Superfluidity and Bose excitations in He^3 films

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The possible existence of two superfluid phases (a and b) in He^3 films is predicted. All branches of the Bose spectrum of the phases (12 in each phase) are calculated. The stability of phonon (Goldstone) modes is proved and the dispersion of nonphonon modes is calculated. The effect of a magnetic field on the number of Goldstone modes in the a- and b-phases is investigated. The behavior of the field correlators that describe the collective excitations is discussed at temperatures $0 < T < T_c$. It is shown that the correlators decrease according to a power law in the case of a system located in a magnetic field.

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1. MODEL OF A He^3 FILM

The experimental investigation of thin He^3 films, and also of monolayers of He^3 adsorbed on a substrate,[1] shows that the behavior of the film becomes "three-dimensional" in the case of thicknesses of three and more layers, and remains "two-dimensional" for one-atom and two-atom layers. Such a conclusion follows, in particular, from the measurement of the specific heat.[2] In addition to the investigation of the thermophysical properties and experiments on NMR, an important method of study of the films is the study of the spectrum of collective excitations, as occurs in three-dimensional He^3.

In the present work, we consider a model of He^3 film—the two-dimensional analog of the three dimensional He^3 model proposed earlier.[3] As is shown in Sec. 2, several superfluid phases turn out to be possible in the model. Two of them, denoted below by a and b, are energetically advantageous and stable relative to small perturbations. The Bose spectrum is studied in Secs. 3 and 4 by the method developed earlier in Refs. 4 and 5 for the three-dimensional case. The spectrum contains both phonon (Goldstone) branches, the number of which is different for different phases, and also nonphonon branches, which have an energy gap.

The He^3 film model is described by the functional of hydrodynamic action

$$S_h=g^{-1}\sum_{p,i,a} c_{ia}^+(p)c_{ia}(p)+{}^1/_2\ln\det \hat{M}(c,c^+)/\hat{M}(0,0), \tag{1.1}$$

where $\hat M$ is the following operator:

$$\hat{M}=\begin{pmatrix} Z^{-1}(i\omega-\xi+\mu(\sigma\mathbf{H}))\delta_{p_1p_2}, & (\beta V)^{-1/2}(n_{1i}-n_{2i})c_{ia}(p_1+p_2)\sigma_a \\ -(\beta V)^{-1/2}(n_{1i}-n_{2i})c_{ia}^+(p_1+p_2)\sigma_a, & Z^{-1}(-i\omega+\xi+\mu(\sigma\mathbf{H}))\delta_{p_1p_2} \end{pmatrix}. \tag{1.2}$$

Here the index i takes on two values, $i=1$ and 2, and it is this which distinguishes the two-dimensional and the three-dimensional system models. In all other respects, the notation used here is identical with that used in Refs. 3–5. In particular, $c_{ia}(p)$ is the Fourier transform of the tensor field $c_{ia}(\mathbf{x},\tau)$ with the vector index i and isotopic index $a=1,2,3$. This field describes the collective Bose excitations of the two-dimensional system. $\mathbf{H}$ is the magnetic field, $\xi=c_F(k-k_F)$ (c_F is the velocity on the Fermi surface, k_F is the Fermi momentum), μ is the magnetic moment of the quasiparticle, $n_i=k_i/k$ is a unit vector, $\beta=T^{-1}$, $\omega=(2n+1)\pi T$ is the Fermi frequency, $p=(k,\omega)$ is the 4-momentum, σ_a ($a=1,2,3$) are the Pauli matrices, $V=S$ is the surface of the system, Z is a normalization constant, g is a negative constant. The momenta $\mathbf{k}$ are located in the layer $|\mathbf{k}-\mathbf{k}_F|<k_0\ll k_F$.

It is known that in two-dimensional systems at room temperature there can be no Bose condensate. However, superfluidity is possible even without the Bose condensate.[6,7] This is connected with long-range correlations, which are damped at $T<T_c$, not exponentially but more slowly. Furthermore, a number of results obtained under the naive assumption of the existence of a condensate remain in force even in the more exact analysis, which takes into account the fact that the Bose condensate is actually "smeared out" by the long-wave fluctuations. This applies, in particular, to the temperature of the phase transition T_c, which can be found for the model (1) from the condition of the appearance of nontrivial solutions of the equation $\delta S_h=0$.

In Sec. 2, we calculate T_c and consider the different nontrivial solutions of the equation $\delta S_h=0$, which correspond to the various superfluid phases. The calculation of the second variation $\delta^2 S_h$ allows us to investigate the stability of the phases relative to small perturbations. These phases differ in the form of the order parameter, which, in the two-dimensional model, is a 2×3 matrix.

The considerations developed in Sec. 2 on the form of the order parameter in the different superfluid phases are certainly valid at $T=0$, when the Bose condensate is actually present. In Secs. 3 and 4, the Bose spectrum of the a and b phases of the two-dimensional system are calculated. We make in S_h (1.1) the following shift

$$c_{ia}(p)\to c_{ia}^{(0)}(p)+c_{ia}(p) \tag{1.3}$$

of the condensate function $c_{ia}^{(0)}(p)$ (which is different for the different phases) and then separate in S_h a quadratic form of the type

$$\sum_p c_{ia}^+(p)c_{jb}(p)A_{ijab}(p)+{}^1/_2\sum_p (c_{ia}(p)c_{jb}(-p)+c_{ia}^+(p)c_{jb}^+(-p))B_{ijab}(p). \tag{1.4}$$

It is this form which determines in the first approximation the Bose spectrum found from the equation

$$\det Q=0, \tag{1.5}$$

where Q is a matrix of quadratic form. Calculating the tensor coefficients $A_{ijab}(p)$ and $B_{ijab}(p)$ (the integrals of products of the Green's functions of the fermions) according to the method developed in Refs. 4 and 5, we obtain all the branches of the Bose spectrum in the a and b phases (12 branches in each phase). In Sec. 5 we discuss the behavior of the correlation functions $\langle c_{ia}(\mathbf{x},\tau)c_{jb}(\mathbf{y},\tau')\rangle$ at finite temperatures $T<T_c$.

