[8]B. A. Zon, Zh. Eksp. Teor. Fiz. 73, 128 (1977) [Sov. Phys. JETP 46, 65 (1977)].
[9]V. M. Buĭmistrov and L. I. Trakhtenberg, *ibid.* 73, 850 (1977) [46, 447 (1977)].
[10]V. M. Buĭmistrov and L. K. Mikhaĭlov, Trudy MFTI, Radiotekhnika i élektronika, No. 8, 158 (1974).
[11]L. I. Trakhtenberg, Opt. Spektrosk. 44, 863 (1978) [Opt. Spectrosc. (USSR) 44, 510 (1978)].
[12]M. Ya. Amus'ya, A. S. Baltenkov, and V. B. Gilerson, Pis'ma Zh. Tekh. Fiz. 3, 1105 (1974) [Sov. Tech. Phys. Lett. 3, 455 (1977)].
[13]B. A. Zon, Zh. Eksp. Teor. Fiz. 77, 44 (1979) [Sov. Phys. JETP 50, 21 (1979)].
[14]M. Gavrila, Proc. 10th Internat. Conf. on Atomic Collisions, Paris, 1977, p. 165.
[15]V. P. Zhdanov Zh. Eksp. Teor. Fiz. 73, 112 (1977) [Sov. Phys. JETP 46, 57 (1977)].
[16]J. E. Rizzo and R. C. Klewe, Brit, J. Appl. Phys. 17, 1137 (1966).
[17]V. A. Kas'yanov and A. N. Starostin, Zh. Eksp. Teor. Fiz. 76, 944 (1979) [Sov. Phys. JETP 49, 476 (1979)].
[18]L. D. Landau and E. M. Lifshitz, Kvantovaya Mekhanika, nerelyativistskaya teoriya (Quantum Mechanics, Nonrelativistic Theory), Nauka, 1974, §§148, 150 (Pergamon, 1977).
[19]V. M. Buimistrov and L. I. Trakhtenberg, Phys. Lett. 69A, 261 (1978).
[20]I. A. Grishaev, G. D. Kovalenko, and V. I. Shramenko, Zh. Eksp. Teor. Fiz. 72, 437 (1977) [Sov. Phys. JETP 45, 229 (1977)].
[21]A. I. Akhiezer, P. I. Fomin, and N. F. Shul'ga, Pis'ma Zh. Eksp. Teor. Fiz. 13, 713 (1971) [JETP Lett. 13, 506 (1971)].

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Theory of wave propagation in nonlinear inhomogeneous media

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A new approach is suggested to the problem of the self-action of waves in a nonlinear medium whose permittivity depends on the intensity of the wave field. The initial boundary-value problem is reduced to the Cauchy problem. The scalar wave equation (plane-layered medium and general three-dimensional case) is considered, as well as the vector problem of propagation of an electromagnetic wave in plane-layered media for two possible polarizations. It is shown that in all cases a closed nonlinear equation holds for the reflection coefficient, and the wave field in the medium can be described by a linear equation. The limiting case of incidence of the wave on a halfspace is considered and asymptotic solutions for the low-intensity waves are found.

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The problem of wave propagation in nonlinear media is of great interest, in particular, for nonlinear optics (see, for example, Refs. 1–3) and electrodynamics of plasma.[4-7] In the simplest formulation of the problem of the self-action of a wave, generation of the various harmonics is not taken into account, and propagation of the wave is studied in a medium whose permittivity is determined by the intensity of the wave field

$$\varepsilon(\mathbf{r})=\varepsilon_0+\varepsilon_1(\mathbf{r},\ I(\mathbf{r})). \tag{1}$$

Here $I(r)=U(r)U^*(r)$ is the intensity of the wave field inside the medium. The wave field itself is described by a nonlinear Helmholtz equation

$$\Delta U+k^2[1+\varepsilon(\mathbf{r},\ I(\mathbf{r}))]U(\mathbf{r})=0, \tag{2}$$

where

$$k^2=\omega^2\varepsilon_0c^{-2},\ \ \varepsilon(\mathbf{r},I(\mathbf{r}))=\varepsilon_0^{-1}\varepsilon_1(\mathbf{r},\ I(\mathbf{r})).$$

Equation (2) describes the stationary self-action of waves in a medium with the permittivity (1) for the scalar problem. The conditions on the boundaries of the medium are given, namely, continuity of the field $U(r)$ and of its derivative $\partial U(\mathbf{r})/\partial n$ in the normal direction.

