Generalized Heisenberg ferromagnet and the Gross-Neveu model

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We use the quantum-mechanical inverse scattering method to diagonalize the Hamiltonian of a many-component quantum system on a one-dimensional lattice. We construct the generating functional of the commuting integrals of motion and the corresponding eigenfunctions and eigenvalues. As an application of the formalism developed here we evaluate the spectrum in the Gross-Neveu model. We enumerate a number of quantum-mechanical systems for which one can obtain an exact solution by this method.

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The development of a quantum-mechanical inverse scattering method enables one to combine in a single scheme the known exact solutions of many different quantum-mechanical systems and to indicate new exactly soluble models (see the review article by Faddeev[1] and Refs. 2–10). Yang's generalization[11] of the Bethe Ansatz to the case of the many-component non-linear Schrödinger equation which describes the interaction of a Bose-field multiplet has been analyzed from the point of view of the quantum-mechanical inverse scattering method (QISM).[5] One of the central objects of the QISM is the R-matrix which determines the commutation relations between the generating functional of the quantum-mechanical integrals of motion and the creation and annihilation operators. The R-matrix satisfies the Yang-Baxter relation (equation for S-matrix factorization) and, in turn, can be considered to be the operator of the auxiliary linear problem connected with the exactly soluble quantum-mechanical system on a lattice. For example, the 4×4 R-matrix of the single-component non-linear Schrödinger equation[2,3] is the operator of the linear problem connected with an isotropic ferromagnetic spin-$\frac{1}{2}$ chain.[4] For the $(M+1)^2\times(M+1)^2$ R-matrix used earlier,[5] such a system is the generalized Heisenberg ferromagnet—a quantal system on a one-dimensional chain with short-range action and the space of states on a site is equal to $\mathbf{C}^M$. The present paper is basically devoted to this system.

The complete space of states of the quantal system $\mathcal{H}$ will be equal to the product of the spaces of the states in all the sites of the chain:

$$\mathcal{H}=\bigotimes_{n=1}^{N}\mathcal{H}_n,\quad \mathcal{H}_n=C^M,\quad \dim C^M=M. \tag{1}$$

The Hamiltonian has the form (assuming the periodicity condition $\mathcal{H}_{N+1}=\mathcal{H}_1$)

$$H=\varepsilon\sum_{n=1}^{N}P\qquad \varepsilon=\pm 1, \tag{2}$$

where $P_{n,n+1}$ is the permutation operator in the $\mathcal{H}_n\otimes\mathcal{H}_{n+1}$ space. If we choose in C^M a base $\{e_i\}$, then the action of $P_{n,n+1}$ on the base in $\mathcal{H}$ will be as follows:

$$P_{n,n+1}e_{i_1}\otimes\ldots\otimes e_{i_n}\otimes e_{i_{n+1}}\otimes\ldots\otimes e_{i_N}=e_{i_1}\otimes\ldots\otimes e_{i_{n+1}}\otimes e_{i_n}\otimes\ldots\otimes e_{i_N}. \tag{3}$$

Sutherland[12] has studied the generalized Heisenberg ferromagnet in the framework of Bethe's coordinate Ansatz. We apply the quantal inverse scattering method to construct a generating functional for the integrals of motion $t(\lambda)$, we get the commutation relations between the $t(\lambda)$ and the creation and annihilation operators of the eigenstates of the Hamiltonian (2), and we recover the transcendental equation for $(M-1)$-st set of quasi-momenta that determine the eigenstates for finite N. In the limit as $N\to\infty$ the system will be a ferromagnet ($\varepsilon=-1$) or an antiferromagnet ($\varepsilon=1$). For those cases we evaluate the spectrum of the elementary excitations and their S-matrix.

The revival of interest in the use of the Bethe Ansatz for solving relativistically invariant sine-Gordon models[6] and the massive Thirring model[8,13] led to interesting results for asymptotically free models with an isotropic fermion multiplet (number of colors $N_c=2$) and with a four-fermion interaction (modification of the Gross-Neveu and Vaks-Larkin models).[14,15] We show that the equations for the quasi-momenta, which determine the eigenfunctions of the Hamiltonians of these models, are the same, apart from the inhomogeneous term, as the corresponding equations of the generalized Heisenberg ferromagnet. This enables us to evaluate the spectrum of the asymptotic states of the model, which is the same as the quasi-classical answer in Ref. 16.

The plan of the paper is as follows. In section 1 we construct the monodromy matrix of the auxiliary linear problem, we evaluate the commutation relations between its elements, and we find the generating functional for the integrals of motion $t(\lambda)$. In section 2 we diagonalize the trace of the transition matrix, and give the transcendental equations for the quasi-momenta which parametrize the eigenstates $t(\lambda)$ for a finite number of sites N on the lattice. We go to the limit as $N\to\infty$ in the ferromagnetic case $\varepsilon=-1$ and calculate the S-matrix of the excitations, and realize the Zamolodchikov algebra in section 3. In section 4 we consider the more complicated case of the antiferromagnetic state $\varepsilon=1$, when the eigenstates of the Hamiltonian (2) are described by integral equations as $N\to\infty$. In the last section the formalism developed here is applied to the Gross-Neveu model and we list a number of quantum models the exact solution of which can be obtained by this method.

1. As in the classical variant of the inverse scattering method, the basic object of study for the QISM*[1] is the auxiliary linear problem connected with the nonlinear equation (in the present case—the Heisenberg equations for quantum operators). We shall give later the Heisenberg equations generated by the Hamiltonian (2), but we start with considering the linear problem on a chain of N sites:

$$\Phi_{n+1}(\lambda)=L_n(\lambda)\Phi_n(\lambda). \tag{4}$$

The operator $L_n(\lambda)$ acts in the space $C^M\otimes\mathscr{H}_n$ [see (1)] and we can represent it as an $M\times M$ matrix in C^M (the auxiliary space) with matrix elements-operators in $\mathscr{H}_n$ (in the space of the quantum states). The operator $L_n(\lambda)$ which interests us can be written as follows:

$$L_n(\lambda)=a(\lambda)I+b(\lambda)P_n, \tag{5}$$

where P_n is the permutation operator in $C^M\otimes\mathscr{H}_n$, λ is the spectral parameter, $a(\lambda)+b(\lambda)=1$, $a(\lambda)=\lambda/(\lambda+i\varepsilon)$, and $\varepsilon=\pm1$.