2. SUPERFLUID PHASES OF A TWO-DIMENSIONAL SYSTEM FOR He^3

We consider a system described by the functional S_h (1.1), first at $|T-T_c|\ll T_c$ (in the Ginzburg-Landau region). In this region, expanding S_h in powers of c_{ia} and c^+_{ia} and limiting ourselves to terms of second and fourth orders, we get

$$S_h=\sum_p A_{ij}(p)c_{ia}^+(p)c_{ja}(p)-\frac{7\zeta(3)Z^2\mu^2H^2k_F}{4\pi^3T^2}\sum_p c_{i3}(p)c_{i3}(p)$$
$$-\frac{7\zeta(3)Z^4k_F}{16\pi^3c_FT^3\beta V}\sum_{p_1+p_2=p_3+p_4}[2c_{ia}^+(p_1)c_{jb}^+(p_2)c_{ia}(p_3)c_{jb}(p_4)$$
$$+2c_{ia}^+(p_1)c_{jb}^+(p_2)c_{ib}(p_3)c_{jb}(p_4)+2c_{ia}^+(p_1)c_{jb}^+(p_2)c_{ja}(p_3)c_{ib}(p_4)$$
$$-2c_{ia}^+(p_1)c_{ja}^+(p_2)c_{ib}(p_3)c_{jb}(p_4)-c_{ia}^+(p_1)c_{ia}^+(p_2)c_{jb}(p_3)c_{jb}(p_4)], \tag{2.1}$$

where

$$A_{ij}(p)=-\frac{\delta_{ij}}{g}+\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}n_{1i}n_{1j}(i\omega_1-\xi_1)^{-1}(i\omega_2-\xi_2)^{-1}. \tag{2.2}$$

Here we take it into account that the magnetic field $\mathbf{H}$ is directed perpendicular to the film—along the $3rd$ axis.

We find the phase transition temperature T_c by equating $A_{ij}(0)$ to zero [the value of the coefficient function $A_{ij}(p)$ at $p=0$]. We obtain the equation

$$\frac{\delta_{ij}}{g}+\frac{4Z^2}{\beta V}\sum_{p_1}\frac{n_{1i}n_{1j}}{\omega_1^2+\xi_1^2}=0. \tag{2.3}$$

Calculating the sum over the frequencies, we rewrite the equation in the form

$$g^{-1}+\frac{Z^2k_F}{\pi c_F}\int_0^{c_Fk_0}\frac{d\xi}{\xi}\,\mathrm{th}\frac{\beta\xi}{2}=g^{-1}+\frac{Z^2k_F}{\pi c_F}\left(C+\ln\frac{2c_F\beta k_0}{\pi}\right)=0, \tag{2.4}$$

where the integral depends logarithmically on k_0. Therefore g^{-1} should also depend logarithmically on k_0:

$$g^{-1}=g_0^{-1}+\frac{Z^2k_F}{\pi c_F}\ln\frac{k_F}{k_0}, \tag{2.5}$$

where g_0 no longer depends on k_0. This leads to a formula for T_c:

$$T_c=\frac{2k_Fc_F}{\pi}\exp\left(C-\frac{\pi c_F}{Z^2k_F|g_0|}\right). \tag{2.6}$$

We now consider the possibilities for a condensate function at $T<T_c$. Substituting

$$c_{ia}(p)=c_{ia}^{(0)}(p)=(\beta V)^{1/2}\delta_{p0}b_{ia} \tag{2.7}$$

in (2.4) and then making the substitution

$$b_{ia}=4\left(\frac{T_c\Delta T}{7\zeta(3)}\right)^{1/2}a_{ia}, \tag{2.8}$$

we get

$$S_h=-\beta V\frac{16\pi^2T_c\Delta T}{7\zeta(3)}\Pi, \tag{2.9}$$

where

$$\Pi=-\mathrm{Sp}\,AA^++\nu\,\mathrm{Sp}\,A^+AP+(\mathrm{Sp}\,AA^+)^2+\mathrm{Sp}\,AA^+AA^+$$
$$+\mathrm{Sp}\,AA^+A^*A^T-\mathrm{Sp}\,AA^TA^*A^+-{}^1/_2\,\mathrm{Sp}\,AA^T\,\mathrm{Sp}\,A^+A^*,$$
$$\nu=7\zeta(3)\mu^2H^2/4\pi^2T_c\Delta T. \tag{2.10}$$

Equation (2.10) is identical in form with that arising in the three-dimensional system. The difference is that the matrix A with elements a_{ia} for the two-dimensional system is a 2×3 matrix. The matrix P in (2.16) is the projector on the third axis:

$$P=\begin{Bmatrix}0&0&0\\0&0&0\\0&0&1\end{Bmatrix}. \tag{2.11}$$

Minimizing Π, we obtain the following equation for the condensate matrix A:

$$-A+\nu AP+2(\mathrm{Sp}\,A^+A)A+2AA^+A+2A^*A^TA$$
$$-2AA^TA^*-A^*\,\mathrm{Sp}\,AA^T=0. \tag{2.12}$$

This equation has several solutions, corresponding to the different superfluid phases. We consider the possibilities:

$$A_1=\frac{1}{2}\begin{pmatrix}1&0&0\\i&0&0\end{pmatrix},\quad A_2=\frac{1}{2}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\quad A_3=\frac{1}{4}\begin{pmatrix}1&i&0\\i&-1&0\end{pmatrix},$$
$$A_4=\frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&0&0\end{pmatrix},\quad A_5=\frac{1}{\sqrt{3}}\begin{pmatrix}0&0&0\\0&1&0\end{pmatrix},\quad A_6=\left(\frac{1-\nu}{3}\right)^{1/2}\begin{pmatrix}0&0&1\\0&0&0\end{pmatrix},$$
$$A_7=\left(\frac{1-\nu}{4}\right)^{1/2}\begin{pmatrix}0&0&1\\0&0&i\end{pmatrix}. \tag{2.13}$$

The corresponding solutions of Π are equal to

$$\Pi_1=-{}^1/_4,\quad \Pi_2=-{}^1/_4,\quad \Pi_3=-{}^1/_8,\quad \Pi_4=-{}^1/_6,\quad \Pi_5=-{}^1/_6,$$
$$\Pi_6=-(1-\nu)^2/6,\quad \Pi_7=-(1-\nu)^2/4. \tag{2.14}$$

For the first five phases, the quantity Π does not depend on H. The minimum value of $\Pi=-\frac{1}{4}$ is reached for phases with matrices A_1 and A_2.

