éffekty w poluprovodnikakh (Symmetry and Strain-Induced Effects in Semiconductors), Nauka, 1972 [Wiley, 1975].
[8]R. G. Wheeler and J. O. Dimmock, Phys. Rev. 125, 1805 (1962).
[9]J. Lagois, Phys. Rev. B16, 1699 (1977).

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Conductivity of two-dimensional systems with macroscopic inhomogeneities

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The conductivity of certain two-dimensional systems with periodic arrangement of the inhomogeneities is considered. An exact solution is obtained for models with dielectric as well as superconducting square inclusions arranged in checkerboard fashion. The field distribution in the conducting region, the fluctuations of the current density, and the effective conductivity are obtained. Taken together with the results obtained by Dykhne [Sov. Phys. JETP 32, 63 (1971)] this makes it possible, when the properties of the components differ drastically, to provide a relatively complete description of such a system both in the vicinity of the metal–insulator transition and far from it. The conductivities of similar systems with inclusions of different shape are qualitatively estimated and the limits of the critical exponent are established. A relation that is valid also in the three-dimensional case is obtained between the permittivity of a medium with metallic inclusion and the conductivity having the same structure with superconducting inclusions.

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1. In the study of the physical properties of inhomogeneous media, principal attention is usually paid to systems with random distribution of the components (see, e.g., Refs. 1 and 2). This is natural, since real systems of this kind are frequently randomly inhomogeneous. Thus, for example, thin films obtained by sputtering on a substrate, have during the initial sputtering stage a strongly inhomogeneous random structure (island films). A theoretical analysis of such systems is fraught with extreme difficulties, which have not been overcome to this day. In essence, the only study in which some exact analytic results were obtained is that of Dykhne[3] (and its generalizations[4–6]). A number of important results in the vicinity of the metal–insulator (MI) phase transition are obtained by similarity theory.[2] To investigate the properties of randomly inhomogeneous media in the entire range of concentrations it is necessary to resort to computer calculations and to model experiments.

Another important class of inhomogeneous media comprises the so-called systems with topological structure (STS), i.e., systems in which all the inclusions are identical and form a periodic lattice. Among the real systems of this type are, for example, thin films with topological structure, used in semiconductor devices.[7] These films, which are also obtained by sputtering, are on the one hand bulky enough to be practically homogeneous, and on the other thin enough to permit the distributions of the electric field and of the current in them to be regarded in a number of cases as two-dimensional. The topological structure, i.e., the periodic distribution of the elements that cause the inhomogeneity, is obtained, for example, by subsequent selective etching. It is important that the dimensions of the inhomogeneities are large in comparison with the mean free path, so that a macroscopic description with a coordinate-dependent conductivity is applicable.

The periodicity of the STS leads to periodicity in the distribution of the electric field (current) and makes it possible by the same token to restrict oneself in the determination of the potential to a single unit cell of the structure. Clearly, this problem is simpler than that of the randomly inhomogeneous medium. The difference in the complexity of the solution of these problems is similar to a considerable degree to the corresponding difference when it comes to finding the phonon spectra of an ideal and a disordered crystal. In two-dimensional STS, when the properties of the components differ greatly, it is possible in a number of cases to make use of the powerful methods of the theory of functions of complex variable.[8,9,10]

In addition to the obvious need for investigating the properties of the STS for various applications, their study is also of fundamental interest since a metal-insulator phase transition can take place in these systems. From this point of view such systems can serve as a relatively simple object for a theoretical and experimental study of the phase-transition problem. The investigation of the STS properties, which is also of independent interest, permits on the one hand to disclose the universality of the conclusions of the general theory of phase transitions and determine the role played in it by the STS. On the other hand, exactly solvable STS models can serve as the touchstones for a variety of possible approximate methods.

We consider in this paper certain STS models. Principal attention is paid to a system in which the inclusions, quadratic in shape, are arranged in checkerboard fashion. The investigation of such a system is

