An exact solution of the nonlinear integro-differential equation for propagation of wave beams in nonlinear media

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An equation for the Winger function is derived from the nonlinear two-dimensional Schrödinger equation. In the geometric optics approximation, it resembles a kinetic equation for which an exact solution is obtained. In particular, the solution describes self-focusing of radiation in a nonlinear medium. It is shown that wave beams with a certain type of initial angular divergence may propagate in the medium without self-focusing. The power of such beams may greatly exceed the critical power in the case of waveguide propagation.

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1. INTRODUCTION

Self-focusing of radiation in a nonlinear medium[1] is in many cases an undesirable phenomenon that restricts the capabilities of instruments and devices. In particular, estimation of the conditions under which we can significantly increase the power of radiation propagating in a nonlinear medium without self-focusing is therefore of interest. It is known that a waveguide regime of propagation is possible,[2-4] in which the self-focusing is compensated by diffraction divergence of the beam. However, in this case the beam power P should have a definite value P_{cr} that depends on the duration of the radiation and the properties of the medium. It is not possible to guarantee a stable (i.e., without focusing and divergence) regime of propagation of beams with power $P>P_{cr}$ due to diffraction divergence.

In this work, we show theoretically that the regime of propagation without self-focusing is possible for beams with $P>P_{cr}$ if the beams have certain angular characteristics at the input to the nonlinear medium. For example, the phase front of the beam should be so modulated that the initial angular divergence of the beam as a whole significantly exceeds the diffraction divergence over the entire aperture of the beam and, in addition, there should be a certain dependence of the angular spectrum of the wave vectors over the transverse cross section of the beam. Then the self-focusing and the initial divergence can cancel each other exactly. The stable regime of propagation that we have discovered is unstable in the sense that the beams with less than the required initial divergence will be self-focused, and those with a greater one will diverge. However, by creating an initial divergence that is sufficiently close to the necessary value, we can significantly increase the distance over which the beam can be propagated without appreciable change in its parameters.

The initial point of our calculations is the well-known parabolic equation. We have written out this equation in Sec. 2 and carried out its simplification, which corresponds to the quasi-classical approximation in quantum mechanics (or to the geometric optics approximation). The most frequently used form of the quasi-classical approximation has involved purely quantum concepts such as ψ functions, eigenvalues, and so on (in optics, wave concepts—the amplitude and the phase of the wave and so forth). For our purposes, however, another form of this same approximation is more convenient, based on the Wigner function and using such purely classical concepts as coordinates, momenta, and particle distribution functions over these variables. Using the approximation mentioned, we obtain a nonlinear integro-differential equation for the Wigner function, which has the form of a kinetic equation.

In Sec. 3, we find the exact solution of this equation and investigate its stability. Section 4 contains the discussion: there, for completeness of the exposition, we formulate several of our results in wave language. In the Appendix, we obtain for the Wigner function in quantum mechanics an equation which is of possible methodological interest.

2. METHOD OF THE KINETIC EQUATION IN THE THEORY OF SELF-FOCUSING

2.1. The electric field $\mathbf{E}$ satisfies the equation

$$\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}-c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{E}=0, \quad (1)$$

where $\varepsilon=\varepsilon_0+\frac{1}{2}\varepsilon_2\mathbf{E}^2$. We seek a solution of (1) in the form of a bounded wave propagating along the Z axis:

$$\mathbf{E}={}^1/_2(\mathbf{A}e^{-i\omega t+ikz}+\text{c.c.}),$$

where the vector $\mathbf{A}$ lies in a plane perpendicular to the Z axis, and the frequency $\omega=ck/\varepsilon_0^{1/2}$. We obtain

$$\left(\frac{\varepsilon_0^{1/2}}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)\mathbf{A}=\frac{i}{2k}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\mathbf{A}+\frac{i\varepsilon_2 k}{8\varepsilon_0}\mathbf{A}|\mathbf{A}|^2 \quad (2)$$

which is a parabolic equation in the form of a two-dimensional nonlinear Schrödinger equation; in the stationary case, when $\partial\mathbf{A}/\partial t=0$, the role of the time is played by the z coordinate.

