# The classical problem of conformal small-angle scattering

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(Submitted 25 June 1980)
Zh. Eksp. Teor. Fiz. **80**, 127–143 (January 1981)

In the small-angle case of classical scattering of high-energy particles or high-energy waves by a nonspherical scatterer, the mapping of the impact-parameter plane on the momentum-transfer plane is conformal, provided only that the scatterer is harmonic (the potential energy satisfies the Laplace equation). This makes it possible to use complex variables in both planes, thereby greatly simplifying the calculations. The rainbow lines (the singularities of the differential cross section) degenerate in this case into focal points. As a result, steplike singularities are preserved in the cross section after averaging over the orientations, and they can be observed in the scattering of protons by molecules or by the crystal surface. The extremal properties of conformal mapping lead to rigorous inequalities for the effective cross section for scattering by a system of Coulomb centers at small and relatively large momentum transfers. The approximation is applicable also to the problem of scattering of a charged particle by a magnetic scatterer, and can thus be used not only in atomic and nuclear problems, but also in electron optics.

PACS numbers: 03.80. + r, 34.10. + x

## 1. INTRODUCTION

Small-angle scattering plays an important role, from both the experimental and theoretical viewpoints, in such problems as scattering of particles by atoms, molecules, or nuclei, as well as in electron optics, in classical scattering of acoustic and electromagnetic waves, and others. On the one hand, the theory becomes greatly simplified and yields the scattering characteristics in the form of integrals. On the other hand, the intensity of scattering of fast particles decreases rapidly with increasing scattering angle, so that large-angle scattering is frequently difficult to observe. In classical scattering, all the trajectories are almost rectilinear inside the scatterer and we obtain directly the well-known result[1] that the product of the scattering angle by the incident-particle energy is a function only of the impact parameter, so that the number of parameters that characterize the scattering is decreased. In the wave approach, small-angle scattering corresponds to the eikonal approximation,[2,3] an important feature of which is that the unitarity relation is satisfied within the framework of this approximation, so that the characteristic and important features of the initial problems are preserved (in contrast from a rougher approach such as, e.g., the Born approximation).

The purpose of the present article is to point out that in the case of a large class of harmonic scatterers, when the potential energy satisfies the Laplace equation, small-angle scattering is closely related to conformal mapping on a plane. This makes it possible to introduce complex variables on the impact-parameter plane and on the momentum-transfer plane, after which all the calculations become exceedingly simple. Besides the simplification of the calculations for harmonic scatterers, it is possible to formulate certain rigorous theorems that permit estimates of classical scattering at small and large momentum transfers, as well as to elucidate the nature of the singularities in the classical differential effective cross section. It turns out, in particular, that rainbow lines are impossible for harmonic scatterers, and only focal points exist.

Harmonic potentials other than the Coulomb potential have no spherical symmetry. The real problems that can be treated by this method are the scattering of charged particles (e.g., protons) by molecules or nonspherical nuclei, when the particle does not penetrate into the molecule or nucleus during the scattering and is deflected in this case by a small angle. The time of flight must be short compared with the time of rotation of the scatterer, so that its rotation during the collision time can be neglected. In addition, the collision should be elastic or almost elastic.

Owing to the presence of focal points, the singularities of the differential cross section do not vanish after averaging over the scatterer orientations, so that the cross section is steplike, in analogy with the atmospheric-halo effect in scattering of light by ice crystals.

Complex variables can be used also in the analysis of the momentum transferred to the scatterer. Here, too, the rainbow lines are replaced by focal points, and this can lead to experimentally observable effects when rotational states of the molecule are excited. In the scattering of ions by a crystal surface it is possible to single out those scattered particles that collide with only two crystal atoms. The results obtained in this case allow us to predict the conditions for grazing scattering, when the reflection coefficient for double scattering is greatly increased as a result of focusing.

The problem of small-angle scattering by an arbitrary combination of electric and magnetic field can also be solved by this method if the trajectory does not pass through a region where field sources (charges and currents) are located.

For the simplest harmonic potential (a single Coulomb center) the scattering is conformal for arbitrary angles (not only small ones) and is connected with stereographic projection.

The introduction of complex variables uncovers new possibilities in various generalizations of the problem of scattering by a harmonic potential, and in particular

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for inelastic scattering and for elastic scattering in the eikonal approximation.

## 2. CLASSICAL HARMONIC SCATTERING. BASIC FORMULAS

We consider the scattering of particles moving along the $z$ axis with momentum $p_0$ by a force center with potential $U(x, y, z)$. We set the particle mass equal to unity. Then, assuming that the transverse momentum transfer $\mathbf{p}_\perp(p_x, p_y)$ is small compared with $p_0$, we obtain

$$\mathbf{p}_\perp=-\int_{-\infty}^{+\infty}\nabla_\perp U\,dt=-p_0^{-1}\nabla_\perp V(x,y),\quad V(x,y)=\int_{-\infty}^{+\infty}U(x,y,z)\,dz. \qquad (1)$$

We introduce the impact-parameter plane $\mathbf{b}(x, y)$. Then the classical small-angle classical can be regarded as a local single-valued mapping of the impact-parameter plane on the momentum-transfer plane. The inverse mapping is not as a rule unique, since cases in which several values of the impact parameter $\mathbf{b}(x, y)$ correspond to the same momentum transfer are perfectly possible, so that the $\mathbf{p}_\perp$ plane is covered several times, and certain values of the momentum transfer may not be reached at all regardless of the value of $\mathbf{b}$, i.e., there can exist regions of $\mathbf{p}_\perp$ that are not covered at all.

The differential effective cross section is defined as the Jacobian of the transformation

$$\sigma(p_x,p_y)=p_0^2\sum\left|\frac{\partial(x,y)}{\partial(p_x,p_y)}\right|,$$

where the summation is over all the values of the multiply valued function $\mathbf{b}(\mathbf{p}_\perp)$ (it can be "zero-valued" for certain $\mathbf{p}_\perp$). In other words, the effective cross section is defined as the ratio of the area elements on the plane $\mathbf{b}$ and the plane $\mathbf{p}_\perp$.

