harmonic is radiated in the direction of the higher plasma density.

The authors thank L. I. Bondarenko for the numerical calculations.

APPENDIX

We present now an algorithm for the calculation of the coefficient Q_{12} of conversion of the energy of the wave incident on the plasma into the second harmonic. First, specifying the parameters u and N_z, we obtain the position of the synchronism point of the harmonics, i.e., v_s, the refractive index $N_s \equiv N_x(\omega, v_s)$, and the rate of divergence q_s of the modes in the vicinity of the synchronism point. We then calculate the functions G_k:

$$G_1=\mu_1^2(1-v_s)[(1-v_s)^2+u(v_s-4)]+\mu_1 N_z u^{1/2}[(1-v_s)(2+v_s)+u(4-v_s)]-(1-u-v_s)^2-3uN_z^2,$$

$$G_2=N_z(1-3u-v_s)+2\mu_1 u^{1/2}(1-v_s),\quad \mu_1=[\sigma+g_1(\sigma^2+1-v_s)^{1/2}]^{-1},$$

$$G_3=\mu_1^2[u-(1-v_s)^2]-2\mu_1 v_s N_z u^{1/2}+(1-u)N_z^2,$$

$$G_4=3\mu_1^2(1-v_s)(1+u-v_s)+\mu_1 N_z u^{1/2}[(1-v_s)(v_s-10)-u(2+v_s)]+(1-u-v_s)^2+u(7-u-v_s)N_z^2$$

and the nonlinear source f:

$$f=\frac{2N_z v_s G_1}{(1-u-v_s)^2(4-u-v_s)}+\frac{2N_s^2 v_s G_2}{(1-v_s)(4-v_s)(1-u-v_s)}+\frac{2N_z v_s(1-v_s)G_3}{(4-v_s)(1-u-v_s)^2}+\frac{2u^{1/2}v_s G_4}{(1-u-v_s)^2(4-u-v_s)}[\sigma-g_2(\sigma^2+4-v_s)^{1/2}],$$

where g_1 and g_2 characterize the polarization of the synchronized harmonics: $g=+1$ for the ordinary waves and $g=-1$ for the extraordinary ones. Given the first-harmonic energy flux S_1 in the synchronism region we now determine the nonlinear transformation coefficient

$$Q_{12}=\frac{(\omega L_s/c)}{q_s}\frac{S_1}{S_\bullet}\frac{\pi^2 f^2}{64N_s}\left(1+\frac{g_1\sigma}{(\sigma^2+1-v_s)^{1/2}}\right)^2\left(1-\frac{g_2\sigma}{(\sigma^2+4-v_s)^{1/2}}\right).$$

The polarization coefficients of the harmonics in the synchronism region are respectively

$$K_1=\frac{i}{N_1}[\sigma+g_1(\sigma^2+1-v_s)^{1/2}],\quad K_2=-\frac{i}{2N_2}[\sigma+g_2(\sigma^2+4-v_s)^{1/2}].$$

[1]N. S. Erokhin, V. E. Zakharov, and S. S. Moiseev, Zh. Eksp. Teor. Fiz. 56, 179 (1969) [Sov. Phys. JETP 29, 101 (1969)].

[2]A. Caruso, A. de Angeles, G. Gatti, R. Gratton, and S. Martellucci, Phys. Lett. 33A, 29 (1970).

[3]Yu. V. Afanas'ev, N. G. Basov, O. N. Krokhin, V. V. Pustovalov, V. P. Silin, G. V. Sklizkov, V. T. Tikhonchuk, and A. S. Shikanov, Itogi nauki i tekhniki, ser. Radiotekhnika (Science and Engineering Summaries, Radio Engineering), Vol. 17, VINITI, 1978.

[4]E. Eidmann and R. Sigel, Phys. Rev. Lett. 34, 799 (1975).

[5]N. G. Basov, V. Yu. Bychenkov, O. N. Krokhin, M. V. Osipov, A. A. Rupasov, V. P. Silin, G. V. Sklizkov, A. N. Starodub, V. T. Tikhonchuk, and A. S. Shikanov, Zh. Eksp. Teor. Fiz. 76, 2094 (1978) [Sov. Phys. JETP 49, 1059 (1978)].

[6]V. Korobkin and R. V. Serov, Pis'ma Zh. Eksp. Teor. Fiz. 4, 103 (1966) [JETP Lett. 4, 70 (1966)].

[7]G. A. Askar'yan, M. S. Rabinovich, A. D. Smirnova, and V. D. Studenov, *ibid*, 5, 116 (1967) [5, 93 (1967)].