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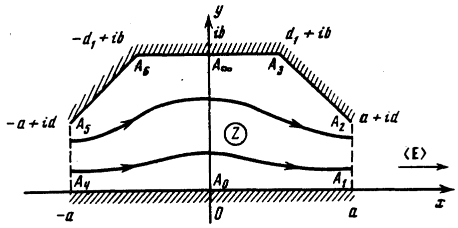
of interest, in particular, for the following reason. Let p be the concentration (fraction) of the conducting component. If $p \to p_c = 1/2$, then the system becomes a checkerboard, and its conductivity σ_e becomes equal to zero, i.e., an MI transition takes place. In similarity theory[2] it is assumed that the vanishing of σ_e proceeds in proportion to a power of the parameter $\tau = (p - p_c)/p_c$ that describes the proximity to the MI transition with respect to concentration: $\sigma_e \sim \tau^t$. Calculation of the critical exponent t makes it possible, within the framework of similarity theory, to describe completely the checkerboard model, since the second critical exponent was obtained by Dykhne.[3] If the conductivities of both components σ_1 and σ_2 differ from zero, then according to Ref. 3 we have $\sigma_e = (\sigma_1\sigma_2)^{1/2}$ at $\tau = 0$ ($p = p_c = 1/2$) for isotropic systems whose macroscopic properties remain unchanged when the substitution $\sigma_1 \rightleftarrows \sigma_2$ is made. In this case the approach to the MI transition is governed by the parameter $h = \sigma_2/\sigma_1 \ll 1$: $\sigma_e \sim h^s$, $s = 1/2$. The critical exponent s is the same both for a randomly inhomogeneous medium and for a checkerboard. However, as will be shown below, with respect to the exponent t there is no such universal behavior. For the checkerboard model σ_e vanishes logarithmically. This is valid for all models in which the inclusions have corners in the region of the contact.

A qualitative estimate was made of the effective conductivity of a system in which the conducting inclusions have a smooth convex shape described by a powerlaw function. This analysis, which is not rigorous, leads to the conclusion that for periodic systems (STS) the critical exponent cannot exceed unity: $0 \leqslant t \leqslant 1$. We note that for randomly inhomogeneous media $t > 1$ ($t \approx 1.4$ in the two-dimensional case[1,2]).

The effective conductivity of STS in the "dielectric" phase $\tau < 0$ was considered, when the conductivities of the components differ greatly. In this case the inclusions have a much higher conductivity than the matrix, and in first-order approximation the problem reduces to a calculation of the effective conductivity of a system with superconducting inclusions. It is possible to apply to the latter problem, which is also of independent interest, the same method as at $\tau > 0$. The result for the checkerboard model coincides with the one that follows from the "reciprocity relation" obtained by Dykhne.[3] The relation between the corresponding exponents is then automatically satisfied.[2]

We consider also the permittivity ε_e of a system with conducting inclusions whose density is lower than critical ($\tau < 0$). We show that ε_e can be connected with σ_e of the problem of conductivity of a system with superconducting inclusions (this is true in both the two-dimensional and the three dimensional case): $\varepsilon_e \propto \sigma_e$. This leads both to the statement that ε_e becomes infinite at the MI transition point, and to the relation between the corresponding critical exponents.[2]

2. We consider an isotropic film of conductivity σ_1 with inclusions in the form of identical nonconducting squares arranged in checkerboard fashion, so that the system as a whole has quadratic symmetry. Since such a system has an isotropic effective conductivity σ_e, the



direction of the average electric field $\langle \mathbf{E} \rangle$ can be chosen arbitrarily in the calculation of σ_e. We direct the coordinate axes x and y parallel to the diagonals of the squares, and $\langle \mathbf{E} \rangle$ along the x axis. In this coordinate system the inclusions are rectangular rhombs located at the points of a square lattice. To find the distribution of the current (field) in the chosen geometry it suffices to consider half the unit cell shown in Fig. 1 in the plane of the complex variable $z = x + iy$. On the shaded sections of Fig. 1, the normal component of the current (field) is zero, while the tangential component vanishes on the dashed sections. To make the analysis general we consider the case of a rectangular lattice with $a \neq b$.

The complex potential $\Phi(z)$, whose derivative yields the components of the electric field

$$\Phi'(z) = E_x - iE_y, \tag{1}$$

can be obtained by the same method as in Refs. 8–10. The conformal mapping of the region of Fig. 1 on the upper half-plane of the complex variable ζ (Fig. 2) is given by the function $\zeta(z)$ determined from the Christoffel-Schwarz integral,[8,9] which takes in this case the form

$$z = C\int_0^\zeta (1-t^2)^{-1/2}[(1-k_1^2t^2)(1-k_2^2t^2)]^{-1/4}\,dt,\qquad 0 \leqslant k_2 \leqslant k_1 \leqslant 1. \tag{2}$$

The appendix contains the equations that follow from the correspondence of the points on Figs. 1 and 2 and yield the parameters k_1, k_2, and C. We now map the upper ζ half-plane on the interior of a rectangle in the plane of the variable w such that the dashed sections a_1a_2 and a_4a_5 of Fig. 2 go over into the vertical side of this rectangle:

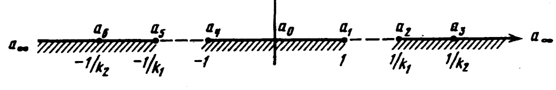
$$w = \int_0^\zeta [(1-t^2)(1-k_1^2t^2)]^{-1/2}\,dt. \tag{3}$$

The problem has thus been reduced to finding the complex potential for flow in a channel of constant cross section, so that

$$\Phi(w) = Aw, \tag{4}$$

where A is a real constant. Formulas (2)–(4) together with the definitions of the parameters they contain yield the solution of our problem. For the derivative of the complex potential we have

$$\Phi'(z) = \frac{A}{C}\left[\frac{1-k_2^2\zeta^2(z)}{1-k_1^2\zeta^2(z)}\right]^{1/4}, \tag{5}$$



where the function $\zeta(z)$ is determined from (2).

Using (5) and (1), we easily verify that the obtained solution satisfies all the boundary conditions in the region of Fig. 1. When the cusp points $z_0 = \pm a + id$ are approached, in accord with Ref. 11, the field increases without limit

$$\Phi'(z) \propto (z-z_0)^{-1/3}, \tag{6}$$

while at the points $z = \pm d_1 + ib$ it vanishes.

From (2)–(4) we can obtain the distribution of the field (current) inside a square with impermeable sides and with a pointlike source and a pointlike sink at opposite vertices. We note for this purpose that as $d \to 0$, as can be seen from Figs. 1 and 2, we have $k_1 \to 1$ and $k_2 \to 0$ $(a = b)$. The complex potential assumes as $d \to 0$ the form

$$\Phi(z) = {}^1/_2 A \ln[(1+\zeta(z))/(1-\zeta(z))], \tag{7}$$

where the functions $\zeta(z)$ are determined from the equation

$$z = C \int_0^\zeta (1-t^2)^{-3/4}\, dt. \tag{8}$$

The integral (8) can be expressed in terms of an incomplete function.[12]

3. To calculate the effective conductivity σ_e, we note that

$$\sigma_e = \frac{a}{b}\frac{I}{U}, \tag{9}$$

where U is the voltage drop on the cell and I is the total current flowing through the cell. U and I are calculated just as in Refs. 8 and 9. As a result we obtain for a quadratic lattice $(a = b)$

$$\sigma_e = \sigma_1 [K(k_1')/K(k_1)], \quad k_1'^2 = 1 - k_1^2, \tag{10}$$

where $K(k)$ is a complete elliptic integral of the first kind. In the case $a \neq b$ the right-hand side of (10) must be multiplied by a/b. It is easy also to calculate the relative quadratic fluctuations of the current density Δ_j^2. As a result it turns out that the expressions for Δ_j^2 and σ_e satisfy the exact relation[6]

$$\sigma_e = \sigma_1/(1+\Delta_j^2), \tag{11}$$

a fact that serves to confirm the validity of the calculations.

Expression (10) yields the effective conductivity of the considered system in the entire region of the existence of conductivity, $0 \leq d \leq a$ or $0 \leq \tau \leq 1$. Here $\tau = (p - p_c)/p_c$ is a parameter that characterizes the proximity to the MI transition, p is the fraction (concentration) of the conducting component, and p_c is the critical concentration at which the effective conductivity vanishes. For the considered checkerboard system we have $p_c = 1/2$, and the connection of τ with d is given by

$$\tau = 1 - [(a-d)^2/a^2]. \tag{12}$$

If the dielectric inclusions are small compared with the size of the cell $(r = a - d \ll a, k_1 \to k_2)$, we get from (19) and (A.9)–(A.12)

$$\sigma_e/\sigma_1 \approx 1 - [K^2(1/\sqrt{2})/\pi](r/a)^2. \tag{13}$$

Comparison of (13) with the formula for the conductivity of a system with low concentration of nonconducting circles (cylinders) allows us to introduce an effective radius of a quadratic insulating inclusion

$$r_e = r\sqrt{2}K(1/\sqrt{2})/\pi \approx 0.8346r. \tag{14}$$

Here r is half the length of the diagonal of the square or the radius of the circle in which the square is inscribed. The effective area is larger than that of the square, but their ratio is close to unity:

$$\pi r_e^2/2r^2 = K^2(1/\sqrt{2})/\pi \approx 1.094.$$

When the system approaches the MI transition point $(d \to 0)$ we have $k_1 \to 1$ and $k_2 \to 0$. From (10), (A.9), (A.13), and (A.14) we get

$$\sigma_e/\sigma_1 \approx (\pi/4)\ln^{-1}(\gamma/\tau); \quad \gamma = 2\pi/K^2(1/\sqrt{2}) \approx 1.83. \tag{15}$$

In expression (15) we have used the connection (12) between the parameter τ and d; as $d \to 0$, this connection takes the form $\tau \approx 2d/a$. The effective conductivity of the system under consideration vanishes logarithmically at the transition point. The MI transition is a phase transition of second order.

