangle. \tag{1.20}$$

The KI threshold is easily determined from (1.18):

$$1=\left\langle\frac{\gamma_{def}(\Omega)}{\gamma_\Omega+\gamma_{def}-\Gamma_\Omega}\right\rangle. \tag{1.21}$$

It is useful to investigate some limiting cases of this formula:

1) In the case of spherical symmetry, in which there is no dependence on Ω, the KI threshold is determined from the condition $\gamma^0=\Gamma$, into which γ_{def} does not enter. The reason for this is simple: in this case the scatter-

ing by the defects does not remove energy from the SeW system.

2) For $\gamma_{def} \gg |\Gamma_\Omega - \gamma_\Omega^0|$, i.e., in the case in which the elastic scattering is intense, it follows from (1.8) and (1.21) that

$$N_\Omega \approx \langle N_\Omega \rangle \gamma_{def}(\Omega)/\gamma_{def}, \tag{1.22}$$

$$\left\langle \frac{\gamma_{def}(\Omega)}{\gamma_{def}}(\gamma_\Omega^0 - \Gamma_\Omega) \right\rangle = \left\langle \frac{\gamma_{def}(\Omega)}{\gamma_{def}^2}(\gamma_\Omega^0 - \Gamma_\Omega)^2 \right\rangle \ll |\Gamma_\Omega - \gamma_\Omega^0|. \tag{1.23}$$

It can be seen that, in the first approximation, the elastic scattering leads to the isotropization of the SeW distribution function N_Ω, with the result that the threshold is given by the condition $\langle\langle \gamma_\Omega^0 \rangle\rangle = \langle\langle \Gamma_\Omega \rangle\rangle$, where $\langle\langle\ \rangle\rangle$ denotes averaging over the angles with the weight $\gamma_{def}(\Omega)/\gamma_{def}$. The right-hand side of (1.23) arises from corrections to N_Ω not written out in (1.22).

3) If the system does not possess spherical symmetry, and γ_{def} is small, then in the first approximation

$$\Gamma_0 = \gamma_0^0 + \gamma_{def}, \tag{1.24}$$

where Γ_0 and γ_0^0 are the minimum values of Γ_Ω and γ_Ω^0 on the surface (1.17). The corrections to (1.24) depend on the symmetry of the problem. For example, in the presence of axial symmetry, when $(\gamma_\Omega^0 - \Gamma_\Omega^0)$ has its minimum value at the equator:

$$\Gamma_0 = \gamma_0 + \gamma_{def} - 2(\pi\gamma_{def})^2/\gamma_{xx}'', \tag{1.25}$$

$$\gamma_{xx}'' = (\partial^2\gamma/\partial x^2)_{x\equiv\cos\theta=0}. \tag{1.26}$$

1.3 The prethreshold heating. In the kinetic equation (1.16), we have neglected the arrival terms from the three- and four-wave processes. In a state close to the state of thermodynamic equilibrium,

$$\pi[\Phi_3(\mathbf{k}) + \Phi_4(\mathbf{k})] = [\gamma_3(\mathbf{k}) + \gamma_4(\mathbf{k})]n_\mathbf{k}. \tag{1.27}$$

Naturally, equilibrium here is assumed not for all $\mathbf{k}$, but only in the region of k space which makes the main contribution to the integrals (1.10), (1.11), and (1.13) for Φ_3 and Φ_4. In the particular case in which $J_{col}^{(2)} = 0$, we obtain from (1.16) and (1.27) an expression for the SeW occupation numbers before the onset of the kinetic instability:

$$N_\mathbf{k} = n_\mathbf{k} - n_\mathbf{k}^0 = \Gamma_\mathbf{k} n_\mathbf{k}^0/(\gamma_\mathbf{k} - \Gamma_\mathbf{k}). \tag{1.28}$$

The deviation of $N_\mathbf{k}$ from zero is due to the thermal fluctuations. In the linear—in $N_\mathbf{k}$—approximation (with allowance for $J_{col}^{(2)}$), $N_\mathbf{k}$ becomes infinite at the KI threshold.

1.4. The kinetic instability of spin waves in ferromagnets. This phenomenon has been experimentally observed,[1] and therefore its investigation is of special interest to us. The SW spectrum in a ferromagnet has the form

$$\omega_\mathbf{k} = \left[\left(\omega_0 + \omega_{ex}(ak)^2 + \frac{\omega_M}{2}\sin^2\theta_\mathbf{k}\right)^2 - \frac{\omega_M^2}{4}\sin^4\theta_\mathbf{k}\right]^{1/2}, \tag{1.29}$$

where $\omega_M = 4\pi gM$, g is the gyromagnetic ratio, M is the equilibrium magnetization, a is the lattice constant, $\omega \approx \Theta_c/\hbar$ (Θ_c is the Curie temperature), $\theta_\mathbf{k}$ is the angle between the wave vector $\mathbf{k}$ and the constant magnetic field $\mathbf{H}$, and ω_0 is the gap in the SW spectrum:

$$\omega_0 = g(H - 4\pi N_z M), \tag{1.30}$$

N_z being the demagnetization factor. It was found that there are excited minimally damped SW lying near the bottom of the spectrum. In determining the SeSW wave vector, we should bear in mind that the decrement of the long SW behaves like

$$\gamma_\mathbf{k} = \gamma_0 + \gamma_1(kl)^{-4}, \quad \gamma_1 \approx 2\pi^3 K\omega_M, \tag{1.31}$$

where l is the sample dimension and K is the space factor of the cavity resonator. The correction $\gamma_1(kl)^{-4}$ describes the radiation damping of the SW. As k increases, the damping intensifies as a result of the three-magnon coalescence processes.[3,4] These processes turn out to be allowed for SW with $k \geq k_1$,

$$k_1 \approx \omega_0/2\omega_{ex}a^2k_{max}, \tag{1.32}$$

where k_{max} is the wave vector at the Brillouin-zone boundary. Thus, it is to be expected that the SeSW wave vector k_0 will be close to k_1. In the experimentally studied situation $k_1 \approx 3\times10^3$ cm^{-1}. It is not difficult to determine on the basis of the dispersion law (1.29) the angle of taper, θ_m, of the cone in which SeSW can exist. Indeed, the waves with $\theta = 0$ and wave vector k_0 can be elastically scattered through an angle that is smaller than θ_m, and is given by the condition $\omega_{k_0,0} = \omega_{0,\theta_m}$. From this we obtain

$$\theta_m = (2\omega_{ex}/\omega_M)^{1/2} ak_0 \approx 10^{-2}. \tag{1.33}$$

Since the three-magnon interaction processes involving the SeSW are forbidden by the conservation law, the kinetic equation for these waves has the form[4,5]

$$dn_\mathbf{k}/2dt + \gamma_\mathbf{k}^{(0)}(n_\mathbf{k} - n_\mathbf{k}^0) + 2\pi\int |T_{\mathbf{k}1,23}|^2[n_\mathbf{k}n_1(n_2 + n_3) - n_2n_3(n_\mathbf{k} + n_1)]\delta(\omega_\mathbf{k} + \omega_1 - \omega_2 - \omega_3)\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3. \tag{1.34}$$