We call the phase with the matrix

$$A_1=\frac{1}{2}\begin{pmatrix}1&0&0\\i&0&0\end{pmatrix} \tag{2.15}$$

the a phase, and the phase with the matrix

$$A_2=\frac{1}{2}\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix} \tag{2.16}$$

the b phase.

We note that the a phase is identical with the superfluid phase considered by Stein and Cross.[8]

We calculate the second variation $\delta^2\Pi$. If it is nonnegative, then the corresponding phase is stable relative to small perturbations. Knowledge of the quadratic form $\delta^2\Pi$ allows us to determine the phonon variables and determine the change in the number of phonon variables upon switching-on the magnetic field.

For the a phase, $\delta^2\Pi$ has the form

$$\delta^2\Pi=\nu(u_{13}^2+u_{23}^2+v_{13}^2+v_{23}^2)+{}^1/_2[2(u_{11}+v_{21})^2+(u_{11}-v_{21})^2$$
$$+(u_{21}+v_{11})^2]+{}^1/_2[2(u_{22}-v_{12})^2+(u_{12}-v_{22})^2+(u_{22}+v_{12})^2]$$
$$+{}^1/_2[2(u_{23}-v_{13})^2+(u_{13}-v_{23})^2+(u_{23}+v_{13})^2], \tag{2.17}$$

where u_{ia} and v_{ia} are the real and imaginary parts of a_{ia}. It follows from (2.17) that the phonon variables in the a phase are

$$u_{21}-v_{11},\quad u_{12}+v_{22},\quad u_{13}+v_{23}\ \text{for}\ H=0\ (v=0);$$
$$u_{21}-v_{11},\quad u_{12}+v_{22}\ \text{for}\ H\neq 0\ (v>0). \qquad (2.18)$$

This means that in the a phase at $H=0$ there exist three phonon (Goldstone) modes, while the remaining nine branches of the Bose spectrum are non-phonon (they have a gap as $\mathbf{k}\to 0$). Upon switching on the field, the phonon branch $u_{13}+v_{13}$ acquires a gap.

Calculation of $\delta^2\Pi$ for the b phase yields

$$\delta^2\Pi=v(u_{13}^2+u_{23}^2)+(v+2)(v_{13}^2+v_{23}^2)+{}^1/_2[3u_{11}^2+3u_{22}^2+2u_{11}u_{22}+(u_{12}+u_{21})^2+(v_{11}-v_{22})^2+3v_{12}^2+3v_{21}^2-2v_{12}v_{21}]. \qquad (2.19)$$

This expression shows that in the b phase the phonon variables will be

$$u_{12}-u_{21},\quad v_{11}+v_{22},\quad u_{13},\quad u_{23}\ \text{for}\ H=0\ (v=0);$$
$$u_{12}-u_{21},\quad v_{11}+v_{22}\ \text{for}\ H\neq 0\ (v>0). \qquad (2.20)$$

In the b phase at $H=0$ there exist four phonon (Goldstone) and eight non-phonon modes. Upon switching on of the magnetic field, the branches u_{13} and u_{23} acquire gaps and become non-phonon.

Formulas (2.17) and (2.10) also demonstrate the stability of the phases a and b relative to small perturbations.

In the considered model, both phases a and b have equal free energies, which does not permit us to give preference to one of them over the other. As is seen from formulas (2.18) and (2.20), the phonon variables in the two phases are essentially different. Calculation of the Bose spectrum in Secs. 3 and 4 also gives results that are significantly different for the a and b phases.

3. THE BOSE SPECTRUM OF THE a PHASE

At $T=0$ a Bose condensate can exist in two-dimensional superfluid systems. Separating it out with the help of the shift (1.3), we consider the quadratic part of the functional S_h in the new variables, c_{ia} and c_{ia}^+—the deviations of the old variables from their condensate values $c_{ia}^{(0)}$ and $c_{ia}^{(0)+}$. This quadratic form (1.4) allows us to find the Bose spectrum in first approximation if we set the determinant of the quadratic form equal to zero [Eq. (1.5)].

For the a phase, the condensate function $c_{ia}^{(0)}(p)$ has the form

$$c_{ia}^{(0)}(p)=(\beta V)^{1/2}\delta_{p0}c\delta_{a1}(\delta_{i1}+i\delta_{i2}). \qquad (3.1)$$

Here c is a constant, determined from the condition of a maximum $S_h[c_{ia}^{(0)}(p),\ c_{ia}^{(0)+}(p)]$. Substituting (3.1) in (1.1), we obtain

$$2\beta V|c|^2g^{-1}+\sum_{p_1}\ln[(\omega_1^2+\xi_1^2+4|c|^2Z^2)/(\omega_1^2+\xi_1^2)]. \qquad (3.2)$$

The condition for a maximum (3.2) is the equation

$$g^{-1}+\frac{2Z^2}{\beta V}\sum_{p_1}(\omega_1^2+\xi_1^2+4|c|^2Z^2)^{-1}=0. \qquad (3.3)$$

It allows us to express the energy gap at $T=0$

$$\Delta=2|c|Z \qquad (3.4)$$

in terms of the transition temperature T_c. It is easy to verify the result [for example, by taking the difference of Eqs. (2.4) at $T=T_c$ and (3.3) at $T=0$] that Δ and T_c are connected by the "universal formula"

$$\Delta=\pi T_c/\gamma, \qquad (3.5)$$

in the same way as in the BCS model ($\gamma=e^C$, C is Euler's constant).