We note that the divergence of the beam because of the Huygens-Fresnel principle is completely determined by the diffraction, which depends on the distribution of the phase and amplitude of the field over the aperture of the beam. This distribution can be rather complicated and include various spatial frequencies (various characteristic scales). In what follows we shall analyze the case in which there are two characteristic groups of spatial frequencies in this distribution: low and high frequencies. For simplification of the further discussion, we shall assume that there is only

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one low spatial frequency. If a plane wavefront corresponds to it, then we have the divergence due to diffraction at the aperture of the beam. If the wavefront has a curvature, then along with diffraction at the aperture, there is divergence due to the curvature of the wavefront, which we shall call regular. The group of high spatial frequencies will cause what we shall arbitrarily call diffusion divergence (divergence of this type arises, for example, in the transmission of a beam through a frosted plate).

Equation (2) is very general. It describes divergences of all type as well as self-focusing. We shall consider below the case in which

$$P \gg P_{cr}, \tag{3}$$

where the power of the beam P is equal to

$$P=\frac{c}{8\pi}\int d^2\rho|\mathbf{A}|^2,$$

the vector $\rho=(x,y)$, $d^2\rho=dxdy$; the critical power, as is well known, is equal to

$$P_{cr}=0.917\frac{c\varepsilon_0^{1/2}}{k^2\varepsilon_2}.$$

It will be shown below (Sec. 4.1) that in the case $P\gg P_{cr}$, the necessary diffusion divergence turns out to be much greater than the diffraction at the aperture and the latter can be neglected. At the same time, there is no term in Eq. (2) whose discard would mean neglect of the diffraction divergence at the aperture. Therefore, the use of (2) carries with it an unjustified complication of the calculations. We develop a method that allows us to avoid the calculation of the diffraction divergence brought about by the low spatial frequencies.[1)]

2.2 On the basis of the analogy with quantum statistics (see Ref. 7, Sec. 5 or Ref. 8, Sec. 7), we introduce the Wigner function

$$W(t,z,\boldsymbol{\rho},\mathbf{s})=\int d^2\xi\exp[-ik\mathbf{s}\boldsymbol{\xi}]A\left(t,z,\boldsymbol{\rho}+\frac{\boldsymbol{\xi}}{2}\right)A^*\left(t,z,\boldsymbol{\rho}-\frac{\boldsymbol{\xi}}{2}\right),$$

where the scalar amplitude A is determined from the condition $\mathbf{A}=\mathbf{e}A$, $\mathbf{e}$ being the unit polarization vector. All the vectors $\boldsymbol{\rho}$, $\mathbf{s}$, $\mathbf{e}$, and $\boldsymbol{\xi}$ lie in the pattern plane perpendicular to the Z axis and crossing it at the point z. The radiation intensity $I(t,z,\rho)$ at the given point ρ of the pattern plane is obviously expressed in terms of the Wigner function

$$I=\frac{c}{8\pi}|A|^2=\frac{ck^2}{8\pi}\int\frac{d^2s}{(2\pi)^2}W(t,z,\boldsymbol{\rho},\mathbf{s}).$$

By definition, $dE=Id^2\rho dt$ is the energy which passes in a time dt across the cross section $d^2\rho$ of the pattern plane z in all directions (toward increasing z).

Using (2), it is not difficult to obtain the equation for $W(\rho,s)$ (we omit the arguments t and z for brevity):

$$\left(\frac{\varepsilon_0^{1/2}}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)W(\boldsymbol{\rho},\mathbf{s})=-s_\alpha\frac{\partial W(\boldsymbol{\rho},\mathbf{s})}{\partial\rho_\alpha}$$
$$+i\frac{\varepsilon_2k^3}{8\varepsilon_0}\int d^2\xi\int\frac{d^2s_1d^2s_2}{(2\pi)^4}e^{ik\xi\mathbf{s}_2}\cdot W(\boldsymbol{\rho},\mathbf{s}+\mathbf{s}_2)\left[W\left(\boldsymbol{\rho}+\frac{\boldsymbol{\xi}}{2},\mathbf{s}_1\right)-W\left(\boldsymbol{\rho}-\frac{\boldsymbol{\xi}}{2},\mathbf{s}_1\right)\right]. \tag{4}$$

Here and below, we carry out summation over repeated indices.