Here $\gamma_\mathbf{k}^{(0)}$ is the SW-relaxation rate due to the interaction with the other quasiparticles and $T_{k1,23}$ is the matrix element of the four-magnon interaction. We shall assume that the SW distribution $n_\mathbf{k}$ under conditions of parametric excitation is the totality of the thermodynamic-equilibrium distribution $n_\mathbf{k}^0$ and the narrow PSW packet $n_\mathbf{k}^p$:

$$n_\mathbf{k} = n_\mathbf{k}^0 + n_\mathbf{k}^p, \tag{1.35}$$

$$n_\mathbf{k}^p = v_\Omega k_\Omega^{-2} n_\Omega^p \delta(\omega_\mathbf{k} - \omega_p/2), \tag{1.36}$$

where ω_p is the pump frequency, Ω is the solid angle at the surface $\omega_\mathbf{k} = \omega_p/2$, k_Ω and v_Ω are the radial wave-vector and group-velocity components in the direction Ω. As a result, we obtain the following linearized kinetic equation for the SeSW occupation numbers $N_\mathbf{k}$:

$$dN_\mathbf{k}/2dt + (\gamma_\mathbf{k}^0 - \Gamma_\mathbf{k})N_\mathbf{k} = J_{col}^{(2)} + \Gamma_\mathbf{k} n_\mathbf{k}^0, \tag{1.37}$$

where $\gamma_\mathbf{k}^0$ is the total SW-relaxation rate in the absence of PSW,

$$\Gamma_\mathbf{k} = 2\pi\int |T_{\mathbf{k}1,23}|^2 n_2^p n_3^p \delta(\omega_\mathbf{k} + \omega_1 - \omega_p) \times \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3, \tag{1.38}$$

$$J_{col}^{(2)} = 2\pi\int\{|T_{\mathbf{k}1,23}|^2 n_1^p n_2^p + \mathrm{Re}\, T_{\mathbf{k}1,23}^* T_{\mathbf{k}\bar{1}\bar{1},3}\sigma_1^*\sigma_2\} \times (N_\mathbf{k} - N_3)\delta(\omega_\mathbf{k} - \omega_3)\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3. \tag{1.39}$$

Here $\sigma_\mathbf{k} = \langle a_\mathbf{k} a_{-\mathbf{k}} \rangle$ is the anomalous PSW correlator. The PSW wave vector $\mathbf{k}_p \gg \mathbf{k}_0$, the SeSW wave vector $\mathbf{k}_0$ being oriented along the magnetic field.[1] In this case the effect of the PSW on the SeSW is described by the matrix elements $T_{0,k'+k'',k',k''}$, entering into the expression (1.38) for Γ_k and $T_{0,k',0,k'}$, entering into the expression (1.39) for $J_{\text{col}}^{(2)}$, k′ and k″ being PSW wave vectors. The quantity $T_{0,k'+k'',k',k''}$ is investigated in Ref. 1: in the experimentally investigated region of the parameters $T_{0,k'+k'',k',k''} \approx 1.4\pi g^2$. As to the matrix element $T_{0,k',0,k'}$, which describes the elastic scattering of the SeSW by the PSW, it is not difficult to compute it in the usual fashion. In the experimentally studied region we have

$$T_{0,k',0,k'} \approx 2\pi g^2 \left\{\cos^2\theta' - \frac{1}{2}\sin^2\theta' - \frac{2\omega_M}{\omega_p-\omega_0}\sin^2 2\theta'\right\}. \qquad (1.40)$$

In particular, for $\theta'=\pi/2$,

$$T_{0,\pi/2,0,\pi/2} = T_1 \approx -\pi g^2. \qquad (1.41)$$

The rate of elastic scattering of the SeSW by the PSW is given in accordance with (1.39) by

$$\gamma_{def} \approx \frac{4\pi^2}{(\Delta\theta)^2}\frac{(T_1 n^p)^2}{(2\omega_{ex}(ak_p)^2\omega_M)^{1/2}}\frac{k_0}{k_p}, \qquad (1.42)$$

where n^p is the total number of PSW and $\Delta\theta$ is the angular dimension of the PSW packet. Comparing γ_{def}, (1.42), with the Γ_k value computed in Ref. (1), we obtain

$$\frac{\gamma_{def}}{\Gamma_0} \approx 2(2\pi)^{1/2}\left(\frac{T_1}{T}\right)^2 \frac{B^{1/2}}{\Delta\theta}\frac{k_0}{k_p} \approx 150\frac{k_0}{k_p}. \qquad (1.43)$$

The quantity k_0/k_p is of the order of the order of 10^{-2}; therefore, in the parameter region of interest to us $\gamma_{\text{def}} \gtrsim \Gamma_k$.