We now construct the quadratic form (1.4) for the a phase, similar to what was done for the three-dimensional model in Refs. 4 and 5. Detailed calculations for the two-dimensional system can be found in Ref. 9.

This form describes both the phonon and the non-phonon Bose excitations in the a phase at $H=0$ and is the sum of three forms, of which the first depends on c_{i1} the second on c_{i2}, and the third on c_{i3}. The second and third forms go over into the first following the substitutions $c_{i2}\to ic_{i1}$ and $c_{i3}\to ic_{i1}$. Therefore, the Bose spectrum of the a phase is triply degenerate, as in the A-phase model of three-dimensional He. We consider one of the three independent forms, for example, the form with $a=1$. It has the form

$$\sum_p c_{i1}^+(p)c_{j1}(p)\left[\frac{\delta_{ij}}{g}+\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}\frac{n_{1i}n_{1j}(i\omega_1+\xi_1)(i\omega_2+\xi_2)}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)}\right]-\frac{1}{2}\sum_p[c_{i1}(p)c_{j1}(-p)+c_{i1}^+(p)c_{j1}^+(-p)]\frac{4\Delta^2Z^2}{\beta V}\times\sum_{p_1+p_2=p}\frac{n_{1i}n_{1j}(n_1\pm in_2)^2}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)}. \qquad (3.6)$$

Here $(n_1\pm in_2)^2$ means that $(n_1+in_2)^2$ is multiplied by $c_{i1}^+(-p)\ c_{j1}^+(-p)$, and $(n_1-in_2)^2$ by $c_{i1}(p)c_{j1}(-p)$. The phonon variable in the form (3.6) is $u_{21}-v_{11}$. This result, which was already obtained in Sec. 2 for the Ginzburg-Landau region, is true for all $T<T_c$. Actually, if we separate out the terms in (3.6) with fixed p and then set $c_{11}=-i$ and $c_{21}=1$ in them, we obtain

$$\frac{2}{g}+\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}\frac{(i\omega_1+\xi_1)(i\omega_2+\xi_2)(n_1^2+n_2^2)}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)}-\frac{4\Delta^2Z^2}{2\beta V}\sum_{p_1+p_2=p}(\omega_1^2+\xi_1^2+\Delta^2)^{-1}(\omega_2^2+\xi_2^2+\Delta^2)^{-1}\times[(n_1-in_2)^2(-n_1^2-2in_1n_2+n_2^2)+(n_1+in_2)^2(-n_1^2+2in_1n_2+n_2^2)]=\frac{2}{g}+\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}\frac{(i\omega_1+\xi_1)(i\omega_2+\xi_2)+\Delta^2}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)}.$$

Equating this expression to zero, we obtain

$$g^{-1}+\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}\frac{(i\omega_1+\xi_1)(i\omega_2+\xi_2)+\Delta^2}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)}=0. \qquad (3.7)$$

At $p=0$, this equation goes over into (3.3). Therefore, (3.7) has the root $p=0$, while the corresponding branch of the spectrum begins from zero. Its calculation is similar to the calculation for the three-dimensional case[4,5] with the replacement of an integral over the Fermi sphere by an integral over neighborhood of the "Fermi circle," and gives the result

$$E=2^{-1/2}c_Fk. \qquad (3.8)$$

The complete phonon spectrum in the a phase at $H=0$ consists of three branches of (3.8), corresponding to the variables $u_{21}-v_{11}$, $u_{12}+v_{22}$, $u_{13}+v_{23}$.

We proceed to the nonphonon branches of the spectrum. We first find all the branches of the spectrum at $\mathbf{k}=0$. For this purpose, we consider the terms in (3.6) with

$\mathbf{k}=0$ and fixed $a\neq 0$, which can be written down in the form

$$(u_{11}^2+u_{21}^2+v_{11}^2+v_{21}^2)f(\omega)-(u_{11}^2-u_{21}^2-v_{11}^2+v_{21}^2+2(u_{11}v_{21}+u_{21}v_{11}))g(\omega), \tag{3.9}$$

where

$$f(\omega)=\frac{1}{g}+\frac{2Z^2}{\beta V}\sum_{p_1}\frac{(i\omega_1+\xi_1)(i\omega_2+\xi_2)}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)},$$

$$g(\omega)=\frac{Z^2\Delta^2}{\beta V}\sum_{p_1}[(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)]^{-1}. \tag{3.10}$$

The quadratic form (3.9) of the variables u_{11}, u_{21}, v_{11}, v_{21} is the sum of two independent forms:

$$(u_{11}^2+v_{21}^2)(f(\omega)-g(\omega))-2u_{11}v_{21}g(\omega),$$
$$(u_{21}^2+v_{11}^2)(f(\omega)+g(\omega))-2u_{21}v_{11}g(\omega). \tag{3.11}$$

Setting the determinants of these forms equal to zero, we obtain the equation

$$\det\begin{pmatrix} f(\omega)-g(\omega), & -g(\omega) \\ -g(\omega), & f(\omega)-g(\omega)\end{pmatrix}=f(\omega)(f(\omega)-2g(\omega))=0,$$
$$\det\begin{pmatrix} f(\omega)+g(\omega), & -g(\omega) \\ -g(\omega), & f(\omega)+g(\omega)\end{pmatrix}=f(\omega)(f(\omega)+2g(\omega))=0.$$

Thus, the quantities $E(0)$ [the values of the Bose spectrum $E(\mathbf{k})$ at $\mathbf{k}=0$] are determined from the equation

$$f(\omega)=0,\quad f(\omega)+2g(\omega)=0,\quad f(\omega)-2g(\omega)=0. \tag{3.12}$$

The equation $f+2g=0$ has the root $\omega=0$, since

$$f(0)+2g(0)=g^{-1}+\frac{2Z^2}{\beta V}\sum_{p_1}(\omega_1^2+\xi_1^2+\Delta^2)^{-1}=0$$

by virtue of (3.1). The phonon branch of the spectrum (3.8) corresponds to this equation, and the non-phonon branches to the remaining two.