Expanding W in a series in $\boldsymbol{\xi}$ and $\mathbf{s}_2$, we get

$$\left(\frac{\varepsilon_0^{1/2}}{c}\frac{\partial}{\partial t}+\frac{\partial}{\partial z}\right)W(\boldsymbol{\rho},\mathbf{s})=-s_\alpha\frac{\partial W(\boldsymbol{\rho},\mathbf{s})}{\partial\rho_\alpha}$$
$$-\frac{\varepsilon_2k^2}{8\varepsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n(2n+1)!k^{2n}}\frac{\partial^{2n+1}W(\boldsymbol{\rho},\mathbf{s})}{\partial\mathbf{s}^{2n+1}}\times\frac{\partial^{2n+1}}{\partial\boldsymbol{\rho}^{2n+1}}\left(\int\frac{d^2s_1}{(2\pi)^2}W(\boldsymbol{\rho},\mathbf{s}_1)\right). \tag{5}$$

The symbol

$$\frac{\partial^nW}{\partial\mathbf{s}^n}\cdot\frac{\partial^nF}{\partial\boldsymbol{\rho}^n}$$

denotes

$$\sum_{\alpha,\beta}\frac{\partial^nW}{\partial s_\alpha\partial s_\beta\ldots}\frac{\partial^nF}{\partial\rho_\alpha\partial\rho_\beta\ldots},$$

where the sum runs over the n different indices $\alpha,\beta,\ldots$.

Equation (4) (or (5)) is equivalent to Eq. (3). But in (5) it is easy to neglect the diffraction effects at the aperture of the entire beam, while continuing to take the diffusion divergence into account. For this purpose, it is necessary to keep only the first term $n=0$ in the sum over n.

We emphasize that such an operation does not mean the limiting transition $k\to\infty$ and total neglect of the diffraction. Actually, the Wigner function depends on k as a parameter, and continuing to take this dependence into account, we treat the diffraction that is due to the high spatial frequencies (i.e., the diffusion divergence) in the same fashion. In the limit as $k\to\infty$ we would be obliged to assume that

$$W(\boldsymbol{\rho},\mathbf{s})=\frac{32\pi^3}{ck^2}I(\boldsymbol{\rho})\delta(\mathbf{s}-\mathbf{s}_0(\boldsymbol{\rho})).$$

The presence of the δ function in the right side of this equation means that in the limit as $k\to\infty$ we only take the regular divergence into account.

In the following, we shall assume that $W(\boldsymbol{\rho},\mathbf{s})$ as a function of $\mathbf{s}$ differs from zero in a finite interval of $\mathbf{s}$, which means allowance for the diffusion divergence. Of course, such an approximation is valid if the characteristic values of the low and high spatial frequencies are widely different from one another.

Keeping only the first term $n=0$ in the right side of (5), we obtain an equation of the type of the kinetic equation of Boltzmann. Introducing the new notation

$$J=\frac{ck^2}{32\pi^3}W,\quad I_2=\frac{c\varepsilon_0}{\pi\varepsilon_2}$$

and setting $\partial J/\partial t=0$, we write down the obtained equation in the form

$$\frac{\partial J}{\partial z}+s_\alpha\frac{\partial J}{\partial\rho_\alpha}=-\frac{1}{I_2}\frac{\partial J}{\partial s_\alpha}\frac{\partial I}{\partial\rho_\alpha}; \tag{6}$$
$$I=\int d^2sJ,\quad d^2s=ds_{x'}ds_{y'}.$$

Equation (6) is similar in form to the kinetic equation which describes the motion of several particles in the pattern plane. Here the function $J(z,\rho,\mathbf{s})$ is similar to

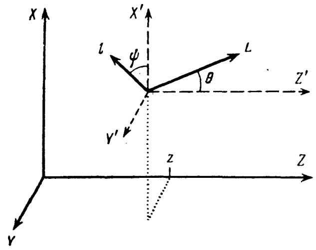

FIG. 1. The dashed lines show the axes of the primed system of coordinates, which has its origin at the point O. The vector l is the projection of the vector L on the (x', y') of the figure.