Let us now determine the effect of the elastic scattering of the SeSW by the PSW on the KI threshold. In the presence of elastic scattering, the threshold is found from the formula (1.21). Here we should bear in mind that the SW spectrum (1.29) is anisotropic, and that the allowed values of the angle lie in the cone $\theta \leqslant \theta_m$, the SeSW wave vector vanishing on the surface of the cone:

$$k(\theta) = k_0(\theta_m^2-\theta^2)^{1/2}. \qquad (1.44)$$

Thus, in accordance with (1.31), the decrement of the SeSW becomes infinite at $\theta=\theta_m$. Nevertheless, we can establish on the basis of (1.21) that the KI threshold changes little when $\gamma_1(kl)^{-4}/\gamma_{\text{def}} \ll 1$, i.e., that

$$\Gamma_0 = \gamma_0 + \gamma_1(k_0 l)^{-4} + \pi[\gamma_{def}/\gamma_1(k_0 l)^{-4}]^{1/2}/2. \qquad (1.45)$$

§2. THE NONLINEAR EQUATIONS FOR THE SECONDARY WAVES

Let us substitute into the basic kinetic equation (1), (1.1)–(1.13) n_k in the form

$$n_k = n_k^0 + N_k \qquad (2.1)$$

and take into account the fact that n_k^0 is the solution to the equation for $\Gamma_k=0$. As a result, we obtain for N_k the nonlinear equation

$$dN_k/2dt = \Gamma_k n_k^0 + (\Gamma_k - \gamma_k^0)N_k + J_{col}^{(2)}\{N_k\} - \gamma_k^{NL}N_k + \pi\Phi_k^{NL}. \qquad (2.2)$$

Here γ_k^0 is the decrement of the waves under conditions of the equilibrium distribution (1.15); $J_{\text{col}}^{(2)}$ describes the elastic scattering of the SeW, and is given by (1.2), γ_k^{NL} is the constant of the nonlinear damping of the SeW; and Φ_k^{NL} is the k-dependent part of the arrival terms Φ_3 and Φ_4.

It is natural to assume at the initial stage of the investigation that N_k is nonzero only in the vicinity of the instability region $|k| \approx k_0$. Then, for $|k| \approx k_0$, only the following processes will make contributions to γ_k^{NL}:

1) the coalescence of two SeW to form a thermal wave [see (1.7)]:

$$\gamma_{3c}^{NL}(k) = 2\pi\int|V_{1,k2}|^2 N_2\delta(\omega_1-\omega_k-\omega_2)\delta(k_1-k-k_2)\,dk_1 dk_2, \qquad (2.3)$$

2) the coalescence of two SeW to form two thermal waves [see (1.12)]:

$$\tilde{\gamma}_{4s}^{NL}(k) = 2\pi\int|T_{k1,23}|^2 N_1(n_2^0+n_3^0+1)\delta(\omega_k+\omega_1-\omega_2-\omega_3)\delta(k+k_1-k_2-k_3)\,dk_1 dk_2 dk_3; \qquad (2.4)$$

3) the processes in which at least three SeW participate:

$$\bar{\gamma}_{4s}^{NL}(k) = 2\pi\int|T_{k1,23}|^2[N_1(N_2+N_3)-N_2N_3] \times\delta(\omega_k+\omega_1-\omega_2-\omega_3)\delta(k+k_1-k_2-k_3)\,dk_1 dk_2 dk_3. \qquad (2.5)$$

The following addends remain in the arrival terms in the vicinity of the instability region:

$$\Phi_{3c}^{NL}(k) = \int|V_{1,k2}|^2 N_2 n_1^0\delta(\omega_1-\omega_k-\omega_2)\delta(k_1-k-k_2)\,dk_1 dk_2, \qquad (2.6)$$

$$\tilde{\Phi}_{4s}^{NL}(k) = 2\int|T_{k1,23}|^2 N_1 n_2^0 n_3^0\delta(\omega_k+\omega_1-\omega_2-\omega_3)\delta(k+k_1-k_2-k_3)\,dk_1 dk_2 dk_3, \qquad (2.7)$$

and, finally,

$$\Phi_{4s}^{NL}(k) = 2\int|T_{k1,23}|^2 N_1 N_2 N_3\delta(\omega_k+\omega_1-\omega_2-\omega_3)\delta(k+k_1-k_2-k_3)\,dk_1 dk_2 dk_3. \qquad (2.8)$$

The remaining contributions to γ^{NL} and Φ^{NL} drop out in the vicinity of the surface $\omega_k=\omega_{k_0}$ of interest to us because of the conservation laws.

The analysis of the relative magnitudes of the various terms in γ^{NL} and Φ^{NL} requires the consideration of the specific form of the dispersion law and the matrix elements V and T. But in the general case we are forced to limit ourselves to the roughest estimates. If the coalescence processes (2.3) are allowed, then $\gamma_{3c}^{NL} \neq 0$:

$$\gamma_{3c}^{NL} \approx \eta_{3c}N, \quad \eta_{3c} \approx \pi V^2/k_0 v. \qquad (2.9)$$

The four-wave processes (2.5) are always allowed, but their contribution contains a temperature smallness:

$$\tilde{\gamma}_{4s}^{NL} \approx \tilde{\eta}_{4s}N, \quad \tilde{\eta}_{4s} \approx T^2 n_{k_0}^0 k_0^3/k_0 v. \qquad (2.10)$$

Here $n_{k_0}^0 k_0^3$ is the total number of thermal waves in a sphere of radius k_0 and T is the characteristic value of the matrix element $T_{k1,23}$ in this region. For example, for spin waves in a ferromagnet, $V \approx \omega_M(g/2M)^{1/2}$, $T \approx \pi g^2$, and

$$\frac{\tilde{\eta}_{4s}}{\eta_{3c}} \approx \frac{\pi^2}{2}\frac{\Theta}{\Theta_c}\left(\frac{\omega_M}{\omega_{ex}}\right)^{1/2}. \qquad (2.11)$$

The processes (2.5) are also always allowed, and, in contrast to (2.3) and (2.6), the expression for $\bar{\gamma}_{4s}^{NL}$ contains the square of the total number N of the SeW:

$$\bar{\gamma}_{4s}^{NL} \approx \bar{\eta}_{4s}N^2, \quad \bar{\eta}_{4s} \approx \pi T^2/k_0 v. \qquad (2.12)$$

In this case

$$\bar{\gamma}_{4s}^{NL}/\tilde{\gamma}_{4s}^{NL} \approx N/n_{k_0}^0 k_0^3. \qquad (2.13)$$

It can be seen that, if the coalescence processes (2.3) are allowed and the magnitude of the three-wave matrix element V is not anomalously small, then $T \approx V^2/k_0 v$, and the nonlinear damping predominates at all admissible values of N. The $\tilde{\gamma}_{4s}^{NL}$ damping is comparable to it when $TN \approx k_0 v$, i.e., at the limit of applicability of the kinetic equation. We shall not be interested in such large N. If, on the other hand, the processes (2.3) are forbidden, then, as can be seen from (2.13), either $\tilde{\gamma}_{4s}^{NL}$ or $\overline{\gamma}_{4s}{}^{NL}$ will predominate, depending on the value of N.