4. The result (15) can be interpreted as follows. Near the MI transition, the cell resistance, which coincides by virtue of the system periodicity with the effective resistivity of the sample, is determined mainly by the region of the contact. As a rough estimate of the contact resistance we use the customary formula for the resistance of a conductor, which take in our case the form

$$\sigma_1/\sigma_e = \int_0^{x_0} dx/y(x), \tag{16}$$

where $x_0 \lesssim a$; $y = y(x)$ specifies the form of the contact, and y likewise does not exceed $\sim a$. For a checkerboard model $\theta = \pi/4, y(x) = d + x$ (see Fig. 3), and from (16) we obtain with logarithmic accuracy

$$\sigma_e \approx \sigma_1 [\ln(a/d)]^{-1}. \tag{17}$$

Expression (17) agrees qualitatively with (15). The difference in the numerical coefficient is connected with the inhomogeneity of the current distribution. To refine the estimate σ_e we obtain the distribution of the current (field) in the region shown in Fig. 3. A similar problem was solved, e.g., in Ref. 9. The transformation $\zeta_1 = \zeta_1(z)$, where $\zeta_1(z)$ is determined from the equation

$$z = iC_1 \int_0^{\zeta_1} (1-t)^{-\beta} t^{-1/2}\, dt, \quad C_1 = d/B({}^1/_2, 1-\beta), \quad \beta\pi = \theta, \tag{18}$$

maps the interior of the region $A_1A_2A_3A_1$ on Fig. 3 on the upper ζ_1 half-plane. The region $\mathrm{Im}\zeta_1 > 0$ is then mapped on a half-strip; the complex potential takes the form[9]

$$\Phi(z) = iA_1 \arcsin[2\zeta_1(z) - 1]. \tag{19}$$

Here A_1 is a real constant, and $\zeta_1(z)$ is determined from (18).

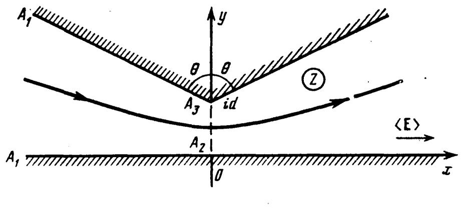
The total current through the contact is

$$I = 2\sigma_1 \mathrm{Im}[\Phi(id) - \Phi(0)] = 2\pi\sigma_1 A_1. \tag{20}$$

The potential difference between the points $z = x_0$ and $z = -x_0$ can be written in the form

$$U = 2\mathrm{Re}[\Phi(0) - \Phi(-x_0)]. \tag{21}$$

Putting $x_0 \sim a \gg d$, where $2a$ is the period of the struc-



ture, we get from (21), (18), and (19)

$$U\approx 4A_1\{(1-2\beta)^{-1}\ln[x_0(1-2\beta)/2C_1]+\ln 2\}. \tag{22}$$

From (20) and (22), taking (9) into account (at $a=b$) we obtain

$$\sigma_e\approx\sigma_1(\pi/2-\theta)\ln^{-1}(\tilde{\gamma}a/d). \tag{23}$$

Here $\tilde{\gamma}=\tilde{\gamma}(\theta)\sim 1$ is a quantity that cannot be calculated by the described method. Equation (23) is valid at angles θ not too close to $\pi/2$: $\pi/2-\theta\gg d/a$. If the inequality is reversed, when $\theta\to\pi/2$ and $\beta\to 1/2$, we have from (18) and (19)

$$\Phi(z)\approx\pi A_1 z/d,$$

so that we obtain for the effective conductivity ($x_0=a$): $\sigma_e\approx\sigma_1 d/a$. The same result is given by formula (16), which is exact at $\theta=\pi/2$, since the current distribution is uniform in this case. At $\theta=\pi/4$ formula (34) agrees with logarithmic accuracy with (15).