the single-particle distribution function in kinetic theory; the coordinate z plays the role of time, the dimensionless vector $\mathbf{s}$ plays the role of the velocity of the particle; the right side of (6) describes the interaction of the particles in the self-consistent-field approximation. Let a ray emerge from some point 0 of the plane in the direction $\mathbf{L}$, and let the polar and azimuthal angles of this direction be θ and ψ (see Fig. 1). Then the components of the vector $\mathbf{s}$ are $s_{x'} = \tan\theta\cos\psi$, $s_{y'} = \tan\theta\sin\psi$. We introduce the function $f(z, \boldsymbol{\rho}, \theta, \psi)$ such that

$$f(z, \boldsymbol{\rho}, \theta, \psi)\, d^2\Omega = J(z, \boldsymbol{\rho}, \mathbf{s})\, d^2 s, \quad d^2\Omega = \sin\theta\, d\theta d\psi;$$

whence

$$f(z, \boldsymbol{\rho}, \theta, \psi) = \frac{1}{\cos^3\theta} J(z, \boldsymbol{\rho}, \mathbf{s}(\theta, \psi)).$$

The function f has a clear physical meaning. $dE = f dt d^2\rho d^2\Omega$ is the energy which passes in a time dt through the area $d^2\rho$ in the direction of the solid angle $d^2\Omega$. Obviously,

$$I = \int d^2\Omega f.$$

It can be shown that $f(z, \boldsymbol{\rho}, \theta, \psi)$ is proportional to the number of rays which emanate from the point $\boldsymbol{\rho}$ of the plane z in the $d^2\Omega$ direction.

2.3. We carry out transformations that are convenient for what follows. We introduce the polar coordinates ρ and φ, in the plane taking as our origin the point of intersection of the plane of the pattern with the Z' axis (see Fig. 2). Let

$$s_{x'} = u\cos\varphi - v\sin\varphi, \quad s_{y'} = u\sin\varphi + v\cos\varphi.$$

Obviously, u and v are the radial and azimuthal components of the vectors (see Fig. 2).

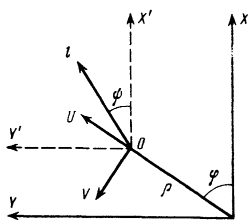

FIG. 2. The polar coordinates of the point O in the plane of the the figure. One should note the difference in the definition of the angles ψ and φ. The letters U and V denote the axes which the corresponding quantities u and v are plotted.

In place of (6), we obtain

$$\frac{\partial J}{\partial z} + u\frac{\partial J}{\partial \rho} + \frac{v}{\rho}\frac{\partial J}{\partial \varphi} = -\frac{1}{I_2}\left(\frac{\partial J}{\partial u}\frac{\partial I}{\partial \rho} + \frac{1}{\rho}\frac{\partial J}{\partial v}\frac{\partial I}{\partial \varphi}\right). \tag{7}$$

In what follows, we limit ourselves to the case in which the entire picture is symmetric relative to the Z axis. Then the function J, and with it also I, does not depend on the angle φ, as a result of which v and the derivative with respect to v drop out of Eq. (7). We are seeking the conditions under which a stable propagation regime is possible when $\partial J/\partial z = 0$. In this case

$$u\frac{\partial J(\rho, u, v)}{\partial \rho} + \frac{1}{I_2}\frac{\partial J(\rho, u, v)}{\partial u} \times \frac{d}{d\rho}\left(\int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} dv_1 J(\rho, u_1, v_1)\right) = 0. \tag{8}$$