Let us now estimate the arrival terms in the kinetic equation. From (2.6) and (2.8) it follows that

$$\Phi_{3c}{}^{NL} \approx V^2 N n_{k_0}{}^0/k_0 v, \qquad \Phi_{4s}{}^{NL} \approx T^2 N k_0{}^3 n_{k_0}{}^{0 2}/k_0 v. \tag{2.14}$$

It is useful to compare these expressions with the corresponding thermodynamic equilibrium values:

$$\Phi_{3c}{}^{NL}/\Phi_{3c}{}^0 \approx N/k_1{}^3 n_1{}^0, \quad \Phi_{4s}{}^{NL}/\Phi_{4s}{}^0 \approx N k_0{}^3 \cdot n_{k_0}{}^0/(k_2{}^3 n_2{}^0)^2. \tag{2.15}$$

Here $k_1^3 n_1^0$ and $k_2^3 n_2^0$ are the total numbers of thermal waves in those regions of k space which respectively make the dominant contributions to the integrals (1.11) and (1.13). For spin waves, we have in the experimental situation of interest to us

$$\Phi_{3c}{}^{NL}/\Phi_{3c}{}^0 \approx N/16\pi k_0{}^3 n_{k_0}{}^0, \tag{2.16}$$

$$\frac{\Phi_{4s}{}^{NL}}{\Phi_{4s}{}^0} \approx \frac{1}{4\pi}\left(\frac{\omega_{ex} a^2 k_0{}^2}{\omega_M}\right)^2 \frac{N}{k_0{}^3 n_{k_0}{}^0}.$$

If the number of SeW and, along with it, the ratios (2.15) and (2.16) are small, then the quantities Φ_{3c}^{NL} and Φ_{4s}^{NL} can be neglected. They are also unimportant in the opposite case, for then the main contribution to Φ_k will be made by the quantity $\overline{\Phi}_{4s}^{NL}$. Indeed, if $N_k \propto \delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}_0})$, then the quantity $\overline{\Phi}_{4s}^{NL}$ is, in accordance with (2.8), also proportional to a delta function, i.e., it is infinitely large in the region where $N_k \neq 0$, i.e., in the region of interest to us. If, on the other hand, the width of the function $N_{\mathbf{k}}$ is finite, and is equal to $\Delta\omega_{\mathbf{k}}$, then it follows from (2.8) that

$$\Phi_{4s}{}^{NL} \approx (TN)^2 N/k_0{}^3 \Delta\omega_{\mathbf{k}}, \tag{2.17}$$

so that the assumption $\overline{\Phi}_{4s}^{NL} \gg \Phi^0$, except in the region of the smallest N values, is quite plausible. We shall prove it in §4 in the course of a self-consistent estimation of $\Delta\omega_k$. Thus, we can assume that in the entire range of N

$$\Phi_{\mathbf{k}}{}^{NL} = \Phi_{4s}{}^{NL}(\mathbf{k}), \quad \Phi_{\mathbf{k}} = \Phi_{\mathbf{k}}{}^0 + \Phi_{4s}{}^{NL}(\mathbf{k}). \tag{2.18}$$

In conclusion of this section, let us note that there can exist still other mechanisms of nonlinear damping that are connected with the deviation of $n_{\mathbf{k}}$ from the equilibrium distribution for $\mathbf{k}$ far from the instability region (see, for example, Ref. 6). But the consideration of their effect falls outside the limits of the present paper.

§3. LIMITATION OF THE KINETIC INSTABILITY BY NONLINEAR DAMPING

At the initial stage of the investigation of Eq. (2.2) we shall allow for only the nonlinear dependence $\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\}$:

$$\gamma_{\mathbf{k}}\{N_{\mathbf{k}}\} = \gamma_{\mathbf{k}}{}^0 + \gamma_{\mathbf{k}}{}^{NL}. \tag{3.1}$$

The role of the nonlinearity in the arrival terms $\Phi_{\mathbf{k}}^{NL}$ will be considered in the following section. For simplicity, we shall also neglect the elastic scattering, i.e., $J_{\text{col}}^{(2)}$. Its role amounts primarily to the isotropization of $N_{\mathbf{k}}$ and the corresponding rise, described in §1, in the KI threshold.

3.1. Strong nonlinear damping. Let us first consider the case in which the SeW-coalescence processes are allowed and the nonlinear damping (2.3) is strong in accordance with the estimate (2.10). For the gap-containing square-law spectrum (12), this means that the SeW-wave vectors k_0 are not too small:

$$2\alpha k_0{}^2 \geqslant \omega_0. \tag{3.2}$$

Thus, in our approximation we have an integral equation, which follows from (2.2):

$$N_{\mathbf{k}} = \Gamma_{\mathbf{k}} n_{\mathbf{k}}{}^0/[\gamma_{\mathbf{k}}{}^0 + \gamma_{\mathbf{k}}{}^{NL}\{N_{\mathbf{k}}\} - \Gamma_{\mathbf{k}}], \tag{3.3}$$

where $\gamma_k^{NL} = \gamma_{3c}^{NL}$. To elucidate the qualitative character of its solution, let us neglect the dependence of γ_k^{NL} and Γ_k on $\mathbf{k}$ in the region where $\Gamma_k \neq 0$, and integrate (3.3) over $\mathbf{k}$. We find that roughly

$$N(\gamma^0 + \eta_{3c} N - \Gamma) = \Gamma n_1{}^0 k_1{}^3. \tag{3.4}$$

Here k_1^3 is the volume of the region where $\Gamma_k \neq 0$:

$$\Gamma k_1{}^3 n_1{}^0 = \int \Gamma_{\mathbf{k}} n_{\mathbf{k}}{}^0 d\mathbf{k}, \tag{3.5}$$