In the general case, a round area element on the b plane is mapped into an elliptic area on the $\mathbf{p}_\perp$ plane. The lines along which the minor semi-axis of this ellipse vanishes are called rainbow lines. All the particles that pass through this area are mapped on an area of higher order of smallness on the momentum-transfer plane, and consequently the effective cross section becomes infinite along these lines. The rainbow lines are demarcation lines, on the $\mathbf{p}_\perp$ plane, between regions having different covering multiplicities. At least one of the values of the Jacobian $\partial(x, y)/\partial(p_x, p_y)$ becomes infinite on these lines. The mapping of the rainbow lines on the impact-parameter plane is defined by the equation $\partial(p_x, p_y)/\partial(x, y)=0$. For a spherically symmetrical potential the scattering is axisymmetric and the rainbow lines are circles on the impact-parameter plane and on the momentum-transfer plane with a center at the origin, and are defined by the equation $dp_\perp/db=0$.

We assume further that the potential $U(x, y, z)$ and hence the potential $V(x, y)$ satisfy respectively the three-dimensional and two-dimensional Laplace equations. The field sources can be point charges, point multipoles, or any spatial charge distribution. We shall show that for such (harmonic) scatterers the rainbow lines degenerate into points that can be named focal points. In fact, the Jacobian of the transformation

$$\frac{\partial(p_x,p_y)}{\partial(x,y)}=\frac{\partial p_x}{\partial x}\frac{\partial p_y}{\partial y}-\frac{\partial p_x}{\partial y}\frac{\partial p_y}{\partial x}$$
$$=-\frac{1}{p_0}\left[\frac{\partial^2V}{\partial x^2}\frac{\partial^2V}{\partial y^2}-\left(\frac{\partial^2V}{\partial x\,\partial y}\right)^2\right]$$

reduces to the so-called Hess determinant, and when account is taken of the Laplace equation the rainbow condition (the vanishing of the Jacobian) takes the form

$$\left(\frac{\partial^2V}{\partial x^2}\right)^2+\left(\frac{\partial^2V}{\partial x\,\partial y}\right)^2=0.$$

Instead of one condition that defines the rainbow line on a plane, we obtain thus two conditions that define in the general case only points on the impact-parameter plane and on the momentum-transfer plane (focal points).

It follows from (1) that the momentum transfer is proportional to the field intensity, so that if $V$ satisfies the Laplace equation in the two-dimensional electrostatic problem, the mapping from $\mathbf{b}$ to $\mathbf{p}_\perp$ is conformal. This leads directly to the impossibility of obtaining rainbow lines, inasmuch as in the vicinity of such a line the mapping is essentially not conformal and small circles on the b plane are mapped as ellipses with vanishingly small semiaxes on the $\mathbf{p}_\perp$ plane, which is perpendicular to the rainbow line.

It is most natural to describe planar conformal mapping with the aid of complex variables. We introduce for this purpose a complex impact parameter and a complex momentum transfer

$$b=x+iy,\quad p=p_x+ip_y$$

as well as a complex potential $V(b)$. We then obtain the following formulas for the momentum transfer, the focal points, and the effective cross section

$$p^*=-p_0^{-1}\frac{dV}{db},\quad \frac{d^2V}{db^2}=0,\quad \sigma=p_0^2\sum\left|\frac{db}{dp}\right|^2,$$

where the summation is over all the values of the multiply valued function $db/dp$.

We consider a system of $N$ point charges $q_j$ located at the points $\mathbf{r}_j(x_j, y_j, z_j)$ $(j=1, 2, \ldots, N)$. The disposition of the charges along the $z$ axis (the values of $z_j$) is of no importance in the calculation of the momentum transfer, and affects only the region in which the approximation is valid. We obtain

$$V=-q_0\sum_{j=1}^{N}q_j\ln(b-b_j),\quad p^*=\frac{q_0}{p_0}\sum_{j=1}^{N}\frac{q_j}{b-b_j},\quad b_j=x_j+iy_j \qquad (2)$$

$q_0/2$ is the charge of the incident particle.

It is easily seen that the multiplicity of the mapping of the $b$ plane on the $p$ plane is equal in this case to the number of $N$ of the charges. Indeed, at sufficiently large momentum transfers we can find in the vicinity of each charge (at $b\approx b_j$) one point each corresponding to a given momentum. Since there are no rainbow lines, the multiplicity of the mapping is the same on the entire plane, and by the same token our statement is proved.

The focal points on the $b$ plane are the roots of the equation

$$\sum_{j=1}^{N}\frac{q_j}{(b-b_j)^2}=0. \qquad (3)$$

After reducing to a common denominator, we obtain an equation of degree $2N-2$, which has according to the Gauss theorem $2N-2$ solutions $b_1^f, b_2^f, \ldots, b_{2N-2}^f$. Some of these can coalesce or go off to infinity. To these points there correspond on the $p$ plane $2N-2$ focal momenta $p_1^f, \ldots, p_{2N-2}^f$, at which the effective cross section is singular. In the vicinity of these points we have $(p-p^f)^* \sim (b-b^f)^2$, and consequently, a circuit around the point $b^f$ on the $b$ plane corresponds to two circuits around $p^f$ on the $p$ plane. In other words, the points $p^f$ are double-valued branch points of the multiply valued function $b(p)$.

The effective cross section

$$\sigma \sim |db/dp|^2 \sim |(p-p^f)^{-1/2}|^2 = |p-p^f|^{-1}$$

has a simple pole at each simple focal point. When $n$ points coalesce, we obtain $(p-p_f)^* \sim (b-b^f)^{n+1}$; we have an $(n+1)$-valued branch point and

$$\sigma \sim |p-p^f|^{-2n/(n+1)},$$

i.e., the singularity becomes stronger and tends to $\sigma \sim |p-p^f|^{-2}$ as $n \to \infty$.

From the point of view of the electrostatic analogy, the electrostatic field is uniform in the vicinity of the focal points, the quadratic terms vanish from the series expansion of the potential, and the curvatures of the field lines and of the equipotential lines vanish. By the same token, all particles that land in the vicinity of this point acquire equal momentum transfers, and it is this which leads to the singularity of the cross section.

## 3. EXAMPLES

We consider first scattering by pointlike $2^n$-pole. Then

$$p^* = \frac{q_0 Q_n}{p_0 b^{n+1}}, \quad b = \left(\frac{q_0 Q_n}{p_0 p^*}\right)^{1/(n+1)},$$

where $Q_n$ is a complex number that characterizes the orientation of the two-dimensional multipole. $Q_0$ (point charge) is a real number at $n=0$. It is obvious that the mapping of the $b$ plane on $p$ has a multiplicity $n+1$. For the effective cross section we obtain

$$\sigma_n = p_0^2 \sum \left|\frac{db}{dp}\right|^2 = \frac{p_0^2}{(n+1)|p|^2}\left|\frac{q_0 Q_n}{p_0 p}\right|^{2/(n+1)}. \tag{4}$$

In particular, for scattering by a point charge and a point dipole ($n=0$ or 1) we obtain

$$\sigma_R = \sigma_0 = \frac{q_0^2 Q_0^2}{|p|^4}, \quad \sigma_1 = \frac{p_0 q_0 |Q_1|}{2|p|^3}. \tag{5}$$

The first equation in (5) is the Rutherford formula. It is interesting that in the classical approximation the differential cross section for scattering by a multipole is axisymmetric even though the potential has no axial symmetry. This symmetry is violated in the quantum approximation.