A large number of researches have been devoted to the study of two aspects of the problem (2). The first aspect is connected with the investigation of the wave reflected from the nonlinear medium and the ambiguities arising therefrom,[4] and with hysteresis phenomena (the hysteretic phenomena were apparently first noted by Silin[5]). On the basis of various considerations, the authors obtain and analyze approximate expressions for the reflection coefficient or for the field inside the medium.

The second aspect of the problem (2) is connected with researches on the effect of nonlinearity on the propagation of wave beams with narrow angular spectrum. Substituting the field

$$U(x,\ \rho)=A(x,\ \rho)e^{-ikx}$$

in the expression (2) (the x axis is directed along the beam, ρ are the coordinates in a plane perpendicular to the x axis) and neglecting the term $\partial^2A/\partial x^2$, we obtain the parabolic equation of nonlinear quasioptics:

$$2ik\partial A(x,\ \rho)/\partial x=\Delta_\rho A(x,\ \rho)+k^2\varepsilon(x,\ \rho;\ |A(x,\ \rho)|^2)A(x,\ \rho) \tag{3}$$

with a specified initial condition at $x=0$. This equation is similar to the nonlinear Schrödinger equation and transforms at $\varepsilon\equiv\varepsilon(x,\rho)$ into a parabolic equation that describes the propagation of waves in linear inhomogeneous media in the approximation of small-angle scattering. The results for the exact integration of Eq. (3) for the simplest statement of the problem are well known (see, for example, Refs. 8 and 9).

Questions arise as to the accuracy of the obtained approximate solutions of problem (2), and also as to the conditions of their applicability and stability. It is clear that, for an answer to these questions, it is impossible to limit ourselves to qualitative considerations, for example, in the transition from (2) to (3). It is necessary to consider the complete problem (2), which is a boundary-value problem.

Making use of the idea of invariant imbedding (see, for example, Ref. 10), it was possible in Refs. 11–13 to develop a method for linear inhomogeneous media that permits a reduction of the initial problem to the Cauchy problem, and enables one to obtain a closed nonlinear equation for the reflected field. The wave field inside the medium is described by a linear equation. This method is generalized below to the case of nonlinear media. The one-dimensional scalar Helmholtz equation is considered in detail; the equations for the three dimensional case and the vector Maxwell equations are discussed more briefly. The results make it possible to derive all the approximate equations enumerated above, under certain assumptions. Moreover, since the initial boundary-value problem is reduced to the Cauchy problem, the equations obtained in the present work are suitable for numerical analysis and solution of statistical problems.[12]

2. We first consider the one-dimensional case, corresponding to a plane-layered medium. Let the layer of the medium occupy part of the space $L_0 \leqslant x \leqslant L$ and let a plane oblique wave

$$U_0(\mathbf{r}) = v\exp[ip(L-x)+i\mathbf{q}\boldsymbol{\rho}],$$

be incident on it at the right. Here ρ is the radius vector in a plane perpendicular to the x axis, $\mathbf{q}$ is the projection of the wave vector on this plane, $p^2 = k^2 - q^2$, $k = \varkappa + i\gamma$ (the quantity γ describes the damping of the wave). For an isotropic medium, we can assume without loss of generality that the vector $\mathbf{q}$ is directed along the y axis. We seek a solution of Eq. (2) in the form

$$U(\mathbf{r}) = vu(x)\exp(iqy).$$

We obtain the following equation for the function $u(x)$

$$d^2u(x)/dx^2 + p^2[1+\tilde{\varepsilon}(x, wI(x))]u(x) = 0, \tag{4}$$

where $w = |v|^2$ is the intensity of the incident wave,

$$I(x) = u(x)u^*(x),\quad \tilde{\varepsilon} = k^2p^{-2}\varepsilon(x, wI(x)).$$