The analog in the QISM of the transition from the potential of the linear problem to the scattering data is the transition from the local operators that act non-trivially only in $\mathscr{H}_n$ to the elements of the transition matrix $T_N(\lambda)$—to operators in $\mathscr{H}$. For Eq. (4) $T_N(\lambda)$ is given as a product:

$$T_N(\lambda)=L_N(\lambda)L_{N-1}(\lambda)\dots L_1(\lambda). \tag{6}$$

The central role in the exact solution of the quantal equations is played by the commutation relations between the matrix elements of $T_N(\lambda)$. These relations can be found because of the special form of the operator $L_n(\lambda)$. Indeed, $L_n(\lambda)$ satisfies the Yang-Baxter relations, which we write down as follows. We consider the product of three spaces $C_a^M\otimes C_b^M\otimes C_n^M$ (the lower indices are used to distinguish operators which act nontrivially in the corresponding spaces). We introduce three operators:

$$L_{ab}(\lambda)=a(\lambda)+b(\lambda)P_{ab} \tag{7}$$

and similarly $L_{an}(\lambda)$ and $L_{bn}(\lambda)$. In that case the following relation holds:

$$L_{ab}(\lambda-\mu)L_{an}(\lambda)L_{bn}(\mu)=L_{bn}(\mu)L_{an}(\lambda)L_{ab}(\lambda-\mu). \tag{8}$$

We now consider two transition matrices $T_{Na}(\lambda)$ and $T_{Nb}(\lambda)$ in the space $C^M\otimes C_b^M\otimes\mathscr{H}$, which act trivially in C_b^M and C_a^M, respectively. Using the fact that $L_{an}(\lambda)$ and $L_{bm}(\lambda)$ commute when $n\neq m$, we get

$$L_{aN}(\lambda)L_{bN}(\mu)\dots L_{a1}(\lambda)L_{b1}(\mu)=T_{Na}(\lambda)T_{Nb}(\mu),$$

and the following equation holds:

$$L_{ab}(\lambda-\mu)T_{Na}(\lambda)T_{Nb}(\mu)=T_{Nb}(\mu)T_{Na}(\lambda)L_{ab}(\lambda-\mu). \tag{9}$$

It is this equation which determines the required commutation relations between the elements of $T_N(\lambda)$.

To write Eq. (9) more compactly it is convenient to rewrite it, using the tensor product

$$R(\lambda-\mu)T_N(\lambda)\otimes T_N(\mu)=T_N(\mu)\otimes T_N(\lambda)R(\lambda-\mu), \tag{10}$$

where $R(\lambda)=b(\lambda)+a(\lambda)P$, while P is the permutation operator in $C^M\otimes C^M$. An obvious consequence of Eq. (10) is that the trace of $T_N(\lambda)$ as matrix in C^M commutes for different λ with itself:

$$[t(\lambda),t(\mu)]=0,\quad t(\lambda)=\mathrm{Sp}\,T_N(\lambda). \tag{11}$$

This enables us to consider $t(\lambda)$ as a generating functional of the higher integrals of motion. Indeed, the operators

$$H^{(l)}=i\left(\frac{d}{d\lambda}\right)^l\ln t(\lambda)t(0)^{-1}|_{\lambda=0} \tag{12}$$

are local for $l\leq N$ and by virtue of (11) commute with one another. The first two of them have the form

$$H^{(1)}=\varepsilon\left(\sum_{n=1}^N P_{n,n+1}-N\right),$$

$$H^{(2)}=i\sum_{n=1}^N[P_{n+1,n+2},P_{n,n+1}].$$

The operator for a cyclic permutation $t(0)$,

$$t(0)e_{i_1}\otimes\dots\otimes e_{i_{N-1}}\otimes e_{i_N}=e_{i_N}\otimes e_{i_1}\otimes\dots\otimes e_{i_{N-1}},$$

is, in the limit $N\to\infty$, connected with the momentum operator, and it is natural to call the logarithm of its eigenvalue the momentum of the state.

The linear problem (4) is connected as follows with the Heisenberg equations of motion. We rewrite the Hamiltonian (2) in terms of the basis matrices e_{ij}:

$$H=\varepsilon\sum_{n=1}^N\sum_{i,j=1}^M e_n^{ij}e_{n+1}^{ji},\qquad (e^{ij})_{kl}=\delta_{ik}\delta_{jl}.$$

The Heisenberg equations of motion for the e^{ij},

$$i\frac{de_n^{ij}}{dt}=\varepsilon\sum_{k,l=1}^M[e_n^{ij},e_n^{kl}](e_{n+1}^{lk}+e_{n-1}^{lk})$$

can be written in the Lax form

$$\frac{dL_n}{dt}=M_{n+1}L_n-L_nM_n,$$

where the operator-valued matrix M_n is given by the equations

$$M_nL_{n-1}=i\varepsilon[P_{n-1,n},L_{n-1}]-iL_{n-1}',$$
$$L_nM_n=-i\varepsilon[P_{n-1,n},L_n]-iL_n'.$$

The proof of this statement follows easily from the Heisenberg equations of motion for $L_n(\lambda)$ and the Yang-Baxter relations.[4,17]

2. To make the formulae clearer and simpler we shall consider the case $M=3$ (we give the final answers for arbitrary M). The 3×3 matrix $L_n(\lambda)$ can be written as follows:

$$L_n(\lambda)=\begin{pmatrix} a(\lambda)+b(\lambda)e_n^{11} & b(\lambda)e_n^{21} & b(\lambda)e_n^{31}\\ b(\lambda)e_n^{12} & a(\lambda)+b(\lambda)e_n^{22} & b(\lambda)e_n^{32}\\ b(\lambda)e_n^{13} & b(\lambda)e_n^{23} & a(\lambda)+b(\lambda)e_n^{33}\end{pmatrix}, \tag{13}$$

where $(e_n^{ij})_{kl}=\delta_{ik}\delta_{jl}$ are the basic matrix-operators in $\mathscr{H}_n$. There exists in $\mathscr{H}_n$ a λ-independent vector $|0\rangle_n$, action on which (element-by-element) makes $L_n(\lambda)$ triangular:

$$L_n(\lambda)|0\rangle_n=\begin{pmatrix} a(\lambda) & 0 & 0\\ 0 & a(\lambda) & 0\\ b(\lambda)e_n^{13} & b(\lambda)e_n^{23} & 1\end{pmatrix}|0\rangle_n,\quad |0\rangle_n=\begin{pmatrix}0\\0\\1\end{pmatrix}\in\mathscr{H}_n. \tag{14}$$