We consider next the problem of scattering by two Coulomb centers $q_1 = q\sin^2(\alpha/2)$ and $q_2 = q\cos^2(\alpha/2)$, whose projections on the impact-parameter planes are located at the points $b_1 = R$ and $b_2 = -R$. This problem was considered earlier in Ref. 4. Then

$$p^* = \frac{qq_0}{p_0}\left(\frac{\sin^2(\alpha/2)}{b-R} + \frac{\cos^2(\alpha/2)}{b+R}\right). \tag{6}$$

Equation (3) for the focal points takes the form

$$\frac{\sin^2(\alpha/2)}{(b^f-R)^2} + \frac{\cos^2(\alpha/2)}{(b^f+R)^2} = 0, \quad b_1^f = Re^{i\alpha}, \quad b_2^f = Re^{-i\alpha}.$$

The focal points on the $b$ plane are thus located on a circle whose diameter is the segment joing both centers, are symmetric about this segment, and are closer to the smaller charge. If the charges are of opposite signs, then $\alpha$ is imaginary and the focal points lie on the straight line joining the center, one inside and the other outside the circle, and inversion with respect to the circle transforms one point into the other. If $q_1 = -q_2$, then one point goes off to infinity and the other goes to the center of the circle (to the origin).

The corresponding momentum-transfer values at which the effective cross section becomes infinite are

$$p_1^f = \frac{qq_0}{2p_0 R}e^{i\alpha}, \quad p_2^f = \frac{qq_0}{2p_0 R}e^{-i\alpha}.$$

To determine the differential effective cross section we solve Eq. (6) with respect to $b$:

$$b = R\left\{\left(\frac{p_1^f p_2^f}{p^{*2}}\right)^{1/2} \pm \left[\left(\frac{p_1^f}{p^*}-1\right)\left(\frac{p_2^f}{p^*}-1\right)\right]^{1/2}\right\},$$

differentiate with respect to $p^*$, and calculate the sum of the squares of the moduli of both derivatives. We get

$$\sigma = \frac{q^2 q_0^2}{2|p|^4}\left[1 + \frac{1}{4}\left|2 + \frac{(p^f/p_1^f)-1}{(p/p_2^f)-1} + \frac{(p/p_2^f)-1}{(p/p_1^f)-1}\right|\right].$$

If the momentum transfer is small compared with $|p^f|$, the the cross section is equal to $q^2 q_0^2 |p|^{-4}$, i.e., we get Rutherford scattering by the summary charge $q$. If $|p| \gg |p^f|$, then the cross section is

$$q^2 q_0^2 |p|^{-4}[\sin^4(\alpha/2) + \cos^4(\alpha/2)]$$

which is the sum of the Rutherford cross sections for each of the charges.

For like charges $q_1 = q_2 = q/2$ we obtain

$$\sigma = \frac{q^2 q_0^2}{2|p|^4}\left(1 + \left|\frac{4R^2 p_0^2}{q^2 q_0^2}p^2 + 1\right|^{-1}\right).$$

For unlike charges $q_1 = -q_2 = q/2$

$$\sigma = \frac{p_0 q_0 q R}{2|p|^3}\left|\frac{Rp_0}{qq_0}p + 1\right|^{-1}.$$

In the latter case we obtain in the limit of small $p$ scattering by the dipole (5) with dipole moment $Q_1 = qR$.

The third example considered is that of $N$ Coulomb centers arranged on the $b$ plane at the vertices of a regular polygon inscribed in a circle of radius $R$. Then

$$q_1 = q_2 = \ldots = q_N = q/N, \quad b_j = R\exp(2\pi ij/N), \quad j = 1, 2, \ldots, N,$$

$$p^* = \frac{q_0 q}{p_0 N}\sum_{j=1}^{N}\left[b - R\exp\left(\frac{2\pi ij}{N}\right)\right]^{-1} = \frac{q_0 q b^{N-1}}{p_0(b^N - R^N)}.$$

The equation for the focal points is of the form

$$(b^f)^{N-2}[(b^f)^N + R^N(N-1)] = 0,$$

from which it is seen that $N-2$ focal point coalesce at $b=0$, and $N$ points

$$b_j^f=R(N-1)^{1/N}\exp[2\pi i(j+{}^1/_2)/N]$$

form a regular polygon with vertices lying halfway between the Coulomb centers. The region $b\approx 0$ contributes to the cross-section singularity at small momentum transfers. In addition, we obtain $N$ focal points at

$$p_j^f=\frac{q_0q(N-1)}{p_0R(N-2)}(N-1)^{-1/N}\exp\left[\frac{2\pi i(j+{}^1/_2)}{N}\right].$$

As $N\to\infty$ we obtain in the limit $p=0$ at $|b|<R$ and $p^*=q_0q/p_0p$ at $|b|>R$, i.e., the whole interior of the circle will not scatter any particles, while the exterior will scatter in the same manner as a point charge at the center of the circle.

## 4. AVERAGING OVER THE ORIENTATIONS

In real problems, the scatterers are frequently randomly oriented relative to the directions of the incident particles. Then the observed quantity is an averaged effective cross section that has axial symmetry and depends only on the modulus $|p|$ of the momentum transfer. It is reasonable to carry out the averaging in two stages—first over the rotations of the system about the $z$ axis, which is equivalent to averaging over the argument of the compiled momentum $p=|p|\exp(i\varphi)$, and then over the inclinations of the axis of the scatterer relative to the $z$ axis—over the angle $\theta$. (For systems that have no axial symmetry it is necessary to average also over a third, Euler angle $\psi$—rotation about the scatterer axis.) We have

$$\langle\sigma\rangle_\varphi=\frac{1}{2\pi}\int_0^{2\pi}\sigma\,d\varphi,\quad \langle\sigma\rangle=\langle\sigma\rangle_{\theta,\varphi}=\frac{1}{2}\int_0^{\pi}\langle\sigma\rangle_\varphi\sin\theta\,d\theta.$$