Γ and n_1^0 are the characteristic values of Γ_k and n_k^0 in in this region. We also used the approximate expression (2.10) for γ_{3c}^{NL}. Going over in (3.4) to the dimensionless variable

$$x = \eta_{3c} N/\gamma^0, \tag{3.6}$$

we obtain

$$x(1 + x - \Gamma/\gamma^0) = A\Gamma/\gamma^0. \tag{3.7}$$

The nature of the solution to this equation is determined by the magnitude of the dimensionless parameter A, which has the simple meaning:

$$A = \eta_{3c} n_1{}^0 k_1{}^3/\gamma^0. \tag{3.8}$$

If $A \ll 1$ (because, for example, of the fact that the region where η_{3c} is defined is anomalously small, as obtains in the vicinity of the point beyond which the process is allowed), then the thermal fluctuations can be neglected, i.e., we can set $A=0$. Then $x = (\Gamma - \gamma^0)/\gamma^0$, or, in the original variables,

$$\eta_{3c} N = \Gamma - \gamma^0. \tag{3.9}$$

Allowing for the fact that A is nonzero, we find from (3.7) that $x = (\Gamma - \gamma^0)/\gamma^0 + A\Gamma/(\Gamma - \gamma^0)$, or

$$\eta_{3c} N = \Gamma - \gamma^0 + \Gamma\eta_{3c} k_1{}^3 n_1{}^0/(\Gamma - \gamma^0). \tag{3.10}$$

Estimates show that the quantity A can easily attain the value one and even exceed it. For example, for spin waves in ferromagnets

$$A \approx \frac{\omega_M}{\gamma^0}\frac{\Theta}{\Theta_c}\left(\frac{\omega_0}{\omega_{ex}}\right)^{1/2} \approx 10^2. \tag{3.11}$$

For $A \gg 1$ the solution (3.7) has the form

$$\eta_{3c} N = (\Gamma\eta_{3c} k_1{}^3 n_1{}^0)^{1/2}. \tag{3.12}$$

As can be seen, the dependence of N on Γ does not have a critical threshold character. The limitation of N

essentially occurs at the prethreshold stage because of the nonlinear damping of the "preheated" SeW.

3.2. Weak nonlinear damping. If the nonlinear damping is weak, then the limitation of the number of SeW occurs at a higher level, and the thermal fluctuations are significantly less important. In particular, the N_k packet will be concentrated near the surface where $\gamma_k - \Gamma_k$ has its minimum value. In the case of spherical symmetry, we find from (3.4) that

$$N_k=\frac{\Gamma_0 n_0^{k0}}{(\gamma_0^0\Gamma_0+\eta N)+\gamma_{kk}''(\Delta k)^2/2}, \qquad (3.13)$$

where γ_{kk}'' is given by the formula (10), and the approximate formula (2.13) for the nonlinear-damping constant has been used. Integrating (3.13) over the modulus of **k**, we obtain a cubic equation for the determination of the dependence of N on Γ_0:

$$N=\frac{2^{1/2}\cdot 4\pi^2\Gamma_0 n_{k_0}^0 k_0^3}{[(\gamma_0^0-\Gamma_0+\eta N)\gamma_{kk}''k_0^2]^{1/2}}. \qquad (3.14)$$

Beyond the KI threshold we have in the case in which $\Gamma_0 > \gamma_0^0$ the equation

$$\gamma_0^0+\eta N-\Gamma_0=A_1^2 2\Gamma_0^2\gamma_0^2/(\Gamma_0-\gamma_0^0)^2\gamma_{kk}''k_0^2, \qquad (3.15)$$

where A_1 is a parameter characterizing the role of the fluctuations:

$$A_1=4\pi^2\eta n_{k_0}^0 k_0^3/\gamma_0^0. \qquad (3.16)$$

For long spin waves in a ferromagnet with the dispersion law (3.2)

$$A_1\approx\frac{1}{(2\pi)^2}\frac{\Theta}{\Theta_c}\frac{\omega_{ex}}{\gamma^0}(ak_0)^2\ll 1. \qquad (3.17)$$

For $A_1 \ll 1$, the width of the SeW packet is small. From (3.13) it follows that

$$\Delta k/k\approx A_1\Gamma_0\gamma_0/(\Gamma_0-\gamma_0^0)\gamma_{kk}''k_0^2. \qquad (3.18)$$

In a narrow region around the threshold (when $|\Gamma_0-\gamma_0^0| < A_1\gamma^0$) the width of the SeW packet is not small, and the expansion (3.13) of the formula (3.4) is no longer valid.

In the case of axial symmetry the role of the thermal fluctuations will be even less important; in particular, the correction to the total number N has an exponential, and not a power-law, smallness with respect to the parameter A_1. In place of (3.15) we find from (3.4) the approximate equation

$$\gamma_0^0+\eta N-\Gamma_0\approx(\gamma_{kk}''k_0^2+\gamma_{zz}'')\exp[-(\Gamma_0-\gamma_0^0)/\Gamma_0 A_1]. \qquad (3.19)$$

§4. THE "COLLISION" MECHANISM OF KINETIC INSTABILITY LIMITATION

As shown in the preceding section, as the supercriticality increases, the SeW-packet width due to the thermal fluctuations decreases in accordance with (3.18), while the number of SeW naturally increases: $\eta N\approx \Gamma_0 - \gamma_0^0$. This means that the arrival term $\bar{\Phi}_{4s}^{NL}$, in accordance with (2.19), rapidly increases and becomes equal to the thermodynamic equilibrium value Φ^0. This occurs when

$$\frac{(TN)^2N^2\gamma_{kk}''k_0^2}{(k_0v)^2(\Gamma_0-\gamma_0)\Gamma_0^2}\approx(4\pi n_{k_0}^0k_0^3)^2. \qquad (4.1)$$