The equation $f=0$ can be reduced to the form

$$(\omega^2+2\Delta^2)F(\omega)=0, \tag{3.13}$$

where

$$F(\omega)=\frac{1}{\omega(\omega^2+4\Delta^2)^{1/2}}\ln\frac{(4\Delta^2+\omega^2)^{1/2}+\omega}{(4\Delta^2+\omega^2)^{1/2}-\omega}. \tag{3.14}$$

Substituting $i\omega\to E$, we find the root

$$E=\Delta\sqrt{2}, \tag{3.15}$$

corresponding to the variables $u_{11}-v_{21}$ and $u_{21}+v_{11}$.

The equation $f-2g=0$ reduces to the form

$$(\omega^2+4\Delta^2)F(\omega)=0 \tag{3.16}$$

and has the root

$$E=2\Delta \tag{3.17}$$

after the substitution $i\omega\to E$. The branch (3.17) corresponds to the variable $u_{11}+v_{21}$.

Taking it into account that in the a phase each of the branches of the Bose spectrum is triply degenerate, we can rewrite the results obtained thus far for the Bose spectrum in the form

$$\begin{aligned} &E=c_Fk/\sqrt{2};\quad u_{21}-v_{11},\quad u_{12}+v_{22},\quad u_{13}+v_{23}\ \text{(3 modes)},\\ &E=\Delta\sqrt{2};\quad \begin{matrix} u_{21}+v_{11}, & u_{12}-v_{22}, & u_{13}-v_{23} \\ u_{11}-v_{21}, & u_{22}+v_{12}, & u_{23}+v_{13}\end{matrix}\ \text{(6 modes)},\\ &E=2\Delta;\quad u_{11}+v_{21},\quad u_{22}-v_{12},\quad u_{23}-v_{13}\ \text{(3 modes)}.\end{aligned} \tag{3.18}$$

Here the branches of the spectrum are written down along with the variables to which they correspond.

The next step is to obtain the corrections of relative order $c_F^2k^2\Delta^{-2}$ to the modes (3.18). We first consider the non-phonon modes. We begin with the mode $u_{11}-v_{21}$. It is not difficult to see that account of corrections $\sim k^2$ gives the following equation in place of (3.13):

$$(\omega^2+2\Delta^2)F(\omega)+\frac{c_F^2k^2}{2}\frac{d}{d\omega^2}[(\omega^2+2\Delta^2)F(\omega)]=0, \tag{3.19}$$

where $F(\omega)$ is the function (3.14). We then get the formula

$$E^2=2\Delta^2+c_F^2k^2/2. \tag{3.20}$$

The same result is obtained for the variable $u_{21}+v_{11}$.

We now consider the branch $E=2\Delta$, which corresponds to the variable $u_{11}+v_{21}$. The equation for it can be reduced to the form

$$\int_0^{2\pi}d\varphi\int_0^1 d\alpha\left[\ln\frac{\Delta^2}{\Delta^2+\alpha(1-\alpha)(\omega^2+c_F^2(\mathbf{nk})^2)}-2\right]=0. \tag{3.21}$$

This equation cannot be solved by expansion in $c_F^2(\mathbf{n}\cdot\mathbf{k})^2$ under the integral. The situation is analogous to that encountered for the branch $E\approx 2\Delta$ in the B phase of a three-dimensional system of the He^3 type.[5] Equation (3.21) itself differs from the corresponding equation of three-dimensional theory by the fact that in place of the integral $\int d\Omega$ over the solid angle, we have in (3.21) the integral $\int d\varphi$ over the planar angle φ. This leads to the result that the dispersion law for the branch $u_{11}+u_{21}$ with $E\approx 2\Delta$, in place of the equation

$$\int_0^1(-z+x^2)^{1/2}dx=0$$

is determined in the three dimensional case[5] by the equation

$$\int_0^1\left(\frac{-z+x^2}{1-x^2}\right)^{1/2}dx=0, \tag{3.22}$$

where

$$z=(E^2-4\Delta^2)/c_F^2k^2. \tag{3.23}$$

The root of this equation turns out to be complex and its value,

$$z=0.500-0.433i \tag{3.24}$$

was obtained on a computer. Knowing z, we can find E from the formula

$$E^2=4\Delta^2+zc_F^2k^2, \tag{3.25}$$

which follows from (3.23). The complex root means that the corresponding Bose excitation is unstable and decays into the fermions of which it is composed. We recall that the branches $E\approx 2\Delta$ had complex increments $\sim k^2$ also in the three-dimensional theory.[5]

For completion of the study of the Bose spectrum of the a phase at small $\mathbf{k}$, it remains to find the dispersion of the phonon branch of the spectrum, which can be done similar to Ref. 4.

The result has the form

$$E=\frac{c_Fk}{\sqrt{2}}\left(1-\frac{5c_F^2k^2}{96\Delta^2}\right). \tag{3.26}$$

It shows the stability of the phonon branch of the Bose spectrum relative to decay.

We write out once again the results obtained in this section for the Bose spectrum in a phase:

$$
\begin{aligned}
&E^2=\frac{c_F^2k^2}{2}\left(1-\frac{5c_F^2k^2}{48\Delta^2}\right); \quad u_{21}-v_{11},\ u_{12}+v_{22},\ u_{13}+v_{23}\ \text{(3 modes)},\\
&E^2=2\Delta^2+c_F^2k^2/2; \quad \begin{matrix} u_{21}+v_{11},\ u_{12}-v_{22},\ u_{13}-v_{23} \\ u_{11}-v_{21},\ u_{22}+v_{12},\ u_{23}+v_{13}\end{matrix}\ \text{(6 modes)}, \qquad (3.27)\\
&E^2=4\Delta^2+(0.500-i0.433)c_F^2k^2; \quad u_{11}+v_{21},\ u_{22}-v_{12},\ u_{23}-v_{13}\ \text{(3 modes)}.
\end{aligned}
$$

These formulas represent a refinement of the formulas (3.18).