[8]L. A. Bol'shov, Yu. A. Dreĭzin, and A. M. Dykhne, *ibid.* 19, 288 (1974) [19, 168 (1974)].

[9]B. A. Al'terkop, E. V. Mishin, and A. A. Rukhadze, *ibid.* 19, 291 (1974) [19, 170 (1974)].

[10]A. Sh. Abdullaev, Yu. M. Aliev, and V. Yu. Bychenkov, *ibid.* 28, 524 (1978) [28, 485 (1978)].

[11]V. V. Denchenko, V. V. Dolgopolov, and A. Ya. Onel'chenko, Izv. vyssh. ucheb. zaved. Radiofizika 14, 1321 (1971).

[12]V. L. Ginzburg, Rasprotranenie élektromagnitnykh voin v plazme (Propagation of Electromagnetic Waves in a Plasma), Nauka, 1967, p. 684 [Pergamon].

Translated by J. G. Adashko

Is renormalization necessary in the quasilinear theory of Langmuir oscillations?

A. A. Galeev, R. Z. Sagdeev, V. D. Shapiro, and V. I. Shevchenko

Institute of Cosmic Studies, Academy of Sciences of the USSR

(Submitted 6 June 1980)
Zh. Eksp. Teor. Fiz. 79, 2167–2174 (December 1980)

We elucidate the conditions for the applicability of the quasilinear approximation for the description of resonance interactions between waves and particles. We show that when the condition for fast phase mixing and collectivization of resonance particles (overlap of neighboring resonances) is satisfied, the nonlinear corrections to the growth rate and to the diffusion coeefficient are negligibly small.

PACS numbers: 52.35.Mw

1. The formalism for the quasi-linear theory for the description of resonance interactions between waves and particles was developed about two decades ago.[1,2] The theory was based upon the assumption that there exists in the plasma a rather broad packet of oscillations in which rapid phase mixing takes place. The equation for the distribution function of the resonance particles has then the form of the Fokker-Planck diffusion equation, and for the evaluation of the appropriate collision integral it is sufficient to restrict oneself, as was assumed earlier,[1,2] to the contribution from the main terms, quadratic in the field amplitude.

 0038-5646/80/121095-05$02.40

The conditions for the applicability of the quasi-linear theory were formulated as follows: if Δv is the width of the packet with respect to the phase velocities, the time for phase mixing in the packet is $t_1 \sim 1/k\Delta v$. This time must be appreciably shorter than the quasi-linear diffusion time

$$t_1 \ll t_D \sim (\Delta v)^2/D_k, \quad k\Delta v \gg (k^2 D_k)^{1/3}; \tag{1}$$

where

$$D_k = \frac{e^2}{m^2 v}|E_k|^2 (kv = \omega_k)$$

is the diffusion coefficients for the particles due to the waves, ω_k is the frequency of the plasma oscillations which is close to the Langmuir frequency $\omega_p \approx (4\pi n_0 e^2/m)^{1/2}$ and n_0 is the plasma density.

Condition (1) can also be written in the form

$$k\Delta v \gg (e\varphi_0/m)^{1/2}, \tag{1'}$$

where

$$\varphi_0 = \left(\sum_k k^{-2}|E_k|^2\right)^{1/2}$$

is the root mean square amplitude of the potential in the packet.

It is necessary for the applicability of the quasi-linear theory that in addition to rapid phase mixing we have also collectivization of the resonance particles when the regions of trapping corresponding to different harmonics in the packet overlap. The width of a separate wave-particle resonance is

$$\tilde{\Delta} v \approx \left| v - \frac{\omega_k}{k} \right| \approx \left(\frac{e^2|E_k|^2}{m^2 k^2} \Delta k \right)^{1/4}.$$

Hence $\tilde{\Delta} v \approx (D_k/k)^{1/3}$ and for the applicability of the quasi-linear theory it is necessary that

$$(k^2 D_k)^{1/3} \gg \max\{k\delta v, \gamma_k\}. \tag{2}$$

Here δv is the distance in phase velocity between neighboring harmonics in the spectrum, and γ_k is the growth rate of the amplitude and determines the broadening of a separate line connected with the non-stationarity of the process.

In Ref. 3 attention was drawn for the first time to the fact that the formal application of perturbation theory when solving the kinetic equation for the distribution function leads to the appearance of divergences in higher orders in the field amplitude as compared to the quasi-linear approximation. These divergences were removed in Ref. 3 using a renormalization procedure based upon the summation of a well-defined class of non-linear terms and leading to a broadening of the resonance $kv = \omega_k$ by an amount $k\tilde{\Delta} v$. By virtue of condition (1), such a renormalization does not change the quasi-linear equations.