If the limitation of the number N occurs as a result of the nonlinear $\bar{\gamma}_{4s}^{NL}$ damping (and this mechanism is always present), then the characteristic number N given by (4.1) is attained when

$$(\Gamma_0-\gamma_0)/\gamma_0\approx A_2\Gamma_0/(k_0v\gamma_{kk}''k_0^2)^{1/2}, \qquad (4.2)$$

where the small parameter A_2 is of the order of A:

$$A_2=4\pi|T|n_{k_0}^0k_0^3/\gamma_0. \qquad (4.3)$$

In experiments on the kinetic excitation of SW in ferromagnets

$$A_2\approx 5\cdot 10^{-5}, \qquad (4.4)$$

so that the characteristic $\Delta\gamma/\gamma \sim 10^{-7}$. At higher supercriticalities, we can neglect the thermal fluctuations, and discard all the arrival terms except $\bar{\Phi}_{4s}^{NL}$. Of the nonlinear-damping mechanisms, we shall explicitly take only the $\bar{\gamma}_{4s}^{NL}$ damping into account. The remaining terms, if they are important in comparison with the $\bar{\gamma}_{4s}^{NL}$ term, will be considered to be included in γ_k (i.e., we shall assume that $\gamma_k=\gamma_k^0+\bar{\gamma}_{4s}^{NL}+\dots$). As a result, Eq. (2.2) can be rewritten in the form

$$dN_k/2dt=(\Gamma_k-\gamma_k)N_k+J_{col}^{(2)}\{N_k\}-\bar{\gamma}_{4s}^{NL}N_k+\pi\Phi_{4s}^{NL}(\mathbf{k}), \qquad (4.5)$$

where $J_{col}^{(2)}$, $\bar{\gamma}_{4s}^{NL}$, and $\bar{\Phi}_{4s}^{NL}$ are given by the expressions (1.2), (2.5), and (2.8).

4.1. The case of spherical symmetry. This is the easiest case to analyze, since N_k does not depend on the orientation of **k**, so that $J_{col}^{(2)}=0$. Choosing N_k in the form

$$N_k=(4\pi k_0^2)^{-1}N_\varkappa,\quad \varkappa=k-k_0, \qquad (4.6)$$

so that

$$N=\int N_\varkappa d\varkappa, \qquad (4.7)$$

and performing the angle integrations in (2.5) and (2.8), we obtain in place of (4.5) the equation

$$N_\varkappa=\frac{F^2}{\Gamma_{eff}+\gamma_{kk}''\varkappa^2/2}\int N_{\varkappa_1}N_{\varkappa_2}N_{\varkappa_3}\delta(\varkappa+\varkappa_1-\varkappa_2-\varkappa_3)d\varkappa_1d\varkappa_2d\varkappa_3. \qquad (4.8)$$

Here γ_{kk}'' is given by the expression (10),

$$\Gamma_{eff}=\gamma_0+\bar{\gamma}_{4s}^{NL}-\Gamma_0, \qquad (4.9)$$

$$F^2=\pi T^2/(k_0v),\quad \bar{\gamma}_{4s}^{NL}=F^2N^2. \qquad (4.10)$$

As shown in Ref. 7, the only stable solution to Eq. (4.8) is

$$N_\varkappa=\frac{N}{2\varkappa_0}\mathrm{ch}^{-1}\left(\frac{\pi\varkappa}{2\varkappa_0}\right), \qquad (4.11)$$

where the effective width, $\varkappa_0$, and the integrated magnitude of the packet are determined from the conditions

$$\gamma_{kk}''\varkappa_0^2/2=\Gamma_{eff}, \qquad (4.12)$$

$$\gamma_{kk}''\varkappa_0^2/2+\Gamma_{eff}=F^2N^2. \qquad (4.13)$$

Here there occurs an exact compensation of the $\bar{\gamma}_{4s}^{NL}$ nonlinear damping term by the corresponding $\bar{\Phi}_{4s}^{NL}$ arrival term, so that (4.13) can be rewritten [with allowance for (4.9) and (4.10)] in the form

$$\Gamma_0=\gamma_0+\gamma_{kk}''\varkappa_0^2/2. \qquad (4.14)$$

Thus, the effective increase of the decrement of the SeW by the amount $\gamma_{kk}''\varkappa_0^2/2$ occurs because of the broadening of the SeW packet as a result of the four-wave processes of scattering of the SeW by each other. Let us, using (4.12), (4.13), and (4.10), represent (4.14) in the form

$$\Gamma_0=\gamma_0+{}^1/_2\bar{\gamma}_{4s}^{NL}, \quad (TN)^2/k_0v=2\pi^{-1}(\Gamma_0-\gamma_0). \tag{4.15}$$

This means that the efficiency of the collision mechanism of kinetic-instability limitation coincides to within a numerical factor (equal to $\frac{1}{2}$ in the spherically symmetric case) with the efficiency of the linear-damping mechanism when the arrival term $\bar{\Phi}_{4s}^{NL}$ is "cut off."

4.2. The axial-symmetry case. We shall consider this case under the assumption that $\gamma_k-\Gamma_k$ first vanishes at the equator, i.e., when $k_z=0$, $|\mathbf{k}_\perp|=k_0$. If the $\Gamma_k-\gamma_k$ anisotropy at the surface is not too high, so that

$$\max(\Gamma_k-\gamma_k)-\min|\Gamma_k-\gamma_k|\leqslant\gamma_{def}, \tag{4.16}$$

then the elastic scattering leads to the isotropization of N_k over this surface, and the theory can be constructed in the same way as in the spherically symmetric case. It is only necessary to replace $\Gamma_0-\gamma_0$ in all the formulas by the quantity $\Gamma_k-\gamma_k$ averaged over the $\omega_k=\omega_{k_0}$ surface, i.e., by $\langle\Gamma_k-\gamma_k\rangle$. If, on the other hand, there is no elastic scattering, then the SeW packet will be concentrated near the equator. Choosing N_k in the form

$$N_k=(2\pi k_0^2)^{-1}N(\varkappa, x),\quad \varkappa=|k|-k_0,\quad x=\cos\theta, \tag{4.17}$$

we obtain in place of (4.5) the equation

$$N(\varkappa,x)=\frac{T^2/(2\pi)^2(k_0v)}{\Gamma_{eff}+\gamma_{kk}''\varkappa^2/2+\gamma_{xx}''x^2/2}\int N(\varkappa_1,x_1)$$
$$\times N(\varkappa_2,x_2)N(\varkappa_3,x_3)\ln\left[\frac{4}{v}\frac{(k_0^3/|k_{\omega\omega}''|)^{1/2}}{|\varkappa\varkappa_1-\varkappa_2\varkappa_3|}\right]$$
$$\times\delta(\varkappa+\varkappa_1-\varkappa_2-\varkappa_3)\,\delta(x+x_1-x_2-x_3)\,d\varkappa_1d\varkappa_2d\varkappa_3dx_1dx_2dx_3. \tag{4.18}$$