4. BOSE SPECTRUM OF THE b PHASE

We now carry out the investigation of the Bose spectrum of the b phase, confining ourselves to the scheme developed for the a phase in Sec. 3.

The condensate function of the b phase has the form

$$c_{ia}^{(0)}(p)=(\beta V)^{1/2}\delta_{p0}c\delta_{ia}, \qquad (4.1)$$

where c is a constant, determined from the condition of a maximum $S_h[c_{ia}^{(0)}(p), c_{ia}^{(0)+}(p)]$. Substituting (4.1) in (1.1) we obtain the same equation (3.2) which we had for the a phase, while the condition for its maximum is identical with Eq. (3.3). The energy gap $\Delta=2|c|Z$ at $T=0$ is given by the same formula $\Delta=\pi T_c/\gamma$, as for the a phase. The calculations, which are similar to those carried out for the a phase, give the following quadratic form for the b phase:

$$
\begin{aligned}
\sum_p c_{ia}^+(p)c_{ja}(p)&\left[\frac{\delta_{ij}}{g}+\frac{4Z^2}{\beta V}\sum_{p_1+p_2=p}\frac{n_{1i}n_{1j}(i\omega_1+\xi_1)(i\omega_2+\xi_2)}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)}\right]\\
&-\frac{1}{2}\sum_p\left(c_{ia}(p)c_{jb}(-p)+c_{ia}^+(p)c_{jb}^+(-p)\right)\\
\times\frac{4\Delta^2Z^2}{\beta V}&\sum_{p_1+p_2=p}\frac{n_{1i}n_{1j}}{(\omega_1^2+\xi_1^2+\Delta^2)(\omega_2^2+\xi_2^2+\Delta^2)}[n_1^2(2\delta_{a1}\delta_{b1}-\delta_{ab})\\
&+n_2^2(2\delta_{a2}\delta_{b2}-\delta_{ab})+2n_1n_2(\delta_{a1}\delta_{b2}+\delta_{a2}\delta_{b1})]. \qquad (4.2)
\end{aligned}
$$

The tensor coefficients A_{ij} and B_{ijab} in (4.2) are real, since the complex expressions $(i\omega_1+\xi_1)(i\omega_2+\xi_2)$ in A_{ij} become real after summation. Therefore the form (4.2) divides into a sum of two independent forms, of which the first depends on $u_{ia}=\mathrm{Re}\,c_{ia}$, and the second on $v_{ia}=\mathrm{Im}\,c_{ia}$. Furthermore, it is seen that the form of the variables $u_{i3}(v_{i3})$ is independent of the form of the variables $u_{ia}(v_{ia})$ with $a=1,2$.

We first investigate the Bose spectrum corresponding to the variables u_{i3}, v_{i3}. Removing from (4.2) the form corresponding to these variables, and carrying out the necessary calculations, we obtain two phonon branches of the spectrum:

$$E=c_Fk\sqrt{3}/2;\quad u_{13};\quad E=c_Fk/2;\quad u_{23} \qquad (4.3)$$

and two non-phonon branches:

$$E=2\Delta;\quad v_{13},\ v_{23}. \qquad (4.4)$$

We proceed to the spectrum for the variables c_{ia}, c_{ia}^+, $a=1,2$. The terms with $p=0$ for these variables in (4.4) can be written in the form

$$-\frac{2Z^2\Delta^2}{\beta V}\left[\sum_{p_1}(\omega_1^2+\xi_1^2+\Delta^2)^{-2}\right]Q_0, \qquad (4.5)$$

where

$$
\begin{aligned}
Q_0=&3u_{11}^2+3u_{22}^2+2u_{11}u_{22}+(u_{12}+u_{21})^2\\
&+3v_{12}^2+3v_{21}^2-3v_{12}v_{21}+(v_{11}-v_{22})^2. \qquad (4.6)
\end{aligned}
$$

It follows from (4.5) and (4.6) that the phonon variables will be $u_{12}-u_{21}$ and $v_{11}+v_{22}$, the same as obtained in Sec. 2 for the Ginzburg-Landau region. To them correspond the branches

$$E=c_Fk/\sqrt{2};\quad u_{12}-u_{21},\quad v_{11}+v_{22}. \qquad (4.7)$$

We now consider the form of the variables c_{ia}, c_{ia}^+ $(a=1,2)$ in the case $\mathbf{k}=0$, in order to find the energies $E(0)$ of the non-phonon modes. We can write the form of the variables u_{ia} as

$$
\begin{aligned}
&(u_{11}^2+u_{12}^2+u_{21}^2+u_{22}^2)f(\omega)+(u_{21}^2+u_{12}^2-u_{11}^2\\
&-u_{22}^2-2u_{11}u_{22}-2u_{12}u_{21})g(\omega), \qquad (4.8)
\end{aligned}
$$

where $f(\omega)$ and $g(\omega)$ are determined by the formulas (3.10). The expression (4.8) is the sum of two independent forms:

$$
\begin{aligned}
&(f(\omega)-g(\omega))(u_{11}^2+u_{22}^2)-2g(\omega)u_{11}u_{22},\\
&(f(\omega)+g(\omega))(u_{12}^2+u_{21}^2)-2g(\omega)u_{12}u_{21}. \qquad (4.9)
\end{aligned}
$$

The matrices of the form (4.9) are the same as for the forms (3.11), obtained in a phase. Using the results of Sec. 3, we can immediately write down the result:

$$
\begin{aligned}
&E=\Delta\sqrt{2};\quad u_{11}-u_{22},\quad u_{12}+u_{21},\\
&E=2\Delta;\quad u_{11}+u_{22}. \qquad (4.10)
\end{aligned}
$$