In recent papers[4-7] it is emphasized that the applicability of the quasi-linear theory is restricted by the condition of very small field amplitudes: $(k^2 D_k)^{1/3} \ll \gamma_k$. The main basis for such statements was the existence of results of numerical simulations of wave-particle resonance interactions which did not fit into the framework of the quasi-linear theory (see Refs. 5, 6). A detailed analysis of these results goes beyond the framework of the present paper but we note merely that they were all obtained under conditions when the strict inequalities (1), (2) corresponding to the applicability of the quasi-linear theory were violated.

The analytical papers in which attempts are made to reconsider the quasi-linear theory can be split into two groups. The first group contains semi-phenomenological models (see, e.g., Ref. 4) which in some form or another start ad hoc from the assumption that there is no complete mixing in the phase plane and that there exist clusters of particles trapped by the waves (clumps). The question of its justification remains obscure, since it is unclear here which analytical procedure for solving the kinetic equation is the basis of this model. In the second group of papers attention is drawn to the fact that also in the perturbation theory which is renormalized by taking the broadening of the particle-wave resonance into account there remain in the higher orders in the field amplitude terms which describe the coupling of harmonics with resonance particles and which are comparable with the quasi-linear terms when $(k^2 D)^{1/3} \gtrsim \gamma_k$.

The non-linear corrections to the diffusion coefficient turn out to be finite only when account is taken of the wave-particle resonance broadening, which itself is expressed in terms of the diffusion coefficient. We show in what follows that under those conditions the non-linear interaction of the harmonics does not change the structure of the quasi-linear equations which are obtained in the main order of perturbation theory.

The equation for the distribution function of the resonance particles has the form of a diffusion equation,

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v}\left(D(v)\frac{\partial f_0}{\partial v}\right) \tag{3}$$

with a diffusion coefficient proportional to the spectral density of the oscillations in the resonance point $kv = \omega_k$:

$$D(v) = q_1 |E_k|^2/v. \tag{3'}$$

The spectral density of the oscillations changes with a growth rate proportional to the derivative of the resonance particle distribution function:

$$\frac{\partial |E_k|^2}{\partial t} = q_2 v^2 \frac{\partial f_0}{\partial v}\left(v = \frac{\omega_k}{k}\right)|E_k|^2 . \tag{4}$$

Taking the non-linear terms into account can affect only the change in the coefficients q_1 and q_2 in the quasi-linear Eqs. (3), and (4). This is in actual fact equivalent to a renormalization of the force of the resonance wave-particle interaction as compared to the usual quasi-linear theory obtained in the main order of perturbation theory (analogous to the charge renormalization in quantum electrodynamics). An attempt to take such a renormalization into account was already made in Ref. 7, but due to the selective nature of the summation of the non-linear terms the conclusion of that paper, that it is necessary to change the quasi-linear equations, seems to us to be premature.

In the present paper we use a perturbation-theory solution of the kinetic equation to evaluate consistently the non-linear corrections to the distribution function in any order in the field amplitude. We use the results to find the non-linear contribution to the resonance particle-wave interaction (growth rate, diffusion coefficient), assuming the phases of the oscillations to be random. It is important that the problem of the non-linear corrections itself occurs only for one-dimensional oscillations as in the three-dimensional case the non-linear terms are clearly negligibly small when condition (1) is satisfied. However, also in the one-dimensional case one can show through direct calculations that the integrals determining the main non-linear contribution to the growth rate, and hence also to the diffusion coefficient, vanish in any order in the field amplitude, so that we get for the coefficients q_1 and q_2 the usual "quasi-linear" values

$$q_1=e^2/m^2; \quad q_2=\pi\omega_k/n_0.$$

2. Just as in Ref. 7, we consider the electron Langmuir oscillations of an isotropic plasma. Our problem consists in establishing the limits of the applicability of the quasi-linear theory, and to solve it we find the non-linear corrections to the quasi-linear equations. For the calculation of the non-linear corrections to the distribution function in the region of the resonance velocities $v=\omega_{\mathbf{k}}/k$ we use a formal perturbation theory in which the solution is written in the form of expansions in powers of the electric field amplitude $E_{\mathbf{k}}$.