We consider first averaging in scattering by an axisymmetric multipole. Under all possible rotations in three-dimensional space, the projection $Q_n$ of the multipole $Q_n^0$ on the $(x,y)$ plane behaves like $Q_n^0(\sin\theta)^n$. In this case the non-averaged effective cross section is already axisymmetric, so that averaging over the azimuthal angle $\varphi$ is unnecessary. The additional factor $c_n$ takes upon averaging over $\theta$, the form

$$c_n=\frac{1}{2}\int_0^{\pi}(\sin\theta)^{2n/(n+1)+1}d\theta=\frac{1}{2}B\left(\frac{1}{2},\,2-\frac{1}{n+1}\right),$$

where $B$ is an Euler integral of the first kind. Using (4), we obtain

$$\langle\sigma_n\rangle=\frac{\sqrt{\pi}\,\Gamma(2-1/(n+1))}{2\Gamma(2+{}^1/_2-1/(n+1))}\frac{p_0^2}{(n+1)|p|^2}\left|\frac{q_0Q_n^0}{p_0p}\right|^{2/(n+1)}.$$

The coefficients $c_n$ vary in small ranges about $c_0=1$, when the averaging, obviously changes nothing; next, $c_1=\pi/4=0.785$, $c_2=0.740$, $c_3=0.719$, . . . and as $n\to\infty$ we have $c_\infty=\frac{2}{3}$. The averaging for the other multipole components, which transform under rotation like products of mutually orthogonal unit vectors, is more complicated and will not be considered here.

We carry out next the averaging for the problem of two Coulomb centers, and confine ourselves to the cases of two unlike and two like charges. We denote by $R_0$ the half-distance between the charges (then $R=R_0\sin\theta$) and introduce the dimensionless parameters $s=Rp_0|p|/qq_0$ and $s_0=R_0p_0|p|qq_0$. We then have for the unlike charges

$$\langle\sigma\rangle_\varphi=\frac{q_0^2q^2}{2|p|^4}\frac{1}{2\pi}\int_0^{2\pi}(1+s^2-2s\cos\varphi)^{-1/2}s\,d\varphi$$

$$=\frac{q_0^2q^2}{|p|^4}\frac{s}{\pi(1+s)}K\left(\frac{2s^{1/2}}{1+s}\right), \tag{7}$$

where $K$ is a complete elliptic integral of the first kind. Averaging over the inclinations yields

$$\langle\sigma\rangle=\frac{q_0^2q^2}{|p|^4}\frac{2}{\pi}\int_0^{\pi/2}K\left[\frac{2(s_0\sin\theta)^{1/2}}{1+s_0\sin\theta}\right]\frac{s_0\sin^2\theta}{1+s_0\sin\theta}d\theta=\sigma_RD(s_0). \tag{8}$$

We see that averaging over $\varphi$ and over $\varphi$ and $\theta$ yields a Rutherford cross section multiplied by a certain form factor, which is a function that depends on one dimensionless parameter. The argument of the function $K$ in expressions (7) and (8) is always less than or equal to unity. Equality is reached at $|p|=|p'|$, in which case the function $K$, and hence $\langle\sigma\rangle_\varphi$, diverges logarithmically.

We obtain similarly for two like Coulomb centers

$$\langle\sigma\rangle_\varphi=\sigma_R\left[\frac{1}{2}+\frac{1}{\pi(1+4s^2)}K\left(\frac{4s}{1+4s^2}\right)\right],$$

$$\langle\sigma\rangle=\sigma_R\left[\frac{1}{2}+\frac{1}{2\pi}\int_0^{\pi}K\left(\frac{4s_0\sin\theta}{1+4s_0^2\sin^2\theta}\right)\frac{\sin\theta\,d\theta}{1+4s_0^2\sin^2\theta}\right],$$

and we again obtain for the Coulomb scattering form factors that depend on one dimensionless parameter.

Let us see now what remains of the focal singularity of the effective cross section after averaging over the scatterer orientations. We can use for this purpose an electrostatic analogy. In fact, the singularities of the cross section at the focal points on the $p$ plane (simple poles) are analogous to the singularities of the three-dimensional potential of pointlike charges located on a plane. Averaging over $\varphi$ means the smearing of the point charges over a circle on the $p$ plane with center at the origin. The singularity in the cross section $\langle\sigma\rangle_\varphi$ is then logarithmic—the same as that of a potential of a uniformly charged ring. It was precisely this result which was obtained in the preceding examples.

Averaging over the inclinations changes $|p^f|$, and for the considered cases we have $|p^f|\sim(\sin\theta)^{-1}$, i.e., $|p^f|$ is minimal for a transverse placement of the scatterer ($\theta=\pi/2$), and $|p^f|$ increases with decreasing $\theta$. Averaging of the singular part of the potential over the inclinations is equivalent to calculation of the potential of an axisymmetric charge distribution on the $p$ plane, with a charge density equal to zero at $p<p_0^f$ and behaving like $(p-p_0^f)^{-1/2}$ at $p>p_0^f$ and $p\to p_0^f$. But this is precisely the charge distribution on the edge of a charged conducting half-plane, and consequently, after averaging over the inclinations, the cross section, just as the potential of a charged half-plane, is smoothly varying at $p>p_0^f$ and varies in proportion to $-(p_0^f-p)^{1/2}$ at $p<p_0^f$, remaining continuous at $p=p_0^f$. It is obvious that the coefficient of the singular part of the cross section is proportional to the residue $g$ of the non-averaged effective cross section at the point $p_0^f$ when the

scatterer is transversely placed, i.e., at the minimum value $p^f = p_0^f$.