We were not able to find the stable exact solution to this equation. But, by analyzing the solution for large values of the arguments x and $\varkappa$, we obtained for the SeW distribution an interpolation formula that is qualitatively valid in the entire $\mathbf{k}$ space.[8] In the dimensionless variables

$$q=(\gamma_{kk}''/2\Gamma_{eff})^{1/2}\varkappa,\quad z=(\gamma_{xx}''/2\Gamma_{eff})^{1/2}x \tag{4.19}$$

it has the form

$$N(q,z)\approx\frac{1}{|T|k_0}\left(\frac{\pi}{2}\frac{k_0v\gamma_{kk}''k_0^2\gamma_{xx}''}{\Lambda\Gamma_{eff}}\right)^{1/2}$$
$$\times(1+q^2+z^2)^{-1/4}\exp[-1.4(1+q^2+z^2)^{1/2}], \tag{4.20}$$

where

$$\Lambda=\ln\left[\frac{4}{v}\left(\frac{k_0^3\gamma_{kk}''}{2|k_{\omega\omega}''|\Gamma_{eff}}\right)^{1/2}\right],\quad k_{\omega\omega}''=\left.\frac{\partial^2k_\perp}{\partial\omega^2}\right|_{k_\perp=k_0}.$$

The total number of SeW is accordingly equal to

$$N\approx\frac{2.2}{|T|}\frac{k_0v\Gamma_{eff}}{\Lambda}. \tag{4.21}$$

It should be noted that the efficiency of the collision mechanism of KI limitation is somewhat higher in the axially symmetric case than in the spherically symmetric case:

$$\Gamma_0\approx\gamma_0+0.8\bar{\gamma}_{4s}^{NL}. \tag{4.22}$$

4.3. If the KI threshold is attained at a pair of points $\pm\mathbf{k}_0$, and if the elastic scattering is weak, then the SeW distribution is localized beyond the threshold near these points. To study this distribution, it is convenient to go over to the spherical coordinates in k space after orienting the z axis along $\mathbf{k}_0$. If the problem is isotropic in the (x,y) plane, then the kinetic equation in this case has the form

$$N(\varkappa,x)=\frac{3T^2/4\pi k_0v}{\Gamma_{eff}+\gamma_{kk}''\varkappa^2/2+\gamma_{\theta\theta}''(1-x)}\int N(\varkappa_1,x_1)N(\varkappa_2,x_2),$$
$$N(\varkappa_3,x_3)\frac{\delta(\varkappa+\varkappa_1-\varkappa_2-\varkappa_3)\,\delta(x+x_1-x_2-x_3)}{[(1-x)(1-x_1)]^{1/2}+[(1-x_2)(1-x_3)]^{1/2}}d\varkappa_1d\varkappa_2d\varkappa_3dx_1dx_2dx_3. \tag{4.23}$$

The corresponding interpolation formula obtained by us for the SeW distribution from the analysis of the limiting cases has the form

$$N(q,\xi)=\frac{1}{|T|k_0}\left(\frac{\gamma_{kk}''k_0^2\gamma_{\theta\theta}''}{\Gamma_{eff}}k_0v\right)^{1/2}$$
$$\times\frac{(1+\xi)^7(1+q^2+\xi)}{(1+q^2+\xi^2)^{17/4}}\exp\{-[(\lambda_1q)^2+(\lambda_2\xi)^2]^{1/2}\}, \tag{4.24}$$

where

$$q=(\gamma_{kk}''/2\Gamma_{eff})^{1/2}\varkappa,\quad \xi=\gamma_{\theta\theta}''(1-x)/\Gamma_{eff},\quad \lambda_1\sim\lambda_2\sim1. \tag{4.25}$$

The total number of SeW in this case is

$$N\sim\frac{1}{|T|}\left(\frac{k_0v\Gamma_{eff}^3}{\gamma_{\theta\theta}''}\right)^{1/2},\quad \gamma_{\theta\theta}''=\left.\frac{\partial^2\gamma}{\partial\theta^2}\right|_{\theta=0}. \tag{4.26}$$

It should be noted that the formula (4.24) is not valid in the region $\xi^2/q<1$, where the asymptotic behavior of the preexponential factor is not scaling invariant. But the number of waves in this region is small compared to N, and, for the majority of estimates, the formula (4.24) can be used.

4.4. Small-k SeW distributions arise if the KI threshold is attained at the point $k_0=0$, or if k_0 is so small that the distribution width $\varkappa_0$ becomes greater in absolute value than k_0 as the supercriticality increases. Here we limit ourselves to the consideration of the simplest model, namely, the isotropic-ferromagnet model. In this case the dispersion law for the SeW is the square-law spectrum with a gap (12), and the decrement of the long SeW is a linear function of the modulus of $\mathbf{k}$ [see (13)].[4,5] The SeW distribution function N_k then depends only on $|\mathbf{k}|$, and the kinetic equation with allowance for (2.5) and (2.9) assumes, after the angle integrations have been performed, the form

$$N_\varepsilon=\frac{\pi^3T^2\alpha^{1/2}}{[\Gamma_{eff}+\gamma_k'(\varepsilon/\alpha)^{1/2}]\varepsilon^{1/2}}\int F_{\varepsilon,\varepsilon_1,\varepsilon_2,\varepsilon_3}N_{\varepsilon_1}N_{\varepsilon_2}N_{\varepsilon_3}$$
$$\times\delta(\varepsilon+\varepsilon_1-\varepsilon_2-\varepsilon_3)\,d\varepsilon_1d\varepsilon_2d\varepsilon_3,\quad \varepsilon=\alpha k^2, \tag{4.27}$$
$$F_{\varepsilon,\varepsilon_1,\varepsilon_2,\varepsilon_3}=\min\{(\varepsilon^{1/2}+\varepsilon_1^{1/2}),\,(\varepsilon_2^{1/2}+\varepsilon_3^{1/2})\}-\max\{|\varepsilon^{1/2}-\varepsilon_1^{1/2}|,\,|\varepsilon_2^{1/2}-\varepsilon_3^{1/2}|\}.$$