To the right and left of the layer, the field has the form

$$\begin{aligned} u(x) &= \exp[ip(L-x)] + R_L(w)\exp[ip(x-L)],\quad x>L,\\ u(x) &= T_L(w)\exp(-ipx),\quad x<L_0, \end{aligned} \tag{5}$$

where $R_L(w)$ and $T_L(w)$ are the complex reflection and transmission coefficients of the wave; for simplicity, it is assumed that the wave number k inside and outside the layer has the same value. From the conditions of continuity of $u(x)$ and $du(x)/dx$ on the boundary of the layer, we obtain the boundary conditions for Eq. (4):

$$\begin{aligned} u(L) &= u_L = 1+R_L(w),\quad u'(L) = -ip[1-R_L(w)] = -ip[2-u_L],\\ u(L_0) &= T_L(w),\quad u'(L_0) = -ipT_L(w) = -ipu(L_0). \end{aligned} \tag{6}$$

The boundary value problem (4) and (6) is equivalent to the integral equation

$$u(x; L, w) = e^{ip(L-x)} + \frac{ip}{2}\int_{L_0}^{L} dx'\, e^{ip|x-x'|}\tilde{\varepsilon}(x', wI(x'))u(x'; L, w), \tag{7}$$

where $I(x') = I(x'; L, w)$ and the following additional variables are introduced in correspondence with the theory of invariant imbedding: L is the location of the right boundary layer and w is the intensity of the incident wave.

We now differentiate Eq. (7) with respect to L:

$$\frac{\partial u(x; L, w)}{\partial L} = ipe^{ip(L-x)} + \frac{ip}{2}e^{ip(L-x)}\tilde{\varepsilon}(L, wI(L))u_L(w) + \frac{ip}{2}\int_{L_0}^{L} dx' e^{ip|x-x'|}$$
$$\times\left[\tilde{\varepsilon}(x', wI(x'))\frac{\partial u(x'; L, w)}{\partial L} + u(x'; L, w)\frac{\partial\tilde{\varepsilon}(x', wI(x'))}{\partial I(x')}\frac{\partial I(x')}{\partial L}\right], \tag{8}$$

where $u_L(w) = u(L; L, w)$ is the field in the plane $x = L$. We set

$$\partial u/\partial L = a(L, w)u(x; L, w) + \psi(x; L, w)$$

and choose the quantity $a(L, w)$ in the form

$$a(L, w) = ip + {}^1\!/_2 ip\tilde{\varepsilon}(L, wI(L))u_L(w). \tag{9}$$

For the function $\psi(x; L, w)$ we obtain the integral equation

$$\psi(x; L, w) = \frac{ip}{2}\int_{L_0}^{L} dx' e^{ip|x-x'|}\tilde{\varepsilon}(x', wI(x'))\psi(x'; L, w)$$
$$+\frac{ip}{2}\int_{L_0}^{L} dx' e^{ip|x-x'|}u(x'; L, w)\frac{\partial\tilde{\varepsilon}(x', wI(x'))}{\partial I(x')}\frac{\partial I(x')}{\partial L}. \tag{10}$$

We differentiate now Eq. (7) with respect to w:

$$\frac{\partial u(x; L, w)}{\partial w} = \frac{ip}{2}\int_{L_0}^{L} dx' e^{ip|x-x'|}\tilde{\varepsilon}(x', wI(x'))\frac{\partial u(x'; L, w)}{\partial w}$$
$$+\frac{ip}{2}\int_{L_0}^{L} dx' e^{ip|x-x'|}u(x'; L, w)\frac{\partial\tilde{\varepsilon}(x', wI(x'))}{w\,\partial I(x')}\left[I(x') + w\frac{\partial I(x')}{\partial w}\right]. \tag{11}$$

Equations (10), (11) are outwardly similar; therefore we can connect their solutions by the linear relation $\psi = b(L, w)\partial u/\partial w$, where the coefficient $b(L, w)$ is described by the formula

$$b(L, w) = w[a(L, w) + a^*(L, w)]. \tag{12}$$

Thus, the solution of Eq. (7) satisfies the equality $(x < L)$

$$\frac{\partial u(x; L, w)}{\partial L} = a(L, w)u(x; L, w) + b(L, w)\frac{\partial u(x; L, w)}{\partial w}, \tag{13}$$

which can be regarded as a differential equation if we add to it the boundary condition as $L \to x$:

$$u(x; L, w)|_{L=x} = u_x(w). \tag{14}$$

For the function $u_L(w)$ we have

$$\frac{\partial u_L(w)}{\partial L} = \frac{\partial u(x; L, w)}{\partial L}\bigg|_{x=L} + \frac{\partial u(x; L, w)}{\partial x}\bigg|_{x=L}. \tag{15}$$