We calculate now the relation between these quantities, assuming $p^f = p_0^f/\sin\theta$. We obtain

$$\sigma_{sing} = \frac{g}{|p-p^f|},$$

$$\langle\sigma_{sing}\rangle = \frac{g}{4\pi}\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi}\left(p^2 + \frac{p_0^{f2}}{\sin^2\theta} - \frac{2pp_0^f}{\sin\theta}\cos\varphi\right)^{-1/2} d\varphi.$$

Changing from integration with respect to $\theta$ to integration with $p^f$ we get

$$\langle\sigma_{sing}\rangle = \frac{g}{2\pi}\int_{p_0^f}^{\infty}\frac{p_0^{f2}p^f dp^f}{p^{f3}(p^{f2}-p_0^{f2})^{1/2}}\int_0^{2\pi}(p^2+p^{f2}-2pp^f\cos\varphi)^{-1/2}\, d\varphi.$$

We see that at $p^f \gtrsim p_0^f$ the equivalent charge density behaves like $(g/\pi)(2p_0^f)^{-3/2}(p^f - p_0^f)^{-1/2}$. From consideration of the potential $\Phi$ of a charged half-plane $x > 0$, $-\infty < y < +\infty$, with a surface charge density $\tau = Ax^{-1/2}$, we obtain $\Phi = -4\pi A(-x)^{1/2}$ at $x < 0$ and $\Phi = 0$ at $x > 0$. Hence

$$\langle\sigma_{sing}\rangle = -g\left(2\frac{p_0^f - p}{p_0^{f3}}\right)^{1/2}, \quad p < p_0^f;$$
$$\langle\sigma_{sing}\rangle = 0, \quad p > p_0^f. \tag{9}$$

Against the background of the decrease with increasing $p$, which is typical of the slowly varying part of the effective cross section, the focal point yields after averaging a peculiar "tooth" with a vertical tangent on the smaller-momentum side (Fig. 1). Formulas (9) are exact for any linear distribution of the charges. All that changes for the projection on the $b$ plane in the case of inclinations is the scale, and the relation (9) remains in force. In the more general case of an axisymmetric scatterer, however, this result is typical. Only the dimensionless coefficient of formula (9) can change.

Thus, even after averaging over the orientations, the cross section $\langle\sigma\rangle$ retains a perfectly noticeable singularity that can be observed in experiment in the corresponding problems. It is clear that it is precisely because of the presence of the focal points (of the stronger singularities) for the harmonic scatterers that the effect remains noticeable also after the averaging. The rainbow lines typical of anharmonic nonspherical scatterers, maxima remain after the averaging and are much more smeared out, more difficult to observe, and more difficult to identify with the focusing phenomenon.

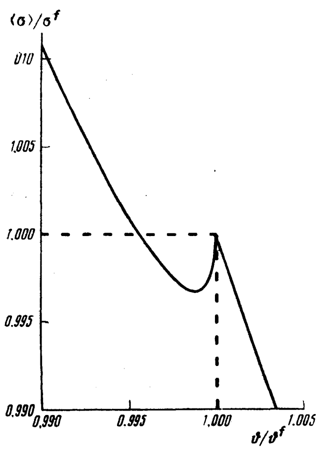


FIG. 1. Halo effect for ion scattering by dipole molecules. Cross section, averaged over radom orientations of the molecules, in the vicinity of the halo angle $\vartheta^f$.

In analogy with atmospheric optics, we can name this maximum of the averaged cross section $\langle\sigma\rangle$ the halo effect. In the atmosphere this effect is due to the refraction of light by randomly oriented ice crystals with 60° angle between the refracting planes. Symmetrical passage of a light ray corresponds to minimum deflection of the ray and, after averaging, to maximum intensity, with an abrupt decrease of the intensity at small angles and with a smoother decrease at larger angles.

The focal points can be treated classically if the change of phase is large for the different trajectories in the impact-parameter region, where the singular part of the effective cross section remains noticeable above the background of the smooth part of the cross section, or, equivalently, the change of action is large compared with Planck's constant. On going over to the wave treatment, the singularity in the nonaveraged cross section and the jump of the derivative of the averaged cross section with respect to the scattering angle are smeared out on the halo circle. Interference maxima and minima appear in the non-averaged cross sections, and their traces may remain in the vicinity of the maximum also after the averaging.

In real physical problems, which will be discussed below, the conditions for the applicability of the classical treatment of scattering are frequently satisfied in a wide range of the parameters of the problems.

## 5. LARGE AND SMALL MOMENTUM TRANSFERS AND EXTREMAL CHARACTER OF CONFORMAL TRANSFORMATION INTO A CIRCLE

We return now to the problem of scattering by $N$ Coulomb centers and consider first small momentum transfers and large impact parameters. We expand formula (2) for the momentum transfer in a series in the multipoles

$$p^* = \frac{q_0}{p_0}\sum_{k=0}^{\infty}\frac{Q_k}{b^{k+1}}, \quad Q_k = \sum_{j=1}^{N} q_j b_j^k. \tag{10}$$

If the first $n$ values $Q_0, \ldots, Q_{n-1}$ vanish, then this property, as well as the value of the multipole $Q_n$, does not depend on the choice of the origin on the $b$ plane. We can then choose the origin such that the multipole $Q_{n+1}$ is zero. We put $Q_0 \neq 0$ and $Q_1 = 0$, and invert the series (10):

$$b = \frac{q_0Q_0}{p_0p^*} + \frac{p_0Q_2}{q_0Q_0^2}p^* + \frac{p_0^2Q_3}{q_0^2Q_0^3}p^{*2} + \frac{p_0^3(Q_4Q_0 - 2Q_2^2)}{q_0^3Q_0^5}p^{*3} + \ldots \tag{11}$$

We calculate now that part of the effective cross section $\langle\sigma\rangle_\varphi^e$, averaged over the azimuthal angle, which corresponds to long distances. We differentiate the series (11) with respect to $p^*$, put $p = |p|\exp(i\varphi)$, square the modulus, and integrate with respect to $\varphi$.

All the cross terms vanish in this case and we get

$$\langle\sigma\rangle_\varphi^e=\frac{q_0^2Q_0^2}{|p|^4}+\frac{|Q_2|^2p_0^4}{q_0^2Q_0^4}+4\frac{|Q_3|^2p_0^6}{q_0^4Q_0^6}|p|^2+9\frac{|Q_4Q_0-2Q_2^2|^2p_0^8}{q_0^6Q_0^{10}}|p|^4\ldots \tag{12}$$

namely a series that consists only of positive terms, so that by retaining in it a finite number of terms we obtain a lower bound for the cross section $\langle\sigma\rangle_\varphi$. The first term of the series does not depend at all on the orientation of the system of Coulomb centers, so that we arrive at the following theorem: The classical effective small-angle scattering cross section, averaged over the azimuthal angle as well as over the various orientations and even over the various configurations of the harmonic scatterer, is always larger than the Rutherford cross section for a summary point charge.

The theorem is valid so long as the series (11) converges, i.e., so long as $|p|$ does not exceed the minimal focal momentum $p^f_{\min}$ for a given assembly of scatterer orientations and configurations, as well as, of course, so long as the small-angle scattering approximation is applicable. We note that in the expansion (12) in even powers of $|p|$ there is no $|p|^{-2}$ term, so that at small $|p|$ the correction to the Rutherford cross section is small.