3. EXACT SOLUTION OF THE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION (8)

3.1. Equations similar in type to (6)–(8) were investigated in plasma theory (see, for example, Refs. 9–11). Acting in the spirit of this thoery, we shall temporarily assume that

$$I(\rho) = \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} dv_1 J(\rho, u_1, v_1) \tag{9}$$

is a known function. Then (8) transforms into a first order linear partial differential equation which can be solved. Writing out the equivalent equation

$$\frac{dI}{d\rho} d\rho = I_2 u\, du,$$

we find its first integral

$$\frac{I(\rho)}{I_2} - \frac{u^2}{2} = \text{const.}$$

Then

$$J(\rho, u, v) = I_2 \Phi\left(\frac{I(\rho)}{I_2} - \frac{u^2}{2}, v\right).$$

The function Φ could be arbitrary were it not for the self-consistency condition (9). We introduce

$$F(x) = \int_{-\infty}^{\infty} dv \Phi(x, v). \tag{10}$$

Then the condition (9) takes the form of an integral equation for the function $F(x)$:

$$\int_{-\infty}^{p} \frac{dx F(x)}{(p - x)^{1/2}} = \frac{p}{\sqrt{2}}, \tag{11}$$

where $p = I(\rho)/I_2$. Since $F(x) \geqslant 0$, it follows from (11) that $F(x<0) = 0$ at $p = 0$ at $p = 0$. Then (11) takes the form of an Abel equation and is easily solved:

$$F(x) = \frac{\sqrt{2}}{\pi} x^{1/2} \chi(x),$$

where $\chi(x<0) = 0$, $\chi(x>0) = 1$ is the Heaviside function. It remains for us to solve the integral equation (10) for the function $\Phi(x, v)$, which takes the form

$$\int_{-\infty}^{\infty} dv \Phi(x, v) = \frac{\sqrt{2}}{\pi} x^{1/2} \chi(x). \tag{12}$$

Even in the class of functions $\Phi \geqslant 0$ this equation has an infinite number of solutions.

3.2. We consider two solutions of (12).

1) Solution of the "fan" type. In this case,

$$\Phi(x,v)=\frac{\sqrt{2}}{\pi}x^{1/2}\chi(x)\delta(v),$$

$$f(\rho,\varphi,\theta,\psi)=I_2\frac{\sqrt{2}\,\delta(\sin(\psi-\varphi))}{\pi\cos^3\theta}\left[\frac{I(\rho)}{I_2}-\frac{\mathrm{tg}^2\theta}{2}\right]^{1/2}\times\chi\left(\frac{I(\rho)}{I_2}-\frac{\mathrm{tg}^2\theta}{2}\right). \qquad (13)$$

All the rays emitting from the arbitrary point O in the plane of the pattern lie in a plane passing through the point O and the Z axis. These rays form an angle with the Z axis that is no longer than

$$\theta_m(\rho)=\left[\mathrm{arctg}\frac{2I(\rho)}{I_2}\right]^{1/2}\approx\left[\frac{2I(\rho)}{I_2}\right]^{1/2} \qquad (14)$$

(it is understood that $2I(\rho)/I_2\ll 1$). The value of the angular divergence of $\theta_m(\rho)$ depends on the total intensity $I(\rho)$ at the given point O.

2) Solution of the "bouquet" type. In this case,

$$\Phi(x,v)=\frac{1}{2\pi}\chi\left(x-\frac{v^2}{2}\right),$$

$$f(\rho,\varphi,\theta,\psi)=\frac{I_2}{2\pi\cos^3\theta}\chi\left(\frac{I(\rho)}{I_2}-\frac{\mathrm{tg}^2\theta}{2}\right). \qquad (15)$$

All the rays passing through an arbitrary point O in plane of the pattern are located symmetrically relative to the axis Z' which passes through this point parallel to the Z axis.

In both cases, the value of the angular divergence $\theta_m(\rho)$ is the same, but the distribution of the rays in the range $0\le\theta\le\theta_m(\rho)$ is different [compare (13) and (15)].

The solutions obtained pertain to the case in which the low spatial frequency corresponds to a plane wavefront. At the same time, the angular divergence of the beam is purely diffusive. We note a beam having only a regular divergence (i.e., a smooth wavefront with curvature) does not satisfy Eq. (8). This can easily be shown by substituting the expression $J\propto I(\rho)\delta(\mathbf{s}-\mathbf{s}(\rho))$ in (8).