We construct in $\mathscr{H}$ the vector $|0\rangle$ which is the product of the vectors $|0\rangle_n$ for all sites ("bare" vacuum, ferromagnetic state),

$$|0\rangle=\prod_{n=1}^{N}|0\rangle_n.$$

Operating on such a vector all $L_n(\lambda)$ are triangular and, hence, the transition matrix $T_N(\lambda)$ is triangular:

$$T_N(\lambda)|0\rangle=\begin{pmatrix} a(\lambda)^N & 0 & 0\\ 0 & a(\lambda)^N & 0\\ C_1(\lambda) & C_2(\lambda) & 1\end{pmatrix}|0\rangle. \tag{15}$$

In correspondence with these results it is natural to write $T_N(\lambda)$ in block form

$$T_N(\lambda)=\begin{pmatrix} A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}. \tag{16}$$

The action of the introduced operators on the generating vector $|0\rangle$ is as follows:

$$A_{ab}(\lambda)|0\rangle=\delta_{ab}a(\lambda)^N|0\rangle,\ D(\lambda)|0\rangle=|0\rangle,$$
$$C_a(\lambda)=|\lambda,a\rangle,\ B_a(\lambda)|0\rangle=0.$$

The vectors $|0\rangle$ and $|\lambda,a\rangle$ are linearly independent, so that it is natural to call $C_a(\lambda)$ creation operators and $B_a(\lambda)$ annihilation operators.

We get from (10) the commutation relations (we write down those which are of most interest for what follows):

$$A_{ab}(\lambda)C_c(\mu)=C_p(\mu)A_{ad}(\lambda)\frac{r_{pd,bc}(\lambda-\mu)}{a(\lambda-\mu)}-\frac{b(\lambda-\mu)}{a(\lambda-\mu)}C_b(\lambda)A_{ac}(\mu), \tag{17}$$

$$D(\lambda)C_a(\mu)=C_a(\mu)D(\lambda)\frac{1}{a(\mu-\lambda)}-\frac{b(\mu-\lambda)}{a(\mu-\lambda)}C_a(\lambda)D(\mu). \tag{18}$$

The difference from the commutation relations for the non-linear Schrödinger equation[3] and the spin-$\frac{1}{2}$ chain[4] is connected with the appearance in (17) of the matrix

$$r(\lambda)=b(\lambda)I+a(\lambda)P,\ P_{ab,cd}=\delta_{ad}\delta_{bc}, \tag{19}$$

where P is the permutation operator in $C^2\otimes C^2$ (in $C^{M-1}\otimes C^{M-1}$ for the general case $M>3$).

In correspondence with the algebraic explanation of the Bethe Ansatz[1] for the construction of the eigenvectors $t(\lambda)$—the trace of the monodromy matrix—we must act upon $|0\rangle$ with the creation operators $C_a(\lambda)$ and, using Eqs. (17) and (18), obtain transcendental equations for the quasi-momenta λ_i. We consider the vector

$$|\lambda_1,\dots,\lambda_n,\ F\rangle=C_{a_1}(\lambda_1)\dots C_{a_n}(\lambda_n)|0\rangle F_{a_n\dots a_1}. \tag{20}$$

In order that this is an eigenvector for $t(\lambda)$ it is necessary and sufficient that F be an eigenvector for the operator $t_1(\lambda)=\mathrm{Tr}\,T_n^{(1)}(\lambda)$ and that the quasi-momenta satisfy the equation

$$a(\lambda_k)^{-N}\prod_{\substack{l\neq k\\ l=1}}^{n}\frac{a(\lambda_k-\lambda_l)}{a(\lambda_l-\lambda_k)}F=t_1(\lambda_k)F. \tag{21}$$

The operators which appear here

$$t_1(\lambda)=\mathrm{Sp}\,T_n^{(1)}(\lambda),\quad T_n^{(1)}(\lambda)=L_n^{(1)}(\lambda-\lambda_n)\dots L_1^{(1)}(\lambda-\lambda_1),$$
$$L_k^{(1)}(\lambda)=r_k(\lambda)P_k \tag{22}$$

completely reproduce our initial construction with the single difference that now we have a chain of n sites and the linear operator $L_k^{(1)}(\lambda-\lambda_k)$ in $C^2\otimes C_k^2$ depends through λ_k on the site number (inhomogeneous chain[18]). The latter, however, does not prevent a relation which is analogous to (10) for $T_N(\lambda)$ from being satisfied for $T_n^{(1)}(\lambda)$:

$$r(\lambda-\mu)T_n^{(1)}(\lambda)\otimes T_n^{(1)}(\mu)=T_n^{(1)}(\mu)\otimes T_n^{(1)}(\lambda)r(\lambda-\mu). \tag{23}$$

Hence it follows that

$$[t_1(\lambda),t_1(\mu)]=0,$$

and Eq. (21) and the requirement that the vector F be an eigenvector of $t_1(\lambda)$ are therefore consistent. In our case ($M=3$) $T_n^1(\lambda)$ is an operator in $C^2\otimes\mathcal{H}^{(1)}$, i.e., a 2×2 matrix and the matrix elements are operators in $\mathcal{H}^{(1)}=\otimes_{k=1}^{n}\mathcal{H}_k^{(1)},\ \mathcal{H}_k^{(1)}\equiv C^2$. Writing $T_n^{(1)}(\lambda)$ in the form

$$T_n^{(1)}(\lambda)=\begin{pmatrix} A^{(1)}(\lambda) & B^{(1)}(\lambda)\\ C^{(1)}(\lambda) & D^{(1)}(\lambda)\end{pmatrix}, \tag{24}$$

we can construct the eigenvector F in the same way as in the preceding case:

$$F=C^{(1)}(\lambda_1^{(1)})\dots C^{(1)}(\lambda_m^{(1)})|0\rangle^{(1)}\equiv|\lambda_i^{(1)}\rangle^{(1)},$$
$$|0\rangle^{(1)}=\bigotimes_{k=1}^{n}|0\rangle_k^{(1)},\quad |0\rangle_k^{(1)}=\begin{pmatrix}0\\1\end{pmatrix}\in\mathcal{H}_k^{(1)}. \tag{25}$$