The first term on the right side of (15) is determined by Eq. (13), in which we must set $x = L$, and the second, by the boundary condition (6). As a result, we obtain the closed nonlinear equation

$$\frac{\partial u_L(w)}{\partial L}=2ip[u_L(w)-1]+\frac{ip}{2}\tilde{\varepsilon}(L, wI(L))u_L^2(w)+b(L,w)\frac{\partial u_L(w)}{\partial w} \tag{16}$$

with the initial condition $U_{L0}(w)=1$ which follows from (7).

Equations (13), (14), and (16) are completely equivalent both to the integral equation (7) and to the initial boundary value problem (4), (6), which is now reduced to the Cauchy problem. These are precisely the equations of the theory of invariant imbedding for the given system. We note that if we set $x=L_0$ in Eq. (13), then we obtain the equations for the transmission coefficient of the wave

$$\frac{\partial T_L(w)}{\partial L}=a(L,w)T_L(w)+b(L,w)\frac{\partial T_L(w)}{\partial w},\quad T_{L_0}(w)=1. \tag{17}$$

The reflection coefficient is equal to $R_L(w)=u_L(w)-1$. If the medium is linear, then the dependence on w disappears and the equations transform into those obtained earlier.[11,12]

If the permittivity does not depend explicitly on x, i.e., if

$$\tilde{\varepsilon}(x, wI(x))\equiv\tilde{\varepsilon}(wI(x)),$$

then we can carry out in Eq. (16) the limiting transition $L_0\to-\infty$, corresponding to incidence of the wave on the halfspace $x<0$. We obtain then a linear differential equation of first order for the reflection coefficient

$$b(w)\frac{\partial R_\infty(w)}{\partial w}=-2ipR_\infty(w)-\frac{ip}{2}\tilde{\varepsilon}(w|1+R_\infty|^2)(1+R_\infty)^2 \tag{18}$$

with the initial condition $R_\infty=R_0$ at $w=0$, and for the quantity R_0 we have from (18) $4R_0+\tilde{\varepsilon}(0)(1=R_0)^2=0$, i.e.,

$$R_0=[2(1+\tilde{\varepsilon}(0))^{1/2}-2-\tilde{\varepsilon}(0)]/\tilde{\varepsilon}(0).$$

The wave field $u(x)$ inside the medium is described here by the linear equation $(L-x\to-x)$

$$\frac{\partial u(x,w)}{\partial x}+a(w)u(x,w)+b(w)\frac{\partial u(x,w)}{\partial w}=0,\quad x<0, \tag{19}$$

with the initial condition $u(0,w)=1+R(w)$, where

$$a(w)=ip+ip\tilde{\varepsilon}(w|1+R_\infty|^2)(1+R_\infty)/2,\quad b(w)=w[a(w)+a^*(w)].$$

At $\tilde{\varepsilon}(0)=0$, we have $R_0=0$ and for a sufficiently low intensity of the incident wave $(w\to0)$ we can set $\tilde{\varepsilon}=\beta k^2p^{-2}w$ and seek a solution of Eq. (18) in the form $R_\infty(w)=\tilde{R}w$; then

$$\tilde{R}=-\frac{\beta k^2}{4p^2}\frac{ip}{ip-\tilde{\gamma}},\quad k=\varkappa+i\gamma. \tag{18'}$$

In this case, $b(w)=-2\tilde{\gamma}w$ and it follows from Eq. (19) that $[\tilde{\gamma}=(p-p^*)/2i]$

$$u(x,w)=(1+\tilde{R}we^{2\tilde{\gamma}x})\exp\left[-ipx+\frac{ik^2\beta w}{4p\tilde{\gamma}}(1-e^{2\tilde{\gamma}x})\right]. \tag{19'}$$