The model considered has the following singularity: the inclusion has in the region of the contact a cusp in which the electric field becomes infinite. To clarify the importance of this circumstance, we consider a model with rounded corners, when the shape of the contacting inclusions is given by the branches of the hyperbola

$$x^2/a_1^2-y^2/b_1^2=1,$$

and the average field is directed along the y axis. The complex potential of this problem is of the form[8]

$$\Phi(z)=\mathrm{const}\cdot\ln[z+(z^2-c_1^2)^{1/2}], \quad c_1^2=a_1^2+b_1^2. \tag{24}$$

An estimate of the conductivity, analogous to that given above, leads to the results (23), with

$$d=2a_1,\quad \theta=\mathrm{arc\,tg}(b_1/a_1).$$

Thus, in the case of rounded contacts with round-off radius $\lesssim d$ the effective conductivity varies logarithmically as before. The logarithmic dependence is "acquired" as usual in the region $d\ll x\ll a$ (a is of the order of the period of the structure), so that if the inclusion has in this region the form of a straight line, then the conductivity takes the form (23).

We return now to Eq. (16). For the contacts considered, Eq. (16) gives the exact result at $\theta=\pi/2$ and a qualitatively correct one at $\theta=\pi/4$. The apparent reason is that for these contacts the current distribution does not differ greatly from a uniform one. At the same time, in the case of sharp obstacles ($\theta\to 0$), which produce a strong perturbation in the distribution of the current, formula (16) yields a result that is far from (23).

The foregoing circumstances raise the hope that for inclusions of sufficiently smooth form, which form in the region of the contact a channel with a smoothly varying cross section, the use of (16) will lead to a result close to the true one. For inclusions whose shape in the region of the contact is given by the equation $y(x)=d+\alpha|x|^{\mu}$ ($\mu\geq 1$), we obtain from (16) $\sigma_e\propto d^t$, where $t=(\mu-1)/\mu$. Since $\mu\geq 1$, it follows that $0\leq t\leq 1$. The value $t=0$ corresponds to a logarithmic behavior of σ_e similar to (23). We can thus expect for inclusions with smooth convex shape

$$\sigma_e\sim\sigma_1\tau^t,\quad 0\leq t\leq 1. \tag{25}$$

It follows from (25) that σ_e vanishes at the MI transition point, with an infinite derivative. At the same time, for randomly inhomogeneous media we have $d\sigma/d\tau\to 0$ as $\tau\to 0$, since $t>1$.[1,2]

We note that the use of Eq. (16) for inclusion of concave shape ($\mu<1$) leads to the conclusion that in this case the MI transition is of first order, since the integral in (16) is finite at $\tau=0$. However, the validity of (16) in this case is doubtful, inasmuch as in the region of the cusp (with zero aperture angle) the perturbations of the current distribution are large.

We have considered so far only dielectric inclusions. If the conductivity σ_2 of the inclusions is different from zero but is low ($\sigma_2\ll\sigma_1$), the results above are valid outside the "smearing" region[2] $\tau\gg\tau_0$. At $\tau\ll\tau_0$, according to Dykhne,[3]

$$\sigma_e=(\sigma_1\sigma_2)^{1/2}. \tag{26}$$

For a system of the checkerboard type, considered in Sec. 3, Eq. (26) becomes equal to (15) at

$$\tau_0=\gamma\exp\left[-\frac{\pi}{4}\left(\frac{\sigma_1}{\sigma_2}\right)^{1/2}\right], \tag{27}$$

i.e., at $\sigma_2\ll\sigma_1$ the region of the smearing of the MI transition is exponentially small. In the general case when σ_e has a power-law dependence on τ [see (25)], the smearing region is determined from the condition $\tau_0^{2t}\approx\sigma_2/\sigma_1$.

5. The methods of Sec. 3 can be used also to investigate the "dielectric phase ($\tau<0, \sigma_2/\sigma_1\ll 1$), also outside the smearing region $|\tau|\gg\tau_0$. We assume that the checkerboard model differs in the region $\tau<0$ from that considered in Sec. 2 ($\tau>0$) by the substitution $\sigma_1\rightleftarrows\sigma_2$, i.e., squares with high conductivity σ_1 are arranged in checkerboard fashion in a matrix of conductivity σ_2. In first-order approximation, outside the region of the smearing, the inclusions can be regarded as "superconducting" ($\sigma_1\to\infty$), so that inside them $E=0$. Consequently the tangential component of the field E_t is zero on the inclusion boundary, and the current (field) has only a normal component. We consider the problem in the same geometry as in Sec. 2. Half of the unit cell is of the same form as in Fig. 1, but now the inclined sections A_2A_3 and A_5A_6, just as the vertical ones, should be drawn as dashed lines ($\mathbf{E}_t=0$ on the dashed lines). Accordingly the sections a_2a_3 and a_5a_6 on Fig. 2 should also be shown dashed. Then the transition to a constant section channel will be given not by (3) but by the expression