In the second order in the field amplitude the non-linear interaction of the harmonics $E_{\mathbf{k}_1-\mathbf{k}_2}$ and $E_{\mathbf{k}_2}$ leads to the appearance of beats at the frequencies $\omega_{\mathbf{k}_1-\mathbf{k}_2}+\omega_{\mathbf{k}_2}$, and the first non-vanishing correction to the distribution function $f_{\mathbf{k}}$ occurs in the third order in $E_{\mathbf{k}}$ and has the form

$$f_{\mathbf{k}}^{NL}=-i\frac{e^3}{m^3}\sum_{\mathbf{k}_1,\mathbf{k}_2}E_{i,\mathbf{k}-\mathbf{k}_1}E_{l,\mathbf{k}_1-\mathbf{k}_2}E_{n,\mathbf{k}_2}\frac{1}{\omega_{\mathbf{k}-\mathbf{k}_1}+\omega_{\mathbf{k}_1-\mathbf{k}_2}+\omega_{\mathbf{k}_2}-\mathbf{kv}}$$
$$\times\left\{\frac{\partial}{\partial v_i}\frac{1}{\omega_{\mathbf{k}_1-\mathbf{k}_2}+\omega_{\mathbf{k}_2}-\mathbf{k}_1\mathbf{v}}\frac{\partial}{\partial v_l}\frac{1}{\omega_{\mathbf{k}_2}-\mathbf{k}_2\mathbf{v}}\frac{\partial f_0}{\partial v_n}-\right.$$
$$-\frac{4\pi e^2}{mk_1^2}k_{1s}\varepsilon^{-1}(\omega_{\mathbf{k}_1-\mathbf{k}_2}+\omega_{\mathbf{k}_2},\mathbf{k}_1)\int d\mathbf{v}'\frac{1}{\omega_{\mathbf{k}_1-\mathbf{k}_2}+\omega_{\mathbf{k}_2}-\mathbf{k}_1\mathbf{v}'}$$
$$\frac{\partial}{\partial v_l'}\left(\frac{1}{\omega_{\mathbf{k}_2}-\mathbf{k}_2\mathbf{v}'}\frac{\partial f_0}{\partial v_n'}\right)\left[\frac{\partial}{\partial v_i}\left(\frac{\partial f_0/\partial v_s}{\omega_{\mathbf{k}_1-\mathbf{k}_2}+\omega_{\mathbf{k}_2}-\mathbf{k}_1\mathbf{v}}\right)\right.$$
$$\left.\left.+\frac{\partial}{\partial v_s}\left(\frac{\partial f_0/\partial v_i}{\omega_{\mathbf{k}-\mathbf{k}_1}-(\mathbf{k}-\mathbf{k}_1)\mathbf{v}}\right)\right]\right\}, \tag{5}$$

where

$$\varepsilon(\omega_{\mathbf{k}},\mathbf{k})=1-\frac{4\pi e^2\mathbf{k}}{mk^2}\int d\mathbf{v}\frac{\partial f_0/\partial\mathbf{v}}{\mathbf{kv}-\omega_{\mathbf{k}}}$$

is the linear dielectric constant.

The structure of the solution given here is very simple. The terms $\propto\varepsilon^{-1}(\omega_{\mathbf{k}_1-\mathbf{k}_2}+\omega_{\mathbf{k}_2},k_1)$ are caused by the appearance of the electric field of the beats, and the first term in the formula for $f_{\mathbf{k}}^{\mathrm{NL}}$ is the result of the solution of the kinetic equation in the given electric field. The application of perturbation theory leads to the occurrence in the resonance point $\mathbf{k}\cdot\mathbf{v}=\omega_{\mathbf{k}}$ of divergences. The divergences are removed by taking into account the non-linear broadening of the resonance. This was done in Ref. 3 but for Langmuir oscillations the procedure is formal because of condition (1), and for us it will be sufficient to use the following representation of the resonance denominators which are based upon Landau's rule for going around singularities (see Ref. 8):

$$\frac{1}{\mathbf{kv}-\omega_{\mathbf{k}}-i\delta_{\mathbf{k}}}=\frac{P}{\mathbf{kv}-\omega_{\mathbf{k}}}+\pi i\delta(\mathbf{kv}-\omega_{\mathbf{k}}), \tag{6}$$

where $\delta_{\mathbf{k}}\approx(k^2D_{\mathbf{k}})^{1/3}$.