In addition to long flights, parts are played in the small-angle scattering by the vicinities of those "equilibrium" values $b=b^g$ for which the momentum transfer vanishes, i.e., by the roots of the equation

$$\sum_{j=1}^{N}\frac{q_j}{b^g-b_j}=0.$$

For $N$ Coulomb centers there are in the general case $N-1$ such points. In analogy with scattering by a central field, we can call these points on the $b$ plane glory points. If the glory points do not coalesce, then in the vicinity of each of them

$$p=g_m(b-b_m^g)+O(|b-b_m^g|^2),\quad m=1,2,\ldots,N-1.$$

The contribution of these points to the effective cross section yields

$$\langle\sigma\rangle_\varphi^g=p_0^2\sum_{m=1}^{N-1}|g_m|^{-2}+Q(|p|^2),$$

so that there are added to the series (12), starting with the second term, also terms connected with the glory effect. In the semiclassical approximation the corresponding amplitudes will interfere, with account taken of the phase shifts of each of them. If $n$ glory points coalesce, then the glory points and the focal points coincide, and an additional singular contribution $\sigma^g\sim|p|^{-2+2/n}$ appears in the small-angle scattering cross section. It is easy to write out also the succeeding terms of this expansion.

If the summary charge $Q_0$ is equal to zero, but the dipole moment $Q_1$ is nonzero, then for the case $Q_0=Q_1=0$ and $Q_2\neq 0$ we obtain correspondingly

$$b=\frac{q_0^{1/2}Q_1^{1/2}}{p_0^{1/2}p^{*1/2}}+\frac{1}{2}\frac{Q_3p_0^{1/2}}{q_0^{1/2}Q_1^{3/2}}p^{*1/2}+\frac{1}{2}\frac{Q_4p_0}{q_0Q_1^2}p^*+O(p^{3/2}),$$

$$b=\frac{q_0^{1/3}Q_2^{1/3}}{p_0^{1/3}p^{*1/3}}+\frac{1}{3}\frac{Q_4p_0^{1/3}}{q_0^{1/3}Q_2^{4/3}}p^{*1/3}+\frac{1}{3}\frac{Q_5p_0^{2/3}}{q_0^{2/3}Q_2^{5/3}}p^{*2/3}+O(p).$$

In these cases the dependence of $b$ on $p$ is doubly and triple-valued, respectively.

On circuiting around the point $p=0$, some values go over into others, so that in the averaging over the azimuthal angle we can, instead of summing over the different values of $|db/dp|^2$, simply extend the integration limit from zero to $4\pi$ and to $6\pi$, respectively. This ensures again the vanishing of the cross terms and we obtain for $\langle\sigma\rangle_\varphi^e$ series of positive terms:

$$\langle\sigma\rangle_\varphi^e=\frac{q_0|Q_1|p_0}{|p|^3}+\frac{|Q_3^2|p_0^3}{8q_0|Q_1^3||p|}+\frac{|Q_4^2|p_0^4}{2q_0^2|Q_1^4|}+O(|p|),$$

$$\langle\sigma\rangle_\varphi^e=\frac{q_0^{2/3}|Q_2^{2/3}|p_0^{4/3}}{3|p|^{8/3}}+\frac{|Q_4^2|p_0^{8/3}}{27q_0^{2/3}|Q_0^{8/3}|}|p|^{-4/3}+\frac{4|Q_5^2|p_0^{10/3}}{27q_0^{4/3}|Q_2^{10/3}|}|p|^{-2/3}+\ldots.$$

This result can be easily generalized to the case when the first $n$ multipoles vanish. The averaged cross section always turns out to be larger than the scattering by the first nonvanishing pointlike equivalent multipole. The larger $n$, the better the convergence of the corresponding series, since the expansion is in terms of ever decreasing fractional powers of $|p|^{2/(n+1)}$.

We proceed now to the case of large momentum transfers. Obviously, we assume here as before that all the momentum transfers to be much less than $p_0$; the concept "large" means simply that in this case the particle passes near one of the Coulomb centers $q_j$ at a distance much smaller than the projections of the distances from $q_j$ to the other Coulomb centers. In the vicinity of the point $b_j$ the momentum transfer is determined primarily by the center $q_j$, and then by the smooth part of the field produced by all the remaining centers. Then

$$p^*=\frac{q_0}{p_0}\left[\frac{q_j}{b-b_j}+A_j+B_j(b-b_j)+\ldots\right],\quad j=1,2,\ldots,N,$$

$$A_j=\sum_{k\neq j}^{N}\frac{q_k}{b_j-b_k},\quad B_j=-\frac{1}{2}\sum_{k\neq j}^{N}\frac{q_k}{(b_j-b_k)^2}.$$

Inverting the series, we obtain an expansion in the inverse powers of $p^*$

$$b=b_j+\frac{q_0q_j}{p_0p^*}+\frac{q_0^2q_jA_j}{p_0^2p^{*2}}+\frac{q_0^3(q_jA_j^2+q_j^2B_j)}{p_0^3p^{*3}}+\ldots$$

and, integrating $\sigma$ up to $\varphi$, we obtain the averaged cross section

$$\langle\sigma\rangle_\varphi=\frac{q_0^2}{|p|^4}\sum_{j=1}^{N}q_j^2+\frac{q_0^4}{p_0^2|p|^6}\sum_{j=1}^{N}q_j^2|A_j|^2+\frac{q_0^6}{p_0^4|p|^8}\sum_{j=1}^{N}|q_j^2A_j^2+q_jB_j|^2+O(|p|^{-10}).$$

We have thus obtained again a series consisting of positive terms, the first of which corresponds to the sum of the cross sections for scattering by each of the centers and does not depend on their arrangement. We arrive at the following theorem: For small-angle scattering by a system of Coulomb centers, at momentum transfers larger than the largest of the focal momenta $p^f_{\max}$ (the condition for the convergence of the series), the classical cross section, averaged over the azimuthal angle, is always larger than the sum of the cross sections for scattering by each of the Coulomb centers.

This result remains valid also after additional averaging over different configurations of the projections of the centers on the plane $b$, connected, for example, with the vibrations of the corresponding molecule. Unfortunately, when averaging over the slopes and orientations of the scatterer one always encounters arrangements such that the projections of two centers are close and coincide in the limit. Then $p^f \to \infty$ and the region of convergence of the series and of the validity of the theorem vanishes.