The case $\tilde{\varepsilon}(0)<0$ corresponds to the problem of the generation of an electromagnetic field in a plasma.[5,6] Equation (18) should describe hysteresis phenomena as a function of the energy and angle of incidence of the wave. For example, in the absence of damping $(\gamma=0)$, it is easy to find a partial solution when $R_\infty(w)$ is real. Under these conditions $b(w)=0$ and a transcendental equation for $R_\infty(w)$ follows from (18):

$$4R_\infty=-\tilde{\varepsilon}[w(1+R_\infty)^2](1+R_\infty)^2.$$

We then get from (19)

$$u(x,w)=(1+R_\infty)\exp\left(-ipx\frac{1-R_\infty}{1+R_\infty}\right).$$

Such a solution was investigated by Bass and Gurevich[6]; it corresponds to a situation in which a plane wave propagates in a nonlinear medium. We note that Eq. (18) is valid for arbitrary damping of the wave and can be used conveniently for numerical analysis.

The incidence of a wave on a layer of the medium (or halfspace) was considered above. Following Ref. 11, we can consider a case in which the source of the field is located inside a layer of the medium. Inasmuch as such a problem is not of special interest, we shall not consider it here.

3. We now turn to the three-dimensional problem. We note that if a bounded beam is incident on a layer of the medium $(L_0<x<L)$, the problem remains three dimensional even in the case in which the permittivity does not depend explicitly on the coordinates. The differential equation (2) is equivalent to the integral

$$U(x,\rho)=U_0(x,\rho)-k^2\int_{L_0}^{L}dx_1\int d\rho_1 g(x-x_1,\rho-\rho_1)\varepsilon(x_1,\rho_1;I(x_1,\rho_1))U(x_1,\rho_1), \tag{20}$$

where $U_0(x,\rho)$ is the given incident field,

$$I(x,\rho)=U(x,\rho)U^*(x,\rho),\quad g(x,\rho)=-(4\pi r)^{-1}\exp(ikr),$$

$g(x,\rho)$ is the Green's function in free space.

We now consider the equation for the Green's function with a source at the point (L,ρ_0) in the case $x\leqslant L$:

$$G(x,\rho;L,\rho_0)=g(L-x,\rho-\rho_0)-k^2\int_{L_0}^{L}dx_1\int d\rho_1 g(x-x_1,\rho-\rho_1)\times\varepsilon(x_1,\rho_1;I(x_1,\rho_1))G(x_1,\rho_1;L,\rho_0). \tag{21}$$

Then the wave field $U(x,\rho)$ is determined by the equation

$$U(x,\rho)=\int d\rho_0 G(x,\rho;L,\rho_0)f(\rho_0), \tag{22}$$

while the function $f(\rho_0)$ corresponds to a distribution of sources in the plane $x=L$. Here

$$U_0(x,\rho)=\int d\rho_0 g(L-x,\rho-\rho_0)f(\rho_0),$$
$$I(x,\rho)=\int d\rho'\,d\rho'' G(x,\rho;L,\rho')G^*(x,\rho;L,\rho'')W(\rho',\rho''), \tag{23}$$

where $W(\rho',\rho'')=f(\rho')f^*(\rho'')$.

The following integral representation is valid for the function $g(L-x,\rho-\rho_0)$ at $x\leqslant L$:

$$g(L-x,\rho-\rho_0)=\int d\mathbf{q}\,g(\mathbf{q})\exp[i(k^2-q^2)^{1/2}(L-x)+i\mathbf{q}(\rho-\rho_0)],\quad g(\mathbf{q})=A/8i\pi^2(k^2-q^2)^{1/2}; \tag{24}$$

it satisfies the equations $(L\geqslant x)$

$$(\partial/\partial L)g(L-x,\rho-\rho_0)=i(k^2+\Delta_\rho)^{1/2}g(L-x,\rho-\rho_0),$$
$$(\partial/\partial L)g(L-x,\rho-\rho_0)=i(k^2+\Delta_{\rho_0})^{1/2}g(L-x,\rho-\rho_0). \tag{25}$$