The commutation relations between $A^{(1)}(\lambda)$, $B^{(1)}(\lambda)$, $C^{(1)}(\lambda)$, and $D^{(1)}(\lambda)$ which follow from (23) enable us to find the eigenvalue $t_1(\lambda)$ and the equations for the auxiliary set of quasi-momenta $\{\lambda_i^{(1)}\}$:

$$t_1(\lambda)|\lambda_i^{(1)}\rangle^{(1)}=\Big\{\prod_{k=1}^{n}a(\lambda-\lambda_k)\prod_{l=1}^{m}\frac{1}{a(\lambda-\lambda_l^{(1)})}$$
$$+\prod_{l=1}^{m}\frac{1}{a(\lambda_l^{(1)}-\lambda)}\Big\}|\lambda_i^{(1)}\rangle^{(1)}, \tag{26}$$

$$\prod_{k=1}^{n}a(\lambda_l^{(1)}-\lambda_k)=\prod_{\substack{k\neq l\\ k=1}}^{m}\frac{a(\lambda_l^{(1)}-\lambda_k^{(1)})}{a(\lambda_k^{(1)}-\lambda_l^{(1)})},\quad l=1\dots m. \tag{27}$$

For the first set of quasi-momenta $\{\lambda_k\}$ we get the scalar equations

$$a(\lambda_k)^N\prod_{l=1}^{m}\frac{1}{a(\lambda_l^{(1)}-\lambda_k)}=\prod_{\substack{l\neq k\\ l=1}}^{n}\frac{a(\lambda_k-\lambda_l)}{a(\lambda_l-\lambda_k)} \tag{28}$$

and the eigenvalues $\nu(\lambda)$ of the operator $t(\lambda)$

$$\nu(\lambda)=a(\lambda)^N\prod_{k=1}^{n}\frac{1}{a(\lambda-\lambda_k)}\Big\{\prod_{k=1}^{n}a(\lambda-\lambda_k)\prod_{l=1}^{m}\frac{1}{a(\lambda-\lambda_l^{(1)})}$$
$$+\prod_{l=1}^{m}\frac{1}{a(\lambda_l^{(1)}-\lambda)}\Big\}+\prod_{k=1}^{n}\frac{1}{a(\lambda_k-\lambda)}. \tag{29}$$

In the case of arbitrary M the matrices $L^{(1)}$ and r will have the dimensionality $(M-1)^2\times(M-1)^2$ and it is convenient to introduce the following notation. Instead of $(\lambda_i)_1^n$ we shall write $(\lambda_i^{(1)})_1^{n_1}$, instead of $(\lambda_i^{(1)})_1^m$ we shall write $(\lambda_i^{(2)})_1^{n_2}$, and so on.

We can regard the operator $T_{n_1}^{(1)}(\lambda)$ in (22) as a monodromy matrix on a lattice with n_1 sites with a space of states C^{M-1} in the site. To construct the eigenvectors of the operator $t_1(\lambda)$ one is obliged to introduce the transition matrix $T_{n_2}^{(2)}(\lambda)$ on a lattice of n_2 sites, and so on, up to $T_{n_{M-2}}^{(M-2)}(\lambda)$. The eigenvectors

$$t_k(\lambda)=\mathrm{Sp}\,T_{n_k}^{(k)}(\lambda)$$

(the trace is taken in the space C^{M-k}) are determined by the set of quasi-momenta

$$(\lambda_i^{(k+1)})_1^{n_{k+1}},\dots,(\lambda_i^{(M-1)})_1^{n_{M-1}}.$$

The required eigenvector $t(\lambda)$ is thus determined by the $(M-1)$-st set of quasi-momenta

$$(\lambda_i^{(k)})_1^{n_k},\quad k=1,\ldots,M-1,$$

satisfying the following equations:

$$a(\lambda_j^{(1)})^N\prod_{i=1}^{n_2}\frac{1}{a(\lambda_i^{(2)}-\lambda_j^{(1)})}=\prod_{\substack{i=1\\i\neq j}}^{n_1}\frac{a(\lambda_j^{(1)}-\lambda_i^{(1)})}{a(\lambda_i^{(1)}-\lambda_j^{(1)})},\ldots$$

$$\prod_{i=1}^{n_{k-1}}a(\lambda_j^{(k)}-\lambda_i^{(k-1)})\prod_{i=1}^{n_{k+1}}\frac{1}{a(\lambda_i^{(k+1)}-\lambda_j^{(k)})}=\prod_{\substack{i=1\\i\neq j}}^{n_k}\frac{a(\lambda_j^{(k)}-\lambda_i^{(k)})}{a(\lambda_i^{(k)}-\lambda_j^{(k)})},\ldots,\tag{30}$$

$$\prod_{i=1}^{n_{M-2}}a(\lambda_j^{(M-1)}-\lambda_i^{(M-2)})=\prod_{\substack{i=1\\i\neq j}}^{n_{M-1}}\frac{a(\lambda_j^{(M-1)}-\lambda_i^{(M-1)})}{a(\lambda_i^{(M-1)}-\lambda_j^{(M-1)})}.$$

The eigenvalue $t(\lambda)$ for given sets $(\lambda_j^{(k)})_1^{n_k}$ is the following:

$$v(\lambda)=a(\lambda)^N\prod_{i=1}^{n_1}\frac{1}{a(\lambda-\lambda_i^{(1)})}\Big\{\prod_{i=1}^{n_1}a(\lambda-\lambda_i^{(1)})\prod_{i=1}^{n_2}\frac{1}{a(\lambda-\lambda_i^{(2)})}\Big\{\ldots$$

$$\ldots\Big\{\prod_{i=1}^{n_{M-2}}a(\lambda-\lambda_i^{(M-2)})\prod_{i=1}^{n_{M-1}}\frac{1}{a(\lambda-\lambda_i^{(M-1)})}+\prod_{i=1}^{n_{M-1}}\frac{1}{a(\lambda_i^{(M-1)}-\lambda)}\Big\}\ldots$$

$$\ldots\Big\}+\prod_{i=1}^{n_2}\frac{1}{a(\lambda_i^{(2)}-\lambda)}\Big\}+\prod_{i=1}^{n_1}\frac{1}{a(\lambda_i^{(1)}-\lambda)}.\tag{31}$$

It is convenient to introduce the variables $\mu_i^{(k)}=\lambda_i^{(k)}+\frac{1}{2}i\varepsilon k$ and then, taking the logarithm of the obtained set (30), we arrive at the equations from Sutherland's paper:[12]

$$2\pi J_j^{(1)}=-N\theta(2\mu_j^{(1)})+\sum_{i=1}^{n_1}\theta(\mu_j^{(1)}-\mu_i^{(1)})-\sum_{i=1}^{n_2}\theta(2\mu_j^{(1)}-2\mu_i^{(1)}),\ldots,$$

$$2\pi J_j^{(k)}=-\sum_{i=1}^{n_{k-1}}\theta(2\mu_j^{(k)}-2\mu_i^{(k-1)})+\sum_{i=1}^{n_k}\theta(\mu_j^{(k)}-\mu_i^{(k)})$$

$$-\sum_{i=1}^{n_{k+1}}\theta(2\mu_j^{(k)}-2\mu_i^{(k+1)}),\ldots,\tag{32}$$

$$2\pi J_j^{(M-1)}=-\sum_{i=1}^{n_{M-2}}\theta(2\mu_j^{(M-1)}-2\mu_i^{(M-2)})+\sum_{i=1}^{n_{M-1}}\theta(\mu_j^{(M-1)}-\mu_i^{(M-1)}).$$

Here the $J_j^{(k)}$ are integers or half-integers (depending on whether the number $n_k-n_{k+1}-n_{k-1}+1$ is even or odd), and $\theta(x)=-2\arctan x$.