Of course, the fact that for a system of Coulomb centers and small and large momentum transfers we obtain in the limit, respectively, scattering by the summary charge or a sum of scattering by individual charges, is quite obvious. However, the fact that the cross section is always larger than these limiting cross section in rigorously defined momentum-transfer regions is far from obvious and is typical only of harmonic scatterers.

The method used here to integrate the square of the modulus of the series, in the phase $\varphi$, of the particular variables in terms of which the expansion is made ($p$ or $p^{-1}$), with vanishing of all the cross terms, which leads to a series of only positive terms, is used in the theory of conformal mapping to prove the extremal character of the conformal mapping of a circle,[5] according to which this mapping always leads to an increase of the area, provided that the center of the circle is mapped with unity scale. The inequalities obtained here are the direct consequence of this variational principle.

## 6. POSSIBILITY OF COMPARISON WITH EXPERIMENT

The simplest and most natural region of application of this theory is for ion-molecule collisions. This raises immediately the question of when and to which degree the potential of the interaction of the ion with the molecule can be regarded as harmonic. We can single out at least two cases when this assumption is natural.

The first is collision of relatively slow ions, but fast enough compared with the speed of rotation of the molecule, i.e., starting with an energy of several eV to several hundred eV. The condition under which the collisions are classical is also well satisfied here, and usually the electronic states of the molecule are not excited here, and the molecule itself can be regarded as immobile during the collision time. It is then possible to regard as harmonic that part of the potential which lies outside the molecule and is described mainly by this part of the interaction potential. In addition to the harmonic multipole terms there are also anharmonic polarization forces that decrease in inverse proportion to the fifth power of the distance, as well as octupole forces. It is natural therefore to confine oneselves in the interaction between a neutral molecule and an ion to two terms—dipole and quadrupole:

$$p^* = \frac{q_0}{p_0}\left(\frac{Q_1}{b^2} + \frac{Q_2}{b^3}\right),$$

and assess the possibility of experimentally observing a focal singularity and the halo effect. The focal impact parameter $b^f$ and the focal scattering angle $\vartheta^f$ can be directly determined:

$$b^f = -\frac{3}{2}\frac{Q_2}{Q_1}, \quad \vartheta^f = \frac{4}{27}\frac{q_0}{p_0^2}\frac{Q_1^3}{Q_2^2} = \frac{1}{3}\frac{q_0 Q_1}{p_0^2 b^{f2}}, \tag{13}$$

and in order that the point $b^f$ lie outside the molecule it is necessary that the dipole moment $Q_1$ be small enough—a condition easily satisfied for many diatomic molecules.

If only the singular and dipole parts are retained in the cross section when averaging over the orientations, we obtain the universal relation

$$\frac{\langle\sigma\rangle}{\sigma^f} = \left(\frac{\vartheta^f}{\vartheta}\right)^3 - \frac{4\sqrt{2}}{9\pi}\left(1 - \frac{\vartheta}{\vartheta^f}\right)^{1/2}, \quad \sigma^f = \frac{\pi q_0 |Q_1|}{8 p_0^2 \vartheta^{f3}},$$

which is valid in the vicinity of the focal angle and is shown in Fig. 1. It is seen that the halo effect for the molecules is small, but its traces can be experimentally observed already at a relative angular and intensity resolution on the order of 1%. The quasiclassicism condition is satisfied if at $b = b'$ the angular momentum of the incident ion is of the order of $10^3\,\hbar$. At an energy on the order of 100 eV this condition is easily satisfied, especially for helium and for heavier ions. According to (13) the halo angle is determined by the ratio of the dipole interaction energy over the distance $b^f$ to the energy of the incident particle. Quantum interference phenomena as well as polarization forces smooth out this effect, but preliminary estimates show that allowance for this factor does not alter the picture qualitatively.

Second, one can expect the presented approximation to be valid if the molecule is located almost exactly along the direction of the incident ions (or atoms), and the fast particle passes in the course of its travel close to both nuclei, i.e., we can neglect in principle the screening in the calculation of the momentum transfer and assume a pure Coulomb interaction. In this case the projections of the molecule nuclei and of the ion on the plane of the impact parameter lie in a region that is small compared with the nuclear size. It was shown in Refs. 6 that the energy lost by a fast particle scattered twice through a given angle can be less than in single scattering by one atom. This makes it possible in principle to single-out double ("cannonball") scattering[7] by measuring scattering with a given energy loss.

Another possibility of singling out the configuration of interest to us for tight binary collisions is to consider glancing scattering of ions by the surface of a single crystal in a direction close to one of the crystal axes. It is then practically impossible to have scattering with small impact parameters with more than two successive atoms on this axis: before and after the double scattering the atom travels notably farther from the nuclei of the other atoms, and screening makes it possible to neglect their influence. For binary tight collisions we can consider the focal points for two Coulomb centers, found by Komarov and Shcherbakov[4] and considered in Sec. 3. The focal scattering should lead

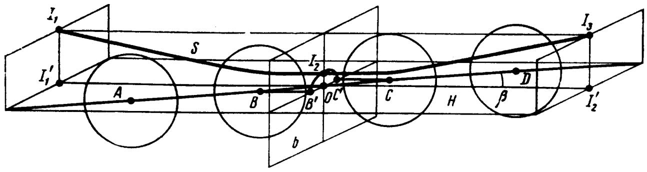


FIG. 2. Focal specular reflection of a grazing ion beam from the surface of a single crystal. $I_1 I_2 I_3$—ion trajectory, $S$—scattering plane, H—crystal surface, $b$—impact-parameter plane; $A$, $B$, $C$, and D—successive atoms along the crystal axis on the surface $H$; $B'$ and $C'$—projections of the nuclei $B$ and $C$ on the impact-parameter plane $b$; $\beta$—angle of inclination of optical axis to the scattering plane. Focusing conditions—the points $B'$, $I_2$ and $C'$ lie on a circle with center at 0, i.e, $OB' = OI_2 = OC'$.

to an increase of the specular reflection coefficient if the crystal is so turned that the crystal axis is slightly tilted away from the scattering plane. The rotation angle is determined such that the projection of the two atoms and of the ion trajectory on the plane of the impact parameters form a right equilateral triangle (Fig. 2).

Many experimental data[8] offer evidence that the specular reflection coefficient of fast ions does indeed have a maximum when the crystal axis is inclined to the scattering plane by angles that satisfy this condition approximately, but the question of the applicability of the theory and of the correct allowance for the various perturbing factors calls for a special investigation.