Our approximate solution to this equation is given in the Introduction: the formula (14). The total number of SeW is

$$N=\frac{2\pi}{\alpha^{3/2}}\int N_\varepsilon\varepsilon^{1/2}d\varepsilon\approx\frac{1}{|T|}(2\alpha\varkappa_1^2\Gamma_{eff})^{1/2}. \tag{4.28}$$

§5. THE DISTRIBUTION OF SeW WITH AN ANISOTROPIC DISPERSION LAW IN THE PRESENCE OF STRONG ELASTIC SCATTERING

It has been experimentally observed[1] that the kinetic instability of SW in a ferrite leads to the formation of a narrow packet of SeW with frequencies close to the bottom of the spectrum. In this case it follows from (1.29) that the surface of constant frequency is

$$\sin^2\theta_k+2\frac{\omega_{ex}(ak)^2}{\omega_M}=2\frac{\omega_{k_0}-\omega_0}{\omega_M}\ll1, \tag{5.1}$$

i.e., the cross section resembles the "figure eight." It should, however, be borne in mind that the damping of the SW intensifies as $k \to 0$ [see (1.31)], and therefore the waves are mainly concentrated near the poles of the surface, where it can be approximated by portions of a sphere with $\theta \leqslant \theta_m = [(\omega_{k_0} - \omega_0)/2\omega_M]^{1/2}$. As shown in §1, large amplitude PSW give rise not only to kinetic instability, but also to elastic scattering, and, what is more, the rate γ_{def} of this elastic scattering is not low compared to $\Gamma_{\mathbf{k}}$. Since the effective relaxation rate is determined by the difference $|\gamma_{\mathbf{k}} - \Gamma_{\mathbf{k}}|$, it can be assumed that $\Gamma_{\text{eff}} \lesssim \gamma_{\text{def}}$ always, and that, in particular, at low supercriticalities $\Gamma_{\text{eff}} \ll \gamma_{\text{def}}$. We shall also assume that the quantities $\Gamma_{\mathbf{k}}$, $\gamma_{\mathbf{k}}$, γ_{4s}^{NL}, and γ_{def} do not depend on the angles inside the cone $\theta \leqslant \theta_m$. Thus, the kinetic equation assumes the form

$$\left(\Gamma_{eff} + \frac{\gamma_{kk}''}{2}\varkappa^2 + \gamma_{def}\right) N(\varkappa, x) = \gamma_{def} N(\varkappa) + F^2 f(x) \int N(\varkappa_1) N(\varkappa_2) N(\varkappa_3) \delta(\varkappa + \varkappa_1 - \varkappa_2 - \varkappa_3) d\varkappa_1 d\varkappa_2 d\varkappa_3, \quad (5.2)$$

where $N(\varkappa, x)$ has been introduced in accordance with (4.17):

$$N(\varkappa) = \frac{1}{\Delta x}\int_0^{x_m} N(\varkappa, x)\, dx, \quad \Delta x = \int_0^{x_m} dx = x_m = \theta_m^2/2. \quad (5.3)$$

The function $f(x)$ arises as a result of the averaging of $\delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$ over the angles within the cone:

$$f(\theta) = 6\pi\{\theta^2/6 + \theta(\theta_m - \theta)/2 + \pi\theta_m^2/8 + 2\cdot 2^{1/2}\theta^3/\theta_m - 2^{1/2}\frac{\theta^2}{\theta_m}(\theta^2 + \theta_m^2)^{1/2} \arcsin\left(\frac{2\theta\theta_m}{\theta^2 + \theta_m^2}\right)\}, \quad (5.4)$$

and F is given by the formula (4.10). Integrating (5.2) over x, we obtain

$$N(\varkappa) = \frac{F^2 f}{\Gamma_{eff} + \gamma_{kk}''\varkappa^2/2}\int N(\varkappa_1) N(\varkappa_2) N(\varkappa_3) \delta(\varkappa + \varkappa_1 - \varkappa_2 - \varkappa_3) d\varkappa_1 d\varkappa_2 d\varkappa_3, \quad (5.5)$$

$$f = \frac{2}{\theta_m^2}\int_0^{\theta_m} f(\theta) \sin\theta\, d\theta \approx 4.41\theta_m^2. \quad (5.6)$$

Equation (5.5) coincides in form with (4.8) to within the subsitution $F^2 \to F^2 f$. As a result, the total number of waves increases by a factor of $f^{-1/2}$, but the shape of the packet coincides with the shape described by (7).

In order to determine the experimentally measurable dependence of the number of SeW on the parametric-pump power, we must also take into account the reaction of the SeSW on the form of the distribution function and the attenuation of the PSW.

[1] A. V. Lavrinenko, V. S. L'vov, G. A. Melkov, and V. B. Cherepanov, Zh. Eksp. Teor. Fiz. **81**, 1022 (1981) [Sov. Phys. JETP **54**, in press (1981)].
[2] V. E. Zakharov, V. S. L'vov, and S. S. Starobinetz, Usp. Fiz. Nauk **114**, 609 (1974) [Sov. Phys. Usp. **17**, 896 (1975)].
[3] V. E. Zakharov, Zh. Prikl. Mekh. Tekh. Fiz. No. 4, 35 (1965); B. B. Kadomtsev and V. M. Kontorovich, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **17**, 511 (1974).
[4] A. G. Gurevich, Magnitnyĭ rezonans v ferritakh i antiferromagnetikakh (Magnetic Resonance in the Ferrites and the Antiferromagnets), Nauka, Moscow, 1973.
[5] A. I. Akhiezer, V. G. Bar'yakhtar, and S. V. Peletminskiĭ, Spinovye volny (Spin Waves), Nauka, Moscow, 1967 (Eng. Transl., North-Holland, Amsterdam; Wiley, New York, 1968).
[6] V. S. L'vov, Preprint 69-72, Nucl. Phys. Inst., Siberian Div. USSR Acad. Sci., Novosibirsk, 1972.
[7] V. S. L'vov and V. B. Cherepanov, Zh. Eksp. Teor. Fiz. **75**, 1631 (1978) [Sov. Phys. JETP **48**, 822 (1978)].
[8] V. S. L'vov and V. B. Cherepanov, Preprint 149, for Automation and Electrometry siberia Div. USSR Acad. Sci., Novosibirsk, 1981.

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