$$\tilde{w}=\int_0^t[(1-t^2)(1-k_1^2t^2)]^{-1/2}dt. \tag{28}$$

The rest of the calculation is analogous to the preceding one: for a quadratic lattice we obtain as a result

$$\tilde{\sigma}_e=\sigma_2 K(k_1)/K(k_1'). \qquad (29)$$

In the derivation of (29) we used the connection (A.8) between k_1 and k_2. From (29) and (10) it follows that

$$\sigma_e\tilde{\sigma}_e=\sigma_1\sigma_2. \qquad (30)$$

Formula (30), obtained for the considered particular case, agrees in form with the exact "reciprocity relation" derived by Dykhne[3] for isotropic two-dimensional two-component systems. The reciprocity relation takes the form (30), where σ_e is the effective conductivity of the "supplementary" system, which differs from the initial one by the substitution $\sigma_1 \rightleftarrows \sigma_2$.

For the checkerboard system considered above, as well as for a randomly inhomogeneous medium, the reciprocity relation takes the form[3]

$$\sigma_e(\tau)\sigma_e(-\tau)=\sigma_1\sigma_2. \qquad (31)$$

Formula (31) is valid for all τ. In particular, at $\tau=0$ we obtain from (31) the relation (26). We consider now the conductivity outside the smearing region $|\tau|\gg\tau_0$. We express σ_e in the form

$$\begin{aligned}\tau>0:\ \sigma_e\approx\sigma_1 f_+(\tau),\\ \tau<0:\ \sigma_e\approx\sigma_2 f_-(\tau),\end{aligned} \qquad (32)$$

From (31) and (32) follows a connection between the functions f_+ and f_-:

$$f_+(\tau)f_-(\tau)=1. \qquad (33)$$

In similarity theory it is assumed that f_+ and f_- are given by

$$f_+(\tau)\sim\tau^t,\quad f_-(\tau)\sim\tau^{-q}, \qquad (34)$$

and it is proved that in the two-dimensional case

$$t=q. \qquad (35)$$

The equality of the critical exponents (35) follows also from expressions (33) and (34). Thus, the reciprocity relation leads to a relation (35) in an independent manner. Moreover, it gives the connection between $\sigma_e(\tau>0)$ and $\sigma_e(\tau<0)$ in the entire range of variation of τ, i.e., even outside the region of the applicability of similarity theory.

6. Directly connected with the considered group of questions is the problem of calculating the effective dielectric constant ε_e of an inhomogeneous medium. An expression for ε_e can be obtained from the formulas for σ_e by making the substitution $\sigma_i\to\varepsilon_i$, so that all the results obtained for the conductivity can be directly applied to the permittivity. Of greatest interest is the behavior of the permittivity of a medium with metallic inclusions, particularly in the vicinity of the MI transition point.[2]

Let the permittivity of the matrix by ε_2 and let the concentration of the metallic component $p<p_c$, so that $\sigma_e=0$, i.e., the medium as a whole is nonconducting. Then $\mathbf{E}=0$ inside the metallic inclusions while on the boundary $\mathbf{E}$ (as well as the induction $\mathbf{D}$) has only a normal component. In the dielectric regions we have

$$\operatorname{rot}\mathbf{E}=0,\ \operatorname{div}\mathbf{D}=0,\ \mathbf{D}=\varepsilon_2\mathbf{E}. \qquad (36)$$

It is easily seen that the calculation of the effective permittivity of such a system is similar to that of the effective conductivity of a medium with superconducting inclusions, considered in the preceding sections. These problems are equivalent subject to the substitutions $\mathbf{D}\rightleftarrows\mathbf{j}$, $\varepsilon_e\rightleftarrows\sigma_e$, $\varepsilon_2\rightleftarrows\sigma_2$. It follows therefore that ε_e is expressed in terms of the same function $f_-(\tau)$ as $\tilde{\sigma}_e=\sigma_e(\tau<0)$:

$$\varepsilon_e=\varepsilon_2 f_-(\tau), \qquad (37)$$

where $f(\tau)$ is defined in (32). The result (37) can be arrived at formally by assuming that $\varepsilon_1\to\infty$ for the first component (the metal). We emphasize that relation (37) is valid in either a two-dimensional or a three-dimensional case, at any shape and arrangement of the inclusions and at all $\tau<0$.

It follows from (37) that when the MI transition is approached from the side of the insulator, ε_e becomes infinite.[2] In similarity theory it is assumed that as $|\tau|\to 0$ (Ref. 2)

$$\varepsilon_e\sim\varepsilon_2|\tau|^{-\tilde{q}} \qquad (38)$$

It follows from (34), (37), and (38) that

$$q=\tilde{q}. \qquad (39)$$

The equality of the critical exponents (39) was established earlier[2] by similarity-theory methods.