3.3. We now investigate the stability of the results. At the entry to the nonlinear medium let the beam be described by the formulas (13) or (15), in which the substitution

$$I_2\to I_2(1+2\varepsilon)$$

has been made. This means that the angular divergence is less than ($\varepsilon<0$) or greater than ($\varepsilon>0$) the correct value (14). The quantity $|\varepsilon|$ is the relative error of the initial angular divergence. Adding the term $\partial J/\partial z$ to the left side of (8) and substituting Eqs. (13) and (15) in (8), we obtain

$$\frac{\partial J}{\partial z}=\frac{\partial J}{\partial a}u\frac{dI}{d\rho}\frac{1}{I_2}\left(1-\frac{1}{1+2\varepsilon}\right),$$

$$a=\frac{I}{I_2(1+2\varepsilon)}-\frac{u^2}{2}.$$

In order of magnitude,

$$u\propto\left(\frac{I}{I_2}\right)^{1/2},\quad \frac{\partial J}{\partial a}\propto\frac{J}{a}\propto J\frac{I_2}{I},\quad \frac{dI}{dr}\propto-\frac{I}{r},$$

where r is the characteristic radius of the beam.

Hence

$$\frac{1}{J}\frac{\partial J}{\partial z}\propto\varepsilon\frac{dI}{d\rho}\frac{1}{(II_2)^{1/2}}\propto-\frac{\varepsilon}{r}\left(\frac{I}{I_2}\right)^{1/2}. \qquad (16)$$

In accord with (16), the function J increases, i.e., the beam is focused if the angular divergence is small ($\varepsilon<0$) and conversely. At a given angular divergence $\theta_m(\rho)$ as a function of ρ we can introduce the characteristic power of the beam

$$P^*=\frac{I_2}{2}\int dxdy\,\theta_m(\rho). \qquad (17)$$

In our case this power is the analog of the critical beam power: if $P>P^*$, then the beam diverges in the medium and conversely. This result is entirely understandable from general considerations. A significant change in the parameters of the beam takes place over a distance

$$Z\propto\frac{r}{|\varepsilon|}\left(\frac{I_2}{I}\right)^{1/2}=\frac{R}{|\varepsilon|}, \qquad (18)$$

where $R=r(I_2/I_1)^{1/2}$ is the self-focusing distance.

4. DISCUSSION

4.1. We have shown that the character of the self-focusing of beams in a nonlinear medium depends essentially on the original divergence.

Let a beam with arbitrary dependence $I(\rho)$ be incident on the medium. This dependence uniquely determines the law of initial angular divergence:

$$\theta_m(\rho)=[2I(\rho)/I_2]^{1/2},$$

at which the critical power P^* in (17) is equal to the beam power. Such a beam will propagate without change in its parameters. We again note that the discussed divergence is purely diffusive.

The diffraction divergence at the aperture, which is equal in order of magnitude to λ/r, turns out in our case to be much smaller than the initial diffusion divergence, θ_m, in as much as we have, by virtue of (3) and (14),

$$\frac{\lambda}{r\theta_m}\approx\frac{\lambda}{r}\left(\frac{I_2}{I}\right)^{1/2}=\left(\frac{P_{cr}}{P}\right)^{1/2}\ll 1. \qquad (19)$$

In addition to (19), the most stringent condition $I(\rho)/I_2\ll 1$ should be satisfied; this means that the diffusion divergence of the beam is much less than $\pi/2$. It is not possible to obtain a beam of the needed type by passing a thin parallel beam through a defocusing lens, since this leads to the appearance of regular divergence only. The appearance of diffusion divergence is easiest to represent as the result of the transmission of the parallel beam through a plate that produces modulation of the wave front. If a is the characteristic amplitude of the modulation, and l is its characteristic scale, then the diffusion divergence that develops is equal to $2a/l$ in order of magnitude. This relation should vary over the cross section of the plate as $[I(\rho)]^{1/2}$. At the exit from the plate the rays will divergence in each elementary cross section of the beam, in accord with the law that we need. A beam of such a type can approach very close to ideal beams considered by us in Sec. 3. The effective value of ε in Eq. (18) for such a beam will be small and the beam travels a large distance without self-focusing and divergence. The divergence that we need can, for example, be given by a frosted (phase) plate and also by an amplitude plate.