The form of the variables v_{ia} differs from (4.8) by the substitution $u_{ia}\to v_{ia}$, $g(\omega)\to -g(\omega)$ and has the form

$$
\begin{aligned}
&(v_{11}^2+v_{12}^2+v_{21}^2+v_{22}^2)f(\omega)\\
&+(v_{11}^2+v_{22}^2-v_{21}^2-v_{12}^2-2v_{11}v_{22}+2v_{12}v_{21})g(\omega). \qquad (4.11)
\end{aligned}
$$

This is the sum of two independent forms:

$$
\begin{aligned}
&(f(\omega)+g(\omega))(v_{11}^2+v_{22}^2)+2g(\omega)v_{11}v_{22},\\
&(f(\omega)-g(\omega))(v_{12}^2+v_{21}^2)+2g(\omega)v_{12}v_{21}. \qquad (4.12)
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
&E=\Delta\sqrt{2};\quad v_{11}-v_{22},\ v_{12}+v_{21},\\
&E=2\Delta;\quad v_{12}-v_{21}. \qquad (4.13)
\end{aligned}
$$

The corrections of relative order k^2 to the branches of the spectrum that have been found can be obtained in a fashion similar to what was done in Sec. 3 for the a phase.

In conclusion, we write down all the branches of the Bose spectrum for the b phase, together with the corresponding variables:

$$
\begin{aligned}
&E^2={}^1/_2c_F^2k^2(1-5c_F^2k^2/48\Delta^2);\quad u_{12}-u_{21},\ v_{11}+v_{22};\\
&E^2={}^3/_4c_F^2k^2(1-c_F^2k^2/72\Delta^2);\quad u_{13};\\
&E^2={}^1/_4c_F^2k^2(1-c_F^2k^2/48\Delta^2);\quad u_{23};\\
&E^2=2\Delta^2+c_F^2k^2/2;\quad u_{11}-u_{22},\ u_{12}+u_{21},\ v_{11}-v_{22},\ v_{12}-v_{21}; \qquad (4.14)\\
&E^2=4\Delta^2+(0.500-i0.433)c_F^2k^2;\quad u_{11}+u_{22},\ v_{12}-v_{21};\\
&E^2=4\Delta^2+(0.152-i0.218)c_F^2k^2;\quad v_{13};\\
&E^2=4\Delta^2+(0.849-i0.216)c_F^2k^2;\quad v_{23}.
\end{aligned}
$$

Here

$$
\begin{aligned}
&0.500-0.433i=z_1,\\
&0.152-0.218i=z_2, \qquad (4.15)\\
&0.849-0.216i=z_3
\end{aligned}
$$

are the roots of the equations

$$\int_0^1 dx\left(\frac{-z+x^2}{1-x^2}\right)^{1/2}\begin{pmatrix}1\\ x^2\\ 1-x^2\end{pmatrix}=0, \qquad (4.16)$$

which arise in consideration of the branches with $E\approx 2\Delta$.

5. THE ASYMPTOTE OF THE CORRELATORS IN THE CASE $0<T<T_c$

The correlation functions

$$\langle c_{ia}(\mathbf{x},\tau)c_{jb}(\mathbf{y},\tau')\rangle \tag{5.1}$$

of the Bose fields introduced above tend in the case $T=0$ and in the limit $r=|\mathbf{x}-\mathbf{y}|\to\infty$ to a constant $\langle c_{ia}\rangle\langle c_{jb}\rangle$. At $T_c>T>0$, the situation should be similar to that existing in the theory of a two-dimensional superfluid, where the correlator of the Bose field $\langle\psi(\mathbf{x},\tau)\bar{\psi}(\mathbf{y},\tau')\rangle$ falls off as the power $r^{-\alpha}$, where $\alpha=m\beta/2\pi\rho_s$, $\beta=T^{-1}$, m is the mass of the Bose particle, ρ_s is the density of the superfluid component.[7] This result for the two-dimensional Bose system is most simply obtained by going over to polar coordinates $\psi(\mathbf{x},\tau)=(\rho(\mathbf{x},\tau))^{1/2}\exp i\varphi(\mathbf{x},\tau)$.[7] In this case it turns out that the decrease $\approx r^{-\alpha}$ is determined by the b phase correlator $\langle\exp i(\varphi(\mathbf{x},\tau)-\varphi(\mathbf{y},\tau))\rangle$, while the correlator $\langle[\rho(\mathbf{x},\tau)\rho(\mathbf{y},\tau)]^{1/2}\rangle$ tends to some constant value.

For our system of tensor Bose fields $c_{ia}(\mathbf{x},\tau)$, it is also natural to introduce the analog of polar coordinates, which turn out to be different for the different phases. Here the number of "angular" variables is equal to the number of phonon modes of the system.

In order to explain the transition to the new coordinates, we note that the expression (2.10) (at $H=0$) is invariant relative to the transformations

$$A\to e^{i\varphi}u_2Au_3, \tag{5.2}$$

where u_2 and u_3 are orthogonal matrices of second and third orders, $e^{i\varphi}$ is the phase factor. For $A=A_1$ (the a phase), multiplication of A_1 on the left by $u_2=\exp(i\sigma_2\psi)$ is equivalent to multiplication by the phase factor $e^{i\psi}$, which can be combined with $e^{i\varphi}$. Multiplication of A_1 on the right by $u_3=u$ gives the matrix

$$A_1u=\frac{1}{2}\begin{pmatrix} u_{11} & u_{12} & u_{13}\\ iu_{11} & iu_{12} & iu_{13}\end{pmatrix}. \tag{5.3}$$

It is expressed in terms of the elements of the first row (u_{11},u_{12},u_{13}) of the orthogonal matrix u, which can be regarded as the components of the unit vector $\mathbf{n}$ ($\mathbf{n}^2=1$). Thus the role of the angular variables in a phase is played by the phase φ and by the unit vector $\mathbf{n}$ (the element of the sphere S^2).