It should be noted that in contrast to Eqs. (30) and (35), which are valid under the restrictions considered above, (37) and (39) are much more general. In particular, a connection of the type (37) between the permittivity and the conductivity holds also in the anisotropic case.

7. It can be concluded from the foregoing results that systems having a topological structure can be fitted within the framework of the general theory of MI transitions. In particular, the relations established between the critical exponents by similarity theory[2] are satisfied. A distinguishing feature of the STS is the nonuniversality of the critical exponent t (in the general case this pertains apparently also to the exponent s). The value of the exponent t is determined by the shape of the inclusions in the region of the contact, whereas for randomly inhomogeneous media the value of t is determined mainly by the topology of an infinite cluster.[1,2]

The shape of the inhomogeneities can be such that the dependence of σ_e on τ is different in different concentration regions. Thus, if the inclusion has in the region of the contact the shape of a wedge with a round-off radius $\sim r_0$, then the effective conductivity in the region $r_0\lesssim d\ll a$ has a logarithmic dependence on d (and on τ), while at $d\ll r_0$ the dependence of σ_e on τ can take a power-law form.

To check on the validity of the qualitative estimates obtained with the aid of (16) it is desirable, in particular, to obtain the solution for a model with round inclusions, or to determine the resistance of a contact made up of a pair of parabolas. An estimate with the aid of (16) yields for such a problem $t=1/2$. It is of interest also to investigate STS with the aid of model experiments similar to those made in Ref. 13.

In conclusion, I am deeply grateful to Corresponding

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APPENDIX

From the correspondence between the points on Figs. 1 and 2 we obtain with the aid of (2) the following equalities:

$$a=CK_1,\ d=CK_2,\ d_1=C(K_1-K_3),\ b=C(K_2+K_3),\ d_1=CK_4. \qquad (A.1)$$

The integrals $K_i=K_i(k_1,k_2)$ take here the form

$$K_1=\int_0^1 (1-t^2)^{-1/2}[(1-k_1^2t^2)(1-k_2^2t^2)]^{-1/4}\,dt,$$
$$K_2=\int_1^{1/k_1} (t^2-1)^{-1/2}[(1-k_1^2t^2)(1-k_2^2t^2)]^{-1/4}\,dt,$$
$$K_3=\frac{1}{\sqrt{2}}\int_{1/k_1}^{1/k_2} (t^2-1)^{-1/2}[(k_1^2t^2-1)(1-k_2^2t^2)]^{-1/4}\,dt, \qquad (A.2)$$
$$K_4=\int_{1/k_2}^{\infty} (t^2-1)^{-1/2}[(k_1^2t^2-1)(k_2^2t^2-1)]^{-1/4}\,dt.$$

According to (A.1), we have five equations with which to determine the three parameters k_1, k_2, and C. For this system to be compatible we must have to identify relations between K_i. The first, $d+a=d_1+b$, which is obvious from the geometry of Fig. 1, turns into an identity when the expressions for a, b, d, and d_1 from (A.1) are substituted. The second follows from (A.1):

$$K_1(k_1,k_2)=K_3(k_1,k_2)+K_4(k_1,k_2). \qquad (A.3)$$

To prove (A.3), we consider the integral

$$J=\int_{C_0} (1-z^2)^{-1/2}[(1-k_1^2z^2)(1-k_2^2z^2)]^{-1/4}\,dz, \qquad (A.4)$$

where the contour C_0 emerges from the point $z=0$ and goes off to infinity in the upper half of the complex z plane. If the contour C_0 is directed along the real positive semiaxis, then

$$J=K_1+iK_2+(i-1)K_3-K_4. \qquad (A.5)$$

We now deform the contour C_0 to make it coincide with the imaginary positive axis. It turns out then that the integral J is purely imaginary. From the vanishing of the real part of (A.5) follows the validity of (A.3). From the equality of the imaginary parts we obtain the relation

$$K_2+K_3=\int_0^{\infty} (1+y^2)^{-1/2}[(1+k_1^2y^2)(1+k_2^2y^2)]^{-1/4}\,dy. \qquad (A.6)$$