As the third example, we consider the scattering of protons by nonspherical nuclei. At an energy of several dozen MeV or more the proton wavelength is substantially less than the size of the nucleus and the classical approximation holds for peripheral collisions. At these velocities we can consider scattering by an immobile nucleus and use formula (12), in which we retain two terms and in which we average over the quadrupole orientations. We see that there is added to the Rutherford cross section a term due to the quadrupole interaction and independent of $p$. This term can be noted at sufficiently large momentum transfers, when the ratio of this term to the Rutherford term is already within the attainable relative measurement accuracy.

## 7. SOME GENERALIZATIONS

We shall show that if the scatterer contains a magnetic as well as an electric field, the scattering remains conformal as before, provided only that the trajectories of the scattered particles do not pass through the field sources—the charges and current. In fact, generalizing (1), we have

$$p_x=-\frac{1}{p_0}\frac{\partial V}{\partial x}+\frac{q_0}{cp_0}\int_{-\infty}^{+\infty}(p_yH_z-p_zH_y)\,dz,$$

$$p_y=-\frac{1}{p_0}\frac{\partial V}{\partial y}+\frac{q_0}{cp_0}\int_{-\infty}^{+\infty}(p_zH_x-p_xH_z)\,dz.$$

Replacing $p_z$ by $p_0$, neglecting $p_x$ and $p_y$ in comparison with $p_0$ in the right-hand side, and changing over from the components of the magnetic field H to the vector potential A, we obtain

$$p_x=-\frac{1}{p_0}\frac{\partial V}{\partial x}-\frac{q_0}{c}\int_{-\infty}^{+\infty}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)dz,$$

$$p_y=-\frac{1}{p_0}\frac{\partial V}{\partial y}+\frac{q_0}{c}\int_{-\infty}^{+\infty}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)dz.$$

If $A_x$ and $A_y$ vanish as $z\to\pm\infty$, the corresponding integrals vanish, and the potential $V$ simply acquires an additional term

$$-\frac{p_0q_0}{c}\int_{-\infty}^{+\infty}A_z\,dz,$$

which also satisfies the two-dimensional Laplace equation outside the field source, so that the scattering remains conformal, and all the previously obtained general results apply also to magnetic scatterers.

The second generalization relates to the possibility of calculating the angular momentum transferred to the scatterer. We return again to the problem of $N$ nucleon centers and assume that each of them acquires in the scattering a transverse momentum $p_j = q_0 q_j p_0^{-1}(b-b_j)^{-1}$. Then the components of the total angular momentum transferred to the scatterer are

$$L_x=-\sum_{j=1}^{N}z_jp_{jy},\quad L_y=\sum_{j=1}^{N}z_jp_{jx},\quad L_z=\sum_{j=1}^{N}(x_jp_{jy}-y_jp_{jx}).$$

Introducing the complex transferred angular momentum $L = L_x + iL_y$ we obtain

$$L_z=\operatorname{Im}(bp^*)=\frac{q_0}{p_0}\operatorname{Im}\left[b\sum_{j=1}^{N}\frac{q_j}{b-b_j}\right]$$

$$L^*=L_x-iL_y=-\sum_{j=1}^{N}z_j(p_{jy}+ip_{jx})$$

$$=-i\sum_{j=1}^{N}z_jp_j^*=-i\frac{q_0}{p_0}\sum_{j=1}^{N}\frac{q_jz_j}{b-b_j},$$

and consequently the mapping of the plane $b$ on the $(L_x, L_y)$ plane is conformal, and all the general results on the momentum transfer are valid also for the $L$ plane. In particular, focal points and predominant values of the momentum transfer appear, and can be connected with selective excitation of the rotational states of molecules. We note a curious cross-symmetry: for two identical Coulomb centers, the function $L(b)$ $(z_1 = -z_2)$ is the same as the function $p(b)$ for two centers with opposite charges, and vice versa.

The third generalization related to the possible extension of the range of angles in which the given approximation is valid. It is noted in Ref. 3 that replacement of the scattering angle $\vartheta=p/p_0$ by $2\sin(\vartheta/2)$ extends this region (in this case the formulas for the Coulomb problem become exact). One can expect that this replacement extends the range of validity also for a sufficiently large class of harmonic scatterers. We note in this connection that the classical Coulomb scattering $[\,|b| = b_0\cot(\vartheta/2)]$ realizes a conformal mapping (a stereographic projection) of the scattering-angle sphere on the impact-parameter plane. This unique property of the Coulomb field is undoubtedly connected with its internal symmetry, and with the role, discovered by Fock,[9] of the transformation of the stereographic projection in momentum space for this problem. This raises the question of whether other potentials exist such that scattering by them realizes conformal mapping of the impact-parameter plane on the scattering sphere, and if they do, what are their properties. In other words, is it possible to generalize the theory considered above to include all angles, disregarding the trivial case of a single Coulomb center.

We consider finally the transition to the limit of an arbitrary anharmonic potential. It is clear that by increasing the number of Coulomb centers and letting the charge of each of them tend to zero, we can obtain any potential. In this case the role of the Coulomb singularities on the $b$ plane will decrease, and the role of the focal points constantly increases. The number of focal singularities in the effective cross section will increase continuously, and their averaging yields in the limit the cross section for the limiting anharmonic potential. This complicated character of the limiting transition can be easily traced by letting the number of charges in the last example of Sec. 3 tend to infinity.

## 8. CONCLUSION

The results show that the processes of small-angle scattering by harmonic scatterers constitute an interesting class, the natural formalism for whose description makes use of complex variables and conformal mapping, and the prospects of further development of the theory continue to improve with further investigations of problems by this method.

The use of complex variables in two-dimensional problems very frequently simplifies to the utmost their solution, but two-dimensional problems are not so frequently encountered in physics, and usually are by way of models. In the small-angle scattering problem one dimension (the direction of motion of the fast particles) drops out, and the problem naturally becomes two-dimensional. Therefore introduction of the complex impact parameter, of the complex momentum transfer, and of conformal mapping serves as an adequate and very fruitful device. Other and possibly more effective applications of the theory than considered in Sec. 6 will undoubtedly be found. This is possible, in particular, in electron optics, where the results are applicable when a thin electromagnetic lens scatters an electron beam through a small angle to dimensions larger than its initial width.

In conclusion I wish to thank I. V. Komarov and A. P. Shcherbakov. This study is the result of a discussion of their discovery that rainbow lines degenerate into focal points for two Coulomb centers.[4] I thank also É. S. Parilis, L. A. Balashov, and V. A. Molchanov for a discussion of the possibility of observing focusing effects in ion scattering by a single crystal.

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Translated by J. G. Adashko