The action functional can depend only on the gradients of the angular variables, but not on the variables themselves. For slowly changing fields, the part of the action that is dependent on the angular variables has the form

$$S_{\varphi,\mathbf{n}}=-\int(a\partial_i\varphi\partial_i\varphi+b\partial_in_a\partial_in_a)\,d^2x. \tag{5.4}$$

We shall assume that φ and $\mathbf{n}$ depend only on $\mathbf{x}$ (but not on τ), assuming that we have integrated over the nonzero Fourier components. We note that the coefficients a and b in (5.4) increase without bound at $T\to0$.

The considerations that have been given show that the significant parts of the correlators (5.1) are the averages

$$\langle\exp\{i(\varphi(\mathbf{x})-\varphi(\mathbf{y}))\}\rangle,\quad\langle(n_a(\mathbf{x}),n_b(\mathbf{y}))\rangle. \tag{5.5}$$

The first of these is characteristic for two-dimensional superfluid systems and decays as $r^{-\alpha}$, $\alpha=(4\pi a)^{-1}$. For the second of the averages (5.5) (the correlator of the n-field of two-dimensional Euclidean theory), Polyakov[10] has obtained a formula (in the notation used here)

$$\langle(\mathbf{n}(\mathbf{x}),\mathbf{n}(\mathbf{y}))\rangle=\left(1-\frac{1}{4\pi b}\ln\frac{r}{a_1}\right)^2, \tag{5.6}$$

where a_1 is some constant with the dimensionality of length. The formula (5.6) loses meaning for large distances, since its right side increases without limit as $r\to\infty$. The reason for the limited applicability of (5.6) is that in its derivation by the method of the renormalization group, the "instantons" existing in the theory, which would correct the behavior of the correlator at large distances, were not taken into account. Up to the present time, the question of the asymptote of the correlator (5.6) as $r\to\infty$ remains open. It is possible that the correlator decays in power-law fashion as $r\to\infty$.

The question of the asymptote of the correlators becomes simplified for a system in a magnetic field. Tuning on the magnetic field reduces the symmetry group which, at $H\neq0$, can be rewritten in the form

$$A\to e^{i\varphi}u_2Au_3,\quad u_3=\begin{pmatrix}\tilde{u}_2 & 0\\ 0 & 1\end{pmatrix}, \tag{5.7}$$

where u_2 and $\tilde{u}_2$ are two orthogonal matrices of second order. In particular, for the a phase, the replacement of u_3 by $\tilde{u}_3$ leads an n-field with two components: $n_1=\cos\psi$, $n_2=\sin\psi$, while the action (5.4) takes the form

$$-\int(a\partial_i\varphi\partial_i\varphi+b\partial_i\psi\partial_i\psi)\,d^2x. \tag{5.8}$$

The correlator of the n-field $(\cos\psi,\ \sin\psi)$ decays in power-law fashion:

$$\begin{aligned}\langle(\mathbf{n}(\mathbf{x}),\ \mathbf{n}(\mathbf{y}))\rangle&=\langle\cos\psi(\mathbf{x})\cos\psi(\mathbf{y})+\sin\psi(\mathbf{x})\sin\psi(\mathbf{y})\rangle\\&=\langle\exp i(\psi(\mathbf{x})-\psi(\mathbf{y}))\rangle\sim r^{-\tilde{\alpha}},\quad\tilde{\alpha}=(4\pi b)^{-1}.\end{aligned} \tag{5.9}$$

A similar consideration can also be carried out for the b phase. Here the multiplication of $A=A_2$ on the left by an orthogonal matrix of second order u_2 is equivalent to multiplication on the right by the third-order matrix

$$\tilde{u}_3=\begin{pmatrix}u_2 & 0\\ 0 & 1\end{pmatrix},$$

which we combine with u_3. Multiplication of A_2 on the right with $u_3=u$ gives the result

$$A_2u=\begin{pmatrix}u_{11} & u_{12} & u_{13}\\ u_{21} & u_{22} & u_{23}\end{pmatrix}, \tag{5.10}$$

which is expressed in terms of the first two rows of the matrix u, which we represent as the components of two mutually orthogonal n-fields $\mathbf{n}_1,\mathbf{n}_2[(\mathbf{n}_1,\mathbf{n}_2)=0]$. The angular coordinates in the b phase are $\varphi,\mathbf{n}_1,\mathbf{n}_2$ $[n_1^2=n_2^2=1,(\mathbf{n}_1,\mathbf{n}_2)=0]$, and the part of the action corresponding to them can be written in the form

$$-\int(a\partial_i\varphi\partial_i\varphi+b(\partial_in_{1a}\partial_in_{1a}+\partial_in_{2a}\partial_in_{2a}))\,d^2x. \tag{5.11}$$

The nontrivial part of the correlators (5.1) is determined by the averages

$$\langle(\mathbf{n}_i(\mathbf{x}),\ \mathbf{n}_j(\mathbf{y}))\rangle,\quad i,j=1,2. \tag{5.12}$$

The question of the asymptote of these averages remains open thus far. However, it is easily resolved for a system in a magnetic field, when, only the first two components of the fields n_1 and n_2 are different from zero, so that $n_{11}=n_{22}=\cos\psi$, $n_{12}=-n_{21}=\sin\psi$. The action (5.11) takes the form

$$-\int (a\partial_i\varphi\partial_i\varphi+2b\partial_i\psi\partial_i\psi)\,d^2x, \qquad (5.13)$$

while the correlator (5.12) decays as $r^{-\gamma}$, $\gamma=(8\pi b)^{-1}$.

Summing up, we can say that the theory of two-dimensional superfluid systems of the He^3 type at $T\neq 0$ is connected with the difficult and interesting problem of the asymptote of the correlators of the n-fields of two-dimensional Euclidean theory. The situation is completely clear only for systems in a magnetic field, where the correlators decay in power-law fashion.

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