We note also that if we use the formalism developed here we can diagonalize intrinsically the transition matrix $T_N\lambda$ from (6). To prove that it is sufficient to note that it can be written in the form of the trace of a new transition matrix:

$$T_N(\lambda)\equiv T_{Na}(\lambda)=\mathrm{Sp}\,L_{ab}(\mu-\lambda)L_{Nb}(\mu)\ldots L_{1b}(\mu)|_{\lambda=\mu}.$$

In this formula the matrices $L_{nm}(\lambda)$ are operators in $C_a^M\otimes C_b^M\otimes C_1^M\otimes\ldots\otimes C_N^M$ and act non-trivially only in $C_N^M\otimes C_m^M$ [cf. (7)], while the trace is taken with respect to the space C_b^M.

3. The transition to the limit of a system with an infinite number of degrees of freedom ($N\to\infty$) is essentially different for $\varepsilon=-1$ and $\varepsilon=1$. The reason is that such a transition $N\to\infty$ is physically of interest when the spectrum of the Hamiltonian H is non-negative. Therefore in the discrete and bounded spectrum of H (when $N<\infty$) we are concerned only with the vicinity of the lowest eigenvalue as $N\to\infty$.

If $\varepsilon=-1$, then the minimum eigenvalue of H corresponds to the bare vacuum

$$H|0\rangle=-N|0\rangle,\tag{33}$$

so that the renormalized energy operator $H+N$ has positive excitation energies as $N\to\infty$.

Proceeding as in the spin-$\frac{1}{2}$ case ($M=2$) and the case of the non-linear Schrödinger equation[1,3,4], i.e., regularizing the operator $T_N(\lambda)$ by using its vacuum average

$$\langle 0|T_N(\lambda)|0\rangle=\begin{pmatrix}a(\lambda)^N I_{M-1} & 0\\ 0 & 1\end{pmatrix},$$

we are led to the following result: the generating function of the integrals of motion as $N\to\infty$ becomes $D(\lambda)$. The commutation relations take the form

$$D(\lambda)C_a(\mu)=C_a(\mu)D(\lambda)\frac{1}{a(\mu-\lambda)},\tag{34}$$

$$D(\lambda)|0\rangle=|0\rangle.\tag{35}$$

Let us say a few words about the region in which the quasi-momenta λ_i vary. As $N\to\infty$ the first of the equations of the set (32) splits off and the following variants are possible. If there are no identical numbers among the J_j, all λ_j lie on the line $\mathrm{Im}\,\lambda_i=\frac{1}{2}$. If, however, some of the numbers J_j are the same, the m quasi-momenta $\tilde{\lambda}_j$ corresponding to them tend, as $N\to\infty$, to the values

$$\tilde{\lambda}_k=\lambda_0+i\left(k-\frac{m+1}{2}\right)+\frac{i}{2}+O(e^{-\alpha N}),\quad k=1,\ldots,m.$$

Such a set $\tilde{\lambda}_k$ describes a bound state of m magnons, and the eigenvalue $D(\lambda)$ has the form

$$D(\lambda)C_{a_1}(\tilde{\lambda}_1)\ldots C_{a_m}(\tilde{\lambda}_m)|0\rangle=\frac{\lambda_0-\lambda-im/2}{\lambda_0-\lambda+im/2}C_{a_1}(\tilde{\lambda}_1)\ldots C_{a_m}(\tilde{\lambda}_m)|0\rangle.\tag{36}$$

The eigenvalues of the Hamiltonian $H+N$ and of the momentum are equal to

$$E_m(\lambda_0)=m(\lambda_0^2+m^2/4)^{-1},\ p_m(\lambda_0)=\theta(m/2\lambda_0),$$
$$E_m(p)=2(1-\cos p)/m.\tag{37}$$

If we consider the operators $A_a(\lambda)$,

$$A_a(\lambda)=C_a(\lambda)D(\lambda)^{-1},\tag{38}$$

we get, using the commutation relations (34),

$$A_a(\lambda)A_b(\mu)=S_{ab}^{cd}(\lambda-\mu)A_c(\mu)A_d(\lambda),\tag{39}$$

$$S_{ab}^{cd}(\lambda)=\frac{a(-\lambda)}{a(\lambda)}r_{ab,cd}(\lambda).\tag{40}$$

The operators $A_a(\lambda)$ realize a Zamolodchikov algebra[19] for the magnons and S is the magnon scattering operator (one can verify this also directly by studying the coordinate representation of the wave function).

We note that we can diagonalize the Hamiltonian (2) when we add to it the operator H_h:

$$H_h=\sum_{n=1}^{N}\sum_{k=1}^{M}e_n^{kk}h_k,\quad \sum_{k=1}^{M}h_k=0.$$

The parameters h_k describe $M-1$) "magnetic fields."

4. We consider now the limit $N\to\infty$ in the case $\varepsilon=1$. The vector $|0\rangle$ corresponds to the largest eigenvalue and we are not interested in it. We denote by $|\Omega\rangle$ the vector corresponding to the lowest eigenvalue ("filled vacuum", antiferromagnetic state).