The non-linear change in the electric field amplitude is found from the equation

$$\mathbf{k}\frac{\partial\varepsilon}{\partial\omega_{\mathbf{k}}}\frac{\partial\mathbf{E}_{\mathbf{k}}}{\partial t}=4\pi e\int f_{\mathbf{k}}^{NL}d\mathbf{v}. \tag{7}$$

One sees easily that the main contribution to the integral over the velocities on the right-hand side of this equation gives the first term in the formula for $f_{\mathbf{k}}^{\mathrm{NL}}$ and that the terms connected with the electric field of the beats are small compared to the main term in the ratios $\gamma_{\mathbf{k}}/\delta_{\mathbf{k}}$ and $\delta_{\mathbf{k}}^2/k^2\Delta v^2$. This is a rather obvious result, since the contribution from the resonant particles to the electric field of the beats contains an extra small factor proportional to the number of these particles, i.e., the growth rate, while, on the other hand, the contribution to the field of the beats from the non-resonant particles is small as it does not contain resonance denominators. We shall therefore in what follows use the approximation of a given electric field when calculating the non-linear corrections to the distribution function also in higher order in $E_{\mathbf{k}}$.

Just as in the quasi-linear theory, we assume the phases of the different harmonics to be random. In that case the non-linear corrections to the growth rate, obtained using Eq. (7) are connected with the processes of induced scattering and decay interaction of the harmonics which are well known from weak turbulence; the only difference with weak turbulence is that we consider processes involving resonant particles. Using Eq. (5) for $f_{\mathbf{k}}^{\mathrm{NL}}$ we get the following non-linear corrections to the growth rate, corresponding to a scattering of the wave $(\omega_{\mathbf{k}},\mathbf{k})$ into the wave $(\omega_{\mathbf{k}_1},\mathbf{k}_1)$:

$$\frac{d^s|E_{\mathbf{k}}|^2}{dt}=\frac{8\pi e^4}{m^3k^2}\frac{\mathbf{k}}{\partial\varepsilon/\partial\omega_{\mathbf{k}}}|E_{\mathbf{k}}|^2$$
$$\times\mathrm{Im}\int\frac{d\mathbf{v}}{(\omega_{\mathbf{k}}-\mathbf{kv})^2}\frac{1}{(2\pi)^3}\int d\mathbf{k}_1|E_{\mathbf{k}_1}|^2\frac{\mathbf{k}_1}{k_1^2}\frac{1}{\omega_{\mathbf{k}}+\omega_{\mathbf{k}_1}-(\mathbf{k}+\mathbf{k}_1)\mathbf{v}}$$
$$\times\frac{\partial}{\partial v_i}\left[\left(\frac{k_ik_{1j}}{\omega_{\mathbf{k}_1}-\mathbf{k}_1\mathbf{v}}+\frac{k_{1i}k_j}{\omega_{\mathbf{k}}-\mathbf{kv}}\right)\frac{\partial f_0}{\partial v_j}\right] \tag{8}$$

and to the four-plasmon decay into a set of waves $(\omega_{\mathbf{k}},\mathbf{k})$, $(\omega_{\mathbf{k}_1},\mathbf{k}_1)$, $(\omega_{\mathbf{k}_2},\mathbf{k}_2)$, $(\omega_{\mathbf{k}_2},\mathbf{k}_3)$ being intermediated by resonance particles:

$$\frac{d^d|E_{\mathbf{k}}|^2}{dt}=\left(\frac{4\pi e^4}{m^3k\partial\varepsilon/\partial\omega_{\mathbf{k}}}\right)^2\frac{1}{(2\pi)^6}\int d\mathbf{k}_1\,d\mathbf{k}_2|E_{\mathbf{k}_1}|^2|E_{\mathbf{k}_2}|^2|E_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2}|^2$$
$$\times\left|\int d\mathbf{v}\frac{k_{1i}}{|\mathbf{k}_1|}\frac{k_{2l}}{|\mathbf{k}_2|}\frac{(k+k_1-k_2)_j}{|\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2|}\frac{1}{\omega_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2}+\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}_1}-\mathbf{kv}}\right.$$
$$\left.\times\frac{\partial}{\partial v_i}\frac{1}{\omega_{\mathbf{k}}+\omega_{\mathbf{k}_1}-(\mathbf{k}+\mathbf{k}_1)\mathbf{v}}\frac{\partial}{\partial v_j}\frac{1}{\omega_{\mathbf{k}_2}-\mathbf{k}_2\mathbf{v}}\frac{\partial f_0}{\partial v_l}\right|^2$$
$$\times\left(\frac{1}{\gamma+i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}-\omega_{\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2})}+\mathrm{c.c.}\right). \tag{9}$$

We analyze first of all the non-linear contribution due to induced scattering. First of all, it is clear than this correction can be appreciable only in the one-dimensional case, while in the case of a three-dimensional