For a square lattice ($a=b, d=d_1$), as follows from (A.1), there should be satisfied one more equality

$$K_1(k_1,k_2)=K_2(k_1,k_2)+K_3(k_1,k_2), \qquad (A.7)$$

which yields the connection between the parameters k_1 and k_2. To find this connection we make in (A.6) the change of variable

$$y=t(1-t^2)^{-1/2}.$$

Comparing the transformed integral (A.6) with the expression for K_1 from (A.2), we verify that to satisfy (A.7) we must stipulate for the square lattice that

$$k_1^2+k_2^2=1. \qquad (A.8)$$

We introduce in lieu of k_1 and k_2 the parameters k and $\varkappa$ defined by

$$k_1^2=k^2(1+2\varkappa),\ k_2^2=k^2(1-2\varkappa). \qquad (A.9)$$

For a square lattice we have from (A.8) and (A.9)

$$k=1/\sqrt{2}. \qquad (A.10)$$

The equation for the determination of the parameter $\varkappa$ at $k=1/\sqrt{2}$ can be written in the form

$$r/a=K_3/K_1. \qquad (A.11)$$

In the case when the inclusions are small compared with the lattice period ($r\ll a$), we have, as seen from Figs. 1 and 2, $k_1\approx k_2$, i.e., $\varkappa\ll 1$. For the integrals K_1 and K_3 at small $\varkappa$ we obtain the following expansions ($k=1/\sqrt{2}$):

$$K_1=K(1/\sqrt{2})(1+{}^1/_3\varkappa^2+\ldots),$$
$$K_3=\sqrt{\varkappa}\pi[K(1/\sqrt{2})]^{-1}(1+{}^4/_5\varkappa^2+\ldots). \qquad (A.12)$$

Here $K(k)$ is a complete elliptic integral of the first kind. In the expansion of K_3 in powers of $\varkappa$ it is convenient to make first in the expression for K_3 in (A.12) the change of variable

$$t^2=[k^2(1+2\varkappa)]^{-1}\{[4\varkappa x^2/(1-2\varkappa)]+1\}.$$

Near the MI transition point we have $d\ll a, k_1\to 1, k_2\to 0, \varkappa\to 1/2$, i.e., $\varkappa=1/2-\lambda$, where $\lambda\ll 1$. The equation for the parameter λ can then be written in the form

$$d/a=K_2/K_1, \qquad (A.13)$$

and for K_1 and K_2 we can use the expansions ($k=1/\sqrt{2}$):

$$K_1=\sqrt{2}K(1/\sqrt{2})-\lambda^{1/4}I+\ldots,$$
$$K_2=\lambda^{1/4}I(1+{}^{11}/_{20}\lambda+\ldots), \qquad (A.14)$$
$$I=\pi/\sqrt{2}K(1/\sqrt{2}),\ \lambda={}^1/_2-\varkappa.$$

When expanding K_2 in powers of λ it is convenient to make first in the expression for K_2 in (A.2) the change of variable

$$t=(1-k_1'^2x^2)^{-1/2},\quad k_1'^2=1-k_1^2.$$

[1]B. I. Shklovskiĭ and A. L. Éfros, Usp. Fiz. Nauk **117**, 401 (1975) [Sov. Phys. Usp. **18**, 845 (1975)].

[2]A. L. Efros and B. I. Shklovskii, Phys. Stat. Sol. (b) **76**, 475 (1976).

[3]A. M. Dykhne, Zh. Eksp. Teor. Fiz. **59**, 110 (1970) [Sov. Phys. Phys. JETP **32**, 63 (1971)].

[4]A. M. Dykhne, *ibid.* p. 641 [348].

[5]B. I. Shklovskiĭ, *ibid.* **72**, 288 (1977) [**45**, 152 (1977)].

[6]B. Ya. Balagurov, Fiz. Tverd. Tela (Leningrad) **20**, 3332 (1978) [Sov. Phys. Solid State **20**, 1922 (1978)].

[7]H. S. Nathanson and J. Goldberg, Russ. transl. in: Fizika tonkikh plenok (Physics of Thin Films), Vol. VIII, Mir, 1978, p. 264.

[8]M. A. Lavrent'ev and B. V. Shabat, Metody teorii funktsiĭ kompleksnogo peremennog (Methods of the Theory of Functions of Complex Variable), Nauka, 1965.

[9]B. A. Fuks and B. V. Shabat, Funktsii kompleksnogo peremnnogo i nekotorye ikh prilozheniya (Complex Variable Functions and Some of Their Applications), Nauka, 1964.

[10]V. I. Smirnov, Kurs vyssheĭ matematiki (A Course of Higher Mathematics), Gostekhizdat, Vol. III, 1957, Part 2.

[11]L. D. Landau and E. M. Lifshitz, Elektrodinamkia sploshnykh sred (Electrodynamics of Continuous Media), Fizmatigiz, 1959 [Pergamon].

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