Following Sutherland[12] one can show that $|\Omega\rangle$ is characterized by the following occupation numbers n_k and quantum numbers $J_i^{(k)}$:

$$n_k=\frac{M-k}{M}N,\quad J_{i+1}^{(k)}=J_i^{(k)}+1;\quad k=1,2,\dots,M-1. \tag{41}$$

In the limit as $N\to\infty$ the numbers $n_k\to\infty$ and the quasi-momenta densely fill the axis. We change to a continuous characteristic of the quasi-momenta—their occupation density:

$$\rho_l(\lambda)=\lim_{N\to\infty}\frac{1}{N(\lambda_{k+1}^{(l)}-\lambda_k^{(l)})},\quad \lambda_k^{(l)}\to\lambda. \tag{42}$$

For the $\rho_l(\lambda)$ which characterize the ground state $|\Omega\rangle$ we get from the set (32) a set of integral equations:[12]

$$\rho_l(\lambda)+\int_{-\infty}^{+\infty}K_{lm}(\lambda-\mu)\rho_m(\mu)\,d\mu=\frac{1}{2\pi}\theta'(2\lambda)\delta_{l1}, \tag{43}$$

$$K_{lm}(\lambda)=\frac{1}{2\pi}[\delta_{lm}\theta'(\lambda)-(\delta_{l\,m+1}+\delta_{l\,m-1})\theta'(2\lambda)]. \tag{44}$$

The Fourier transform $\tilde{\rho}_l(k)$ of the solution $\rho_l(\lambda)$ has the form

$$\tilde{\rho}_l(k)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\rho_l(\lambda)e^{ik\lambda}\,d\lambda=\frac{\operatorname{sh}(k(M-l)/2)}{\operatorname{sh}(kM/2)}. \tag{45}$$

We show that in the limit $N\to\infty$ the generating functional of the integrals again becomes $D(\lambda)$. Indeed, for sufficiently small λ the contribution to the eigenvalue $t(\lambda)$ from $a(\lambda)^N$ will be of order $\exp[N\,\mathrm{Re}\ln a(\lambda)]$ and all terms in $\nu(\lambda)$ are exponentially small compared to the last one:

$$\nu(\lambda)=\exp\Big(-\sum_{i=1}^{n}\ln a(\lambda_i-\lambda)\Big)(1+O(e^{-\alpha N})).$$

In the limit as $N\to\infty$ the sum changes into an integral with density $\rho_1(\lambda)$:

$$I(\lambda)=\lim_{N\to\infty}\frac{1}{Ni}\ln\nu(\lambda)=i\int_{-\infty}^{+\infty}\ln a\Big(\mu-\lambda-\frac{i}{2}\Big)\rho_1(\mu)\,d\mu.$$

The function $I(\lambda)$ obtained here is the generating function for the values of the densities of the higher integrals in the ground state $|\Omega\rangle$. As in a normal antiferromagnet,[20] the excitations are described by a change in the distribution of the quantum numbers $J_i^{(k)}$. The simplest excitation

$$J_{j_0+1}^{(k)}=J_{j_0}^{(k)}+2,\quad \lambda_{j_0}^{(k)}=\lambda_0$$

leads, as $N\to\infty$, to an addition to (43) in the form of an inhomogeneous term $N^{-1}\delta(\lambda-\lambda_0)\delta_{lk}$, and for $\rho_{lk}(\lambda,\lambda_0)$ we have

$$\rho_{lk}(\lambda,\lambda_0)=\rho_l(\lambda)+N^{-1}R_{lk}(\lambda-\lambda_0), \tag{46}$$

where

$$R_{lk}(\lambda)=((1+K)^{-1})_{lk}(\lambda)$$

is the resolvent of the integral Eq. (43). The energy and momentum of such an excitation are the following:

$$\varepsilon_k(\lambda_0)=\int_{-\infty}^{+\infty}\theta'(2\lambda)R_{k1}(\lambda-\lambda_0)\,d\lambda, \tag{47}$$

$$p_k(\lambda_0)=\int_{-\infty}^{+\infty}\theta(2\lambda)R_{k1}(\lambda-\lambda_0)\,d\lambda. \tag{48}$$

There are thus $M-1$ excitation branches.

We now turn to the scattering of excitations. In the limit $N\to\infty$, $n_k\to\infty$ we are unable to study the coordinate representation of the wave function. However, the right-hand sides of the periodicity conditions, Eqs. (32), have the meaning of the scattering phase of a magnon of type l with momentum $\lambda_j^{(l)}$ by the other magnons. Proceeding as in Korepin's paper[8] we get for the phase shift of the scattering of the l-th excitation by the m-th

$$\varphi_{lm}(\lambda-\mu)=2\pi\int_0^{\lambda-\mu}(\delta_{lm}\delta(\nu)-R_{lm}(\nu))\,d\nu,$$

$$\varphi_{lm}(\lambda)=\varphi_{ml}(\lambda)=i\sum_{k=0}^{m-1}\ln\Big\{\Gamma\Big(1+\frac{2k-l-m}{2M}-i\frac{\lambda}{M}\Big)\Gamma\Big(\frac{l-m+2k}{2M}+i\frac{\lambda}{M}\Big)\Big/\Gamma\Big(1+\frac{2k-l-m}{2M}+i\frac{\lambda}{M}\Big)\Gamma\Big(\frac{l-m+2k}{2M}-i\frac{\lambda}{M}\Big)\Big\}. \tag{49}$$

For the corresponding S-matrix we can also construct a Zamolodchikov algebra[19]:

$$A_l(\lambda)A_m(\mu)=A_m(\mu)A_l(\lambda)\exp[i\varphi_{lm}(\lambda-\mu)]. \tag{50}$$

To do this we determine the operators $a_j(\lambda)$, $d_j(\lambda)$; $j=1,\dots,M-1$ and how they operate upon the state with a fixed set of quasi-momenta:

$$a_k(\lambda)|(\lambda_j^{(1)})\dots(\lambda_j^{(M-1)})\rangle=\prod_{j=1}^{n_{k-1}}a(\lambda-\lambda_j^{(k-1)})\prod_{j=1}^{n_k}\frac{1}{a(\lambda-\lambda_j^{(k)})}|(\lambda_j^{(1)})\dots(\lambda_j^{(M-1)})\rangle, \tag{51}$$

$$d_k(\lambda)|(\lambda_j^{(1)}),\dots,(\lambda_j^{(M-1)})\rangle=\prod_{j=1}^{n_k}\frac{1}{a(\lambda_j^{(k)}-\lambda)}|(\lambda_j^{(1)}),\dots,(\lambda_j^{(M-1)})\rangle. \tag{52}$$