Langmuir oscillation spectrum this contribution is reduced by a factor $\delta_{\mathbf{k}}^2/k^2(\Delta v)^2$ due to the fact that the phase volume of the wave vectors (or particle velocities) entering resonance is small. In the one-dimensional case an order of magnitude estimate of the integral on the right-hand side of (6) for the resonance region of velocities $v \approx \omega_{\mathbf{k}}/k \approx \omega_{\mathbf{k}_1}/k_1$ gives the following result:

$$\gamma_L \frac{e^2|E_k|^4 k^2}{m^2 v \delta_k^3}, \tag{10}$$

where

$$\gamma_L = \frac{2\pi^2 e^2}{mk^2}\omega_p \frac{\partial f_0}{\partial v}$$

is the growth rate used in the quasi-linear theory, and $\delta_{\mathbf{k}} \approx (k^2 D_{\mathbf{k}})^{1/3}$ is the width of the resonance.

Using the formula for $D_{\mathbf{k}}$ we find that the correction considered is of the order $\gamma_L|E_{\mathbf{k}}|^2$ and in principle taking into account similar types of term without changing the form of the quasi-linear equations may affect the renormalization of the force of the resonance wave-particle interaction compared with the usual quasi-linear theory, as was already mentioned in the first section. It turns out, however, that when one evaluates the integral on the right-hand side of (8) exactly in the main order in the parameter $\delta_{\mathbf{k}}/k\Delta v$ it vanishes and there is no such renormalization. Indeed, this integral can be split into two terms, the first of which has the form

$$\int dv \varphi(v) \frac{1}{(\omega_k - kv)^4},$$

where

$$\varphi(v) \sim \int dk_1 \frac{|E_{k_1}|^2}{\omega_{k_1} - k_1 v}$$

is a smooth function of v. This term can be transformed to the form

$$\frac{1}{k^3}\int dv \frac{1}{\omega_k - kv}\frac{\partial^3 \varphi}{\partial v^3}$$

and does not at all contain singularities $\propto \delta_k^{-1}$.

The second term is proportional to the integral

$$\int dv \frac{\partial f_0/\partial v}{(\omega_k - kv)^3}\int dk_1 \frac{|E_{k_1}|^2}{(\omega_k + \omega_{k_1} - (k+k_1)v)(\omega_{k_1} - k_1 v)}$$

and as in the integral over k_1 all singularities of the integrand lie in the upper half-plane, we find, substituting approximately $|E_{k_1}|^2 \approx |E_k|^2$, that the integral over k_1 vanishes, i.e., the main singularity $\propto \delta_k^{-3}$ is also absent from that term.

One proves completely analogously also the absence of singularities corresponding to corrections comparable with the quasi-linear growth rate in arbitrary $(n+1)$-st order in $|E_k|^2$. In the random phase approximation for the Fourier harmonics of the electric field the formula for the non-linear correction to the growth rate describing the n-fold scattering of a wave by resonance particles has the form

$$\frac{d^s|E_k|^2}{dt} = -\frac{8\pi e^{2(n+1)}}{km^{2n+1}\partial\varepsilon/\partial\omega_k}|E_k|^2 \operatorname{Im}\int dv \frac{1}{\omega_k - kv}\frac{1}{2\pi}$$
$$\times \int d\varkappa_1 |E_{\varkappa_1}|^2 \frac{\partial}{\partial v}\frac{1}{\omega_k + \omega_{\varkappa_1} - (k+\varkappa_1)v}\sum_{k_1 = k, \varkappa_1}\frac{\partial}{\partial v}\frac{1}{\omega_{k_1} - k_1 v}\cdots$$
$$\times \frac{1}{2\pi}\int d\varkappa_n |E_{\varkappa_n}|^2 \frac{\partial}{\partial v}\frac{1}{\omega_{k_{n-1}} + \omega_{\varkappa_n} - (k_{n-1} + \varkappa_n)v}$$
$$\times \sum_{k_n = k_{n-1}, \varkappa_n}\frac{\partial}{\partial v}\frac{1}{\omega_{k_n} - k_n v}\frac{\partial f_0}{\partial v}. \tag{11}$$

The integral over $\varkappa_n$ in this formula contains two terms, one of which, proportional to

$$\frac{\partial}{\partial v}\int d\varkappa_n \frac{|E_{\varkappa_n}|^2}{\omega_{k_{n-1}} + \omega_{\varkappa_n} - (k_{n-1} + \varkappa_n)v}\frac{\partial}{\partial v}\frac{1}{\omega_{\varkappa_n} - \varkappa_n v},$$

by complete analogy with the analysis given above does not contain singularities δ^{-3} which can make a contribution to the growth rate comparable to the quasi-linear one.