4.2. Upon passage of a beam through a plate which creates diffusion divergence, fluctuations of the intensity can develop over the cross section of the beam. However, in regions with increased intensity, the self-focusing will not develop. Actually, the characteristic size of such a region is connected with the divergence θ_m by the relation $d\theta_m \propto \lambda$. But in our case, by virtue of the condition (14), we obtain

$$Id^2 \approx I_2\lambda^2 \approx P_{cr}.$$

Thus, each region contains a power not exceeding the critical value.

In addition, if the regions with increased intensity arose as the result of interference, then the location of these regions over the cross section of the beam will change rapidly with increase in z. Therefore, fine-scale self-focusing will not occur.

4.3. Suppression of self-focusing after transmission of a parallel beam through an etched phase plate was observed experimentally in Ref. 12. The authors of Ref. 12 connected the suppression with the splitting of the beam into several regions for each of which the condition $P < P_{cr}$ holds. We note that this circumstance can be not the only and not even the principal reason for the observed phenomenon. Upon transmission of the beam through the etched phase plate, diffusion divergence arises inevitably, and, as we have shown in this paper, can cause the observed suppression of the self-focusing.

4.4. We obtained Eqs. (6)–(8) neglecting the diffraction of the entire beam by the aperture in the initial equations (2), (4) and (5). In this connection, we note that Eqs. (6)–(8) can be written down simply from heuristic considerations taking into account, in the approximation of geometric optics, not the trajectory of a single ray, as is usually done, but the motion of a ray ensemble characterized by a certain ray distribution function. It is therefore possible that the region of applicability of Eqs. (6)–(8) turns out to be broader than the region of applicability of Eq. (2). Thus the method that we have developed allows us to consider the diffusion divergence of the beam in the approximation of geometric optics.

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APPENDIX

We consider the one-dimensional Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi. \qquad \text{(A.1)}$$

Elementary calculation shows that, by virtue of this equation, the Wigner function, which is equal to

$$W(t,p,x) = \int_{-\infty}^{\infty} d\xi\, e^{-ip\xi/\hbar}\,\psi(t, x+\xi/2)\,\psi^*(t, x-\xi/2),$$

satisfies the equation

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial x} = \frac{1}{i\hbar}\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty}\frac{dp_1}{2\pi\hbar}e^{ip_1\xi/\hbar}\cdot$$

$$\times W(t, p+p_1, x)\left[V\left(x+\frac{\xi}{2}\right) - V\left(x-\frac{\xi}{2}\right)\right],$$

or

$$\frac{\partial W}{\partial t} + \frac{p}{m}\frac{\partial W}{\partial z} - \frac{dV}{dx}\frac{\partial W}{\partial p} = \sum_{n=1}^{\infty}\frac{(-1)^n\hbar^{2n}}{(2n+1)!4^n}\frac{\partial^{2n+1}W}{\partial p^{2n+1}}\frac{d^{2n+1}V}{dx^{2n+1}}. \qquad \text{(A.2)}$$

Equations (A.1) and (A.2) are equivalent, but it is clearly seen from Eq. (A.2) how quantum mechanics transforms into classical kinetics in the limiting case $\hbar \to 0$. Actually, in this case, the right side of (A.2) can be neglected and $W(t,p,x)$ can be identified with the one-dimensional distribution function $f(t,p,x)$ of kinetic theory.

The Wigner function contains all the information on the quantum system and is a very convenient tool, for example, in the investigation of the statistics of an ensemble of oscillators located in a thermostat and excited by external action,[13-15] or in the determination of the first quantum corrections to the classical equations of motion.[16]

Equation (A.2) for the Wigner function is, in our view, of definite methodological interest, since (A.2) is an exact quantum equation, and at the same time, in contrast with (A.1), there is a smooth transition to the classical theory.

[1)]In the one-dimensional case, when $\delta^2A/\delta y^2 = 0$ there is probably no need of such a method, since Eq. (2) can be solved exactly by the method of the inverse problem.[5,6]

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