The commutation relations between the operators creating the excitations $C(l,\lambda)$ and the operators $a_k(\mu)$ have the form

$$a_k(\mu-ik/2)C(l,\lambda)=\sigma_{k,l}(\mu-\lambda)C(l,\lambda)a_k(\mu-ik/2), \tag{53}$$

$$\ln\sigma_{k,l}(\mu)=\int_{-\infty}^{+\infty}\ln a\Big(\mu-\lambda-\frac{i}{2}\Big)R_{k-1\,l}(\lambda)\,d\lambda-\int_{-\infty}^{+\infty}R_{kl}(\lambda)\ln a(\mu-\lambda)\,d\lambda. \tag{54}$$

The generators of the Zamolodchikov algebra are the following operators:

$$A_l(\lambda)=C(l,\lambda)a_l^{-1}(\lambda-il/2)\langle\Omega|a_l(\lambda-il/2)|\Omega\rangle. \tag{55}$$

5. The formalism developed in the preceding sections can be naturally applied to multi-component systems. As an example we consider the chiral Gross-Neveu model. This is a relatively invariant field-theory model in a two-dimensional space-time with four-fermion interactions. The model is determined by the Langrangian

$$\mathscr{L}=\int dx\,(i\bar{\psi}\gamma^\mu\partial^\mu\psi+g((\bar{\psi}\psi)^2-(\bar{\psi}\gamma_5\psi)^2)).$$

The operators $\psi_{\alpha a}(x)$, $\psi^+_{\beta b}(y)$ have not only a spinor index $\alpha,\beta=1,2$, but also an isotopic (color) index $a,b=1,2,\dots,M$ and satisfy the anti-commutation relations

$$\{\psi_{\alpha a}(x),\psi_{\beta b}^+(y)\}=\delta_{\alpha\beta}\delta_{ab}\delta(x-y).$$

The Hamiltonian of this model

$$H=\int dx\,(i\psi_{1a}^+\partial_x\psi_{1a}-i\psi_{2a}^+\partial_x\psi_{2a}+4g\psi_{1a}^+\psi_{2b}^+\psi_{1b}\psi_{2a}) \tag{56}$$

was diagonalized in Refs. 14, 15 by means of the coordinate Bethe Ansatz for the case of the isotopic number of colors $M=2$ (Andrej and Lowenstein[21] have also considered the case of arbitrary M). The eigenstates of H are parametrized by the momenta $k\equiv(k_1, k_2, \ldots, k_N)$, the spirality $\sigma\equiv(\sigma_1, \sigma_2, \ldots, \sigma_N)$, where $\sigma_i=\pm1$, and the isotopic vector F which will be defined below. The eigenstates are constructed from the non-physical vacuum $|0\rangle: \psi_{\alpha a}|0\rangle=0$ and have the form

$$|k, \sigma, F\rangle=\int dx_1\ldots dx_N\, \chi^{a_1\ldots a_N}_{\alpha_1\ldots\alpha_N}(\{x_i\}; k, \sigma, F)\, \psi^+_{\alpha_1 a_1}(x_1)\ldots\psi^+_{\alpha_N a_N}(x_N)|0\rangle.$$

The wave function χ is a superposition of $N!$ plane waves

$$\chi=\sum_p F(p)\exp\left(i\sum_{i=1}^{N} x_i k_{p_i}\right)\prod_{i=1}^{N}\delta_{\alpha_i\sigma_{p_i}},$$

where p is a permutation of N numbers, p: $(1,2,\ldots,N)\to(p_1, p_2, \ldots, p_N)$, and the coefficients $F(p)$ describe the way χ depends on the isotopic indices and are connected with one another through the two-particle S-matrix.

$$F(p_1,\ldots,p_i,p_{i+1},\ldots,p_N)=P_{i\,i+1}S_{i\,i+1}F(p_1,\ldots,p_{i+1},p_i,\ldots,p_N).$$

The S-matrix depends only on the spiralities $\sigma_i=\pm1$ of the colliding particles:[14,15]

$$S_{kl}(\lambda)=\begin{cases}(a(\lambda)+b(\lambda)P_{kl})e^{i\lambda\varphi}, & \lambda=\pm2;\\ P_{kl}, & \lambda=0\end{cases} \tag{57}$$

$$a(\lambda)=\frac{\lambda}{\lambda+ic},\quad a(\lambda)+b(\lambda)=1,\quad \varphi=\operatorname{arctg}\frac{c}{2},\quad c=4g(1-g^2)^{-1},$$

P_{kl} is the permutation operator [see (3)].

Denoting the coefficient corresponding to the identity permutation simply by F we get from the periodicity condition for the wave function vector equations for the set of momenta k (L is the size of the section where the periodic problem is considered)

$$\exp(ik_jL)F=Z_jF. \tag{58}$$

The operators $Z_j=S_{j+1j}\ldots S_{Nj}S_{1j}\ldots S_{j-1j}$ are constructed from the two-particle S-matrices and they are, as was noted in Ref. 5, a particular value of the trace of the transition matrix for an inhomogeneous lattice of N sites (see section 2). Hence, the eigenvectors of Z_j can be constructed according to the proposed scheme for any M.

For finite L and N we shall have $M-1$ sets of quasi-momenta $\lambda^{j(l)}$, $j=1,2,\ldots,n_l$; $l=1,2,\ldots,M-1$ which satisfy the set of Eqs. (32) with small changes in the first equations:

$$k_jL=2\pi n_j-\varphi\sum_{i=1}^{N}(\sigma_j-\sigma_i)-\sum_{i=1}^{n_1}\theta\left(\frac{2\sigma_j}{c}-2\lambda_i^{(1)}\right),$$

$$2\pi J_j^{(1)}=-\sum_{i=1}^{N}\theta\left(2\lambda_j^{(1)}-\frac{2\sigma_i}{c}\right)+\sum_{i=1}^{n_1}\theta(\lambda_j^{(1)}-\lambda_i^{(1)})-\sum_{i=1}^{n_2}\theta(2\lambda_j^{(1)}-2\lambda_i^{(2)}),\ldots, \tag{59}$$

$$2\pi J_j^{(M-1)}=-\sum_{i=1}^{n_{M-2}}\theta(2\lambda_j^{(M-1)}-2\lambda_i^{(M-2)})+\sum_{i=1}^{n_{M-1}}\theta(\lambda_j^{(M-1)}-\lambda_i^{(M-1)}).$$