The second has the form

$$\frac{\partial^2}{\partial v^2}\left(\frac{1}{\omega_{k_{n-1}} - k_{n-1}v}\right)\varphi_n(v, k_{n-1}),$$

where

$$\varphi_n \sim \int d\varkappa_n \frac{1}{\omega_{k_{n-1}} + \omega_{\varkappa_n} - (k_{n-1} + \varkappa_n)v}$$

is a smooth function of v. Similarly we evaluate the integrals over $\varkappa_{n-1}$, $\varkappa_{n-2}$, and so on. In final reckoning we get, dropping during the calculations terms whose contribution is clearly small compared to the quasi-linear one, the following integral:

$$\int \frac{dv}{(\omega_k - kv)^{3n+1}}\varphi(v),$$

where $\varphi(v)$ is a function without singularities, so that the integral given here does not contain any singularities at all.

3. We now turn to an evaluation of the non-linear corrections to the growth rate which are connected with the decay interaction of the harmonics. As in the case of the induced scattering process by resonance particles considered above in the three-dimensional case the non-linear corrections are reduced in that case by a factor $(k\Delta v/\delta_{\mathbf{k}})^4$ due to the fact that the phase volume of the wavevectors and velocity vectors entering the resonance $\omega_{\mathbf{k}} = \mathbf{k}\cdot\mathbf{v}$ is small. The one-dimensional case is degenerate—the non-linear corrections are in that case anomalously large and their contribution to the growth rate may become appreciable. It was just to this kind of non-linear corrections caused by the decay interaction of the harmonics intermediated by resonance particles that the necessity to renormalize the quasi-linear theory was attributed in Ref. 7.

We consider the one-dimensional case in more detail. As in Ref. 7, we shall assume that the condition

$$\frac{d^2\omega_k}{dk^2}k^2\frac{\delta_k^2}{\omega_k^2} \sim k^2 r_D^2 \frac{\delta_k^2}{\omega_k} \ll \gamma_k$$

is satisfied which corresponds to the smallness of the dispersion of the frequencies of the waves which interact with one another in the case when their wavevectors are incident in the resonance region $k_i \approx \omega_{k_i}/v$.

When this condition is satisfied we can in Eq. (9) put approximately

$$\omega_k+\omega_{k_1}=\omega_{k_2}+\omega_{k+k_1-k_2}.$$

Changing to the one-dimensional case we write this formula in the form

$$\frac{d^d|E_k|^2}{dt}=\left(\frac{4\pi e^4}{km^3\partial\varepsilon/\partial\omega_k}\right)^2\frac{2}{\gamma}\int dv\,dv'\frac{1}{kv-\omega_k}\left(\frac{1}{kv'-\omega_k}\right)^*\frac{\partial}{\partial v}\frac{\partial}{\partial v'}$$
$$\times\frac{1}{2\pi}\int dk_1\frac{|E_{k_1}|^2}{(k+k_1)v-\omega_k-\omega_{k_1}}\left(\frac{1}{(k+k_1)v'-\omega_k-\omega_{k_1}}\right)^*\frac{\partial}{\partial v}\frac{\partial}{\partial v'}$$
$$\times\frac{1}{2\pi}\int dk_2|E_{k_2}|^2|E_{k+k_1-k_2}|^2\frac{1}{k_2v-\omega_{k_2}}\left(\frac{1}{k_2v'-\omega_{k_2}}\right)^*\frac{\partial f_0}{\partial v}\frac{\partial f_0}{\partial v'}. \quad (12)$$

An order of magnitude estimate of the integral on the right-hand side of (12) gives

$$\left(\frac{4\pi e^4}{m^3k\partial\varepsilon/\partial\omega_k}\right)^2\left(\frac{\partial f_0}{\partial v}\right)^2\frac{k^2|E_k|^6}{\gamma_k v^2\delta_k^6}\sim\gamma_L|E_k|^2\frac{e^4k^4|E_k|^4}{m^4v^2\delta_k^6}. \quad (13)$$