Using this set to determine (for given L, N, σ_j, n_j, $J_j^{(l)}$) the momenta k_j we find the energy and momentum of the state:

$$E=\sum_{j=1}^{N}\sigma_jk_j,\quad P=\sum_{j=1}^{N}k_j.$$

To get a Hamiltonian which is semi-bounded from below we introduce an ultra-violet cut-off Λ ($|k_i|\leq\Lambda$) and we define the physical vacuum as the state with the lowest energy with this cut-off. The number of fermions in the ground state will then be connected with Λ and L: $N\sim\Lambda L$. We are interested in the field-theoretic limit: $N, L, \Lambda\to\infty$. In that limit the ground state is described by the densities $\rho_l(\lambda)$ of the quasi-momentum $\lambda_j^{(l)}$ distribution. We can find this from the set of integral equations:

$$\rho_l(\lambda)+\int_{-\infty}^{+\infty}K_{lm}(\lambda-\mu)\rho_m(\mu)\,d\mu=\frac{\delta_{l1}}{2\pi}\left[\theta'\left(2\lambda-\frac{2}{c}\right)+\theta'\left(2\lambda+\frac{2}{c}\right)\right], \tag{60}$$

$$K_{lm}(\lambda)=\frac{1}{2\pi}[\theta'(\lambda)\delta_{lm}-\theta'(2\lambda)(\delta_{l\,m+1}+\delta_{l\,m-1})]. \tag{61}$$

As in the case of an antiferromagnet, one possible kind of excitation is connected with the variation of some set of numbers $J^{j(l)}$, and this leads in the limit considered to the appearance of a δ-function in the inhomogeneous term in Eq. (60). The difference between the quasi-momentum density in the excited and the ground state is, as before, determined by Eq. (46), the resolvent in which we can evaluate explicitly:

$$R_{lj}(\lambda)=R_{jl}(\lambda)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ik\lambda+|k|/2}\frac{\operatorname{sh}(k(M-j)/2)\operatorname{sh}(kl/2)}{\operatorname{sh}(kM/2)\operatorname{sh}(k/2)}\,dk,\quad j\geq l. \tag{62}$$

The energy and momentum of the excitation are determined as the difference between their values in the excited and the ground states:

$$\varepsilon_l(\lambda)=\frac{N}{L}\int_{-\infty}^{+\infty}\left[\theta\left(2\mu-\frac{2}{c}\right)-\theta\left(2\mu+\frac{2}{c}\right)\right]R_{l1}(\mu-\lambda)\,d\mu,$$

$$p_l(\lambda)=\frac{N}{L}\int_{-\infty}^{+\infty}\left[\theta\left(2\mu-\frac{2}{c}\right)+\theta\left(2\mu+\frac{2}{c}\right)\right]R_{l1}(\mu-\lambda)\,d\mu.$$

The model studied is asymptotically free, i.e., when we remove the cut-off $g\to0$ and in the spectrum of the model there appear massive excitations (mass generation):[14,15]

$$\varepsilon_l(\theta)=m_l\operatorname{ch}\theta,\quad p_l(\theta)=m_l\operatorname{sh}\theta,\quad \theta=2\pi\lambda/M, \tag{63}$$

$$m_l=m_0\frac{\sin(\pi l/M)}{\sin(\pi/M)},\quad m_0\sim\Lambda\exp\left(-\frac{2\pi}{cM}\right),\quad l=1,2,\ldots,M-1, \tag{64}$$

there is also a massless branch corresponding to non-isotopic excitations.

The phases for the scattering of the massive excitations with one another are, as before, given by Eq. (49):

$$\varphi_{lm}(\theta)=\varphi_{ml}(\theta)=i\ln\prod_{k=0}^{m-1}\left\{\Gamma\left(1+\frac{2k-l-m}{2M}-i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{l-m+2k}{2M}+i\frac{\theta}{2\pi}\right)\Big/\Gamma\left(1+\frac{2k-l-m}{2M}+i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{l-m+2k}{2M}-i\frac{\theta}{2\pi}\right)\right\}. \tag{65}$$

Here $\theta=\theta_1-\theta_2$ is the difference in the speeds of the colliding particles. When $l, m=1, M-1$ the answer is the same as the one obtained earlier[22,23] by the method of factorizing the S-matrix.

In conclusion it seems to us to be appropriate to enumerate relatively simple generalizations of the models considered to which one can apply completely the formalism developed above.

1. The generalized Heisenberg ferromagnet with fermions. The operator of the linear problem on the lattice will be

$$L_n(\lambda) = a(\lambda) + b(\lambda) P_n,$$

where P_n is the permutation operator in the Z_2-spaces $C^M \otimes C^M$. The first $M-1$ elements of the base in C^M are even and the last element is odd, i.e., the gauge is $(M-1, 1)$.

2. The non-linear matrix Schrödinger equation with gauge (see also Ref. 5)

$$i\psi_t = -\psi_{xx} - 2g\psi\psi^+\psi,$$

where ψ is an $n \times m$ matrix, all rows of which are even and a number p of columns is odd (i.e., those columns consist of anti-commuting elements). The R-matrix has the usual form (10), but P is the permutation operator in $C^M \otimes C^M$ where $M = n + m$, the gauge of the space is $(n+m-p, p)$. Apart from the initial momenta k_j, the eigenfunctions of the trace of the monodromy matrix are parametrized by two sequences of quasi-momenta.

3. The matrix generalizations of the Belavin[14] and Andrej and Lowenstein[15] models:

$$\mathscr{L} = i \,\mathrm{Sp}\, \bar{\psi}\gamma^\mu\partial_\mu\psi + \mathscr{L}_{int},$$

$$\mathscr{L}_{int} = \mathrm{Sp}\,\{g_1\bar{\psi}\gamma_\mu\psi\bar{\psi}\gamma_\mu\psi + g_2\bar{\psi}\psi\bar{\psi}\psi + g_3\bar{\psi}\gamma_5\psi\bar{\psi}\gamma_5\psi + g_4\bar{\psi}\gamma_\mu\lambda^\alpha\psi\bar{\psi}\gamma_\mu\lambda^\alpha\psi\},$$

where the $\psi(x,t)$ are spinors with respect to the Lorentz group in the $1+1$ space and are $n \times m$ matrices, and the matrices λ^α are generators of the fundamental representation of the $SU(n)$ group. The Lagrangian is invariant under the transformations $\psi \to U\psi V$, $U \in SU(n)$, $V \in SU(m)$. The conditions for the factorization of the two-particle S-matrix, guaranteeing the possibility of applying the formalism developed here, are fulfilled if one of the equations: 1) $g_1 = g_4 = 0$; 2) $g_2 = g_3 = g_1 = 0$; 3) $g_2 = g_3 = g_4 = 0$ is valid.

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1) We do not dwell on a description of the general scheme of the quantal inverse scattering method which has been expounded in detail in Faddeev's review.[1]

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