Substituting $\delta_k\sim(k^2D_k)^{1/3}$ we find that the non-linear correction is of the order of the quasi-linear growth rate. The authors of Ref. 7 restricted themselves to just such an estimate and they based upon that estimate the incorrect conclusion that it is necessary to take the non-linear corrections to the quasi-linear theory into account. In fact, the integrals on the right-hand side of Eq. (12) can be evaluated exactly without particular difficulties. Putting approximately $|E_{k_i}|^2\approx|E_k|^2$ and using the residue theorem to evaluate the integrals over k_1 and k_2 (the poles of the integrand are on both sides of the real k_1, k_2 axis) we get the following result:

$$\frac{d^d|E_k|^2}{dt}=\left(\frac{8\pi^2e^4}{m^3k\partial\varepsilon/\partial\omega_k}\right)^2\frac{1}{\gamma}\int dv\,dv'\frac{1}{kv-\omega_k-i\delta_k}$$
$$\times\frac{1}{\omega_k-i\delta_k-kv'}\frac{\partial f_0}{\partial v}\frac{\partial f_0}{\partial v'}\left(\frac{\partial}{\partial v}\frac{\partial}{\partial v'}\frac{1}{\omega_p(v'-v)+2i\delta_k v}\right)^2. \quad (14)$$

In each of the integrals over v and v' all singularities of the integrand are on one side of the real axis (in the upper half-plane in the integral over v and in the lower half-plane in the integral over v'). Replacing approximately the derivative $\partial f_0/\partial v$ by its value for $v=\omega_k/k$ one shows easily that the integrals over v and v' do in fact not contain the higher-order singularities which lead to the estimate (13). The non-linear corrections to the growth rate (12) are thus small compared to the quasi-linear value. This conclusion is obtained for four-plasmon decays but a completely analogous proof can be given also for higher-order decays.

When conditions (1) and (2) which were first formulated in Ref. 1 as the conditions for the applicability of the quasi-linear theory are satisfied the non-linear corrections to the quasi-linear equations turn out to be indeed small. In the present paper we showed that the non-linear corrections to the growth rate are small; however, it follows from the energy conservation law that in that case the non-linear corrections to the diffusion coefficient are also small.

[1]A. A. Vedenov, E. P. Velikhov, and R. Z. Sagdeev, Usp. Fiz. Nauk 73, 701 (1961) [Sov. Phys. Usp. 4, 332 (1961)]; Nuclear Fusion, Suppl., 2, 465 (1962).

[2]W. E. Drummond and D. Pines, Nuclear Fusion, Suppl. 3, 1049 (1962).

[3]T. H. Dupree, Phys. Fluids 9, 1773 (1966).

[4]A. S. Bakaĭ, Dokl. Akad. Nauk SSSR 237, 1069 (1977) [Sov. Phys. Dokl. 22, 753 (1977)].

[5]A. S. Bakaĭ and Yu. S. Sigov, Dokl. Akad. Nauk SSSR 237, 1326 (1977) [Sov. Phys. Dolk. 22, 734 (1977)].

[6]J. C. Adam, G. Laval, and D. Pesme, Equipe de Recherche du CNRS no 74 1978.

[7]J. C. Adam, G. Laval, and D. Pesme, Phys. Rev. Lett. 43, 1671 (1979).

[8]L. A. Artsimovich and R. Z. Sagdeev, Fizika plazmy dlya fizikov (Plasma physics for physicists) Atomizdat, 1979.

Translated by D. ter Haar

Dynamo of small-scale fields

S. I. Vaĭnshteĭn

Siberian Institute of Terrestrial Magnetism, Ionosphere, and Radio Wave Propagation of the Siberian Division of the USSR Academy of Sciences

(Submitted 14 June 1980)
Zh. Eksp. Teor. Fiz. 79, 2175–2188 (December 1980)

An equation for the dynamics of small-scale magnetic fields is derived for a non-Markov model by using the Lagrangian statistical characteristics of turbulence. It is shown that the behavior of the fields is closely connected with the correlation characteristics of the scalar admixture. In the case of extremely low magnetic viscosity $D \ll \nu$ (ν is the kinematic viscosity), the dynamics of the fields is described at large wave numbers by a universal equation. It is shown in this case that a dynamo solution, i.e., a solution that increases without limit, exists. The problem of the dynamo of a small scale field is thus solved for the case $D \ll \nu$.

PACS numbers: 41.10.Hv

While the question of the dynamics of large-scale magnetic fields in a turbulent medium is now regarded as solved in the main outline, the problem of pulsation fields remains open despite its large urgency. Indeed, it is regarded as established that turbulence leads to the onset of turbulent diffusion and to generation (in the case of reflection non-invariance) of a large-scale field.[1,2] The question lies here only in the accuracy of

0038-5646/80/121099-09$02.40