We now introduce the function $H(L,\rho,\rho_0)=G(L,\rho;L,\rho_0)$, which describes the field of the wave in the plane of the source. Equation (21) is analogous to Eq. (7) considered above, but in place of the wave number b there is the operator $(k^2+\Delta_\rho)^{1/2}$, and in place of

w, the function $W(\rho', \rho'')$; therefore, we can repeat the derivation of the equations of the theory of invariant imbedding. The analog of the quantity $a(L, w)$ is an integro-differential operator, and the analog of the derivative $\partial/\partial w$ is the variational-differentiation operator $\delta/\delta W(\rho', \rho'')$. As a result, we obtain the relation

$$\frac{\partial G(x,\rho;L,\rho_0)}{\partial L}=\hat{a}_{\rho_0}G(x,\rho;L,\rho_0)+\int d\rho' d\rho'' \hat{b}(\rho',\rho'')\frac{\delta G(x,\rho;L,\rho_0)}{\delta W(\rho',\rho'')}, \tag{26}$$

which can be regarded as the equation for the quantity $G(x,\rho;L,\rho_0)$, if we add the initial condition

$$G(x,\rho;L,\rho_0)|_{L=x}=H(x,\rho,\rho_0). \tag{27}$$

The operators $\hat{a}_\rho$ and $\hat{b}$ are defined by the equations

$$\hat{a}_{\rho_0}G(x,\rho;L,\rho_0)=i(k^2+\Delta_{\rho_0})^{1/2}G(x,\rho;L,\rho_0)$$
$$-k^2\int d\rho_1 G(x,\rho;L,\rho_1)\varepsilon(L,\rho_1;I(L,\rho_1))H(L,\rho_1,\rho_0),$$
$$\hat{b}(\rho',\rho'')=W(\rho',\rho'')(\hat{a}_{\rho'}+\hat{a}_{\rho''}^*). \tag{28}$$

For the quantity $H(L,\rho_1,\rho_0)$ we have

$$\frac{\partial H(L,\rho_1,\rho_0)}{\partial L}=\frac{\partial G(x,\rho;L,\rho_0)}{\partial L}\Big|_{x=L}+\frac{\partial G(x,\rho;L,\rho_0)}{\partial x}\Big|_{x=L}. \tag{29}$$

The first term in the right side of (29) is determined by Eq. (26) at $x=L$, and to find the second term we must differentiate Eq. (21) with respect to x and set $s=L$. Then, with account of (25), we obtain, denoting $g(0,\rho-\rho_0)\equiv g(\rho-\rho_0)$,

$$\frac{\partial G(x,\rho;L,\rho_0)}{\partial x}\Big|_{x=L}=i(k^2+\Delta_\rho)^{1/2}[H(L,\rho,\rho_0)-2g(\rho-\rho_0)].$$

Consequently, (29) takes the form of a closed equation

$$[\partial/\partial L-i(k^2+\Delta_{\rho_0})^{1/2}-i(k^2+\Delta_\rho)^{1/2}]H(L,\rho,\rho_0)=-2i(k^2+\Delta_\rho)^{1/2}g(\rho-\rho_0)$$
$$-k^2\int d\rho_1 H(L,\rho,\rho_1)\varepsilon(L,\rho_1;I(L,\rho_1))H(L,\rho_1,\rho_0)$$
$$+\int d\rho' d\rho''\hat{b}(\rho',\rho'')\frac{\delta H(L,\rho,\rho_0)}{\delta W(\rho',\rho'')} \tag{30}$$

with the initial condition [which follows from (21)]:

$$H(L,\rho,\rho_0)|_{L=L_0}=g(\rho-\rho_0). \tag{31}$$

Equations (26) and (30), with the initial conditions (27) and (31), as well as the relations (22) and (28), correspond to the theory of invariant imbedding for the initial three-dimensional boundary problem. For a linear medium the dependence on W vanishes, and we arrive at the equations obtained in Ref. 13.

We note that, in accord with (24),

$$g(\rho)=\frac{1}{2i}(k^2+\Delta_\rho)^{-1/2}\delta(\rho). \tag{31'}$$

If the quantity ε does not depend explicitly on the coordinates, we can, as above, consider the case in which the medium fills the halfspace $x<0$. This is achieved by the limiting transition $L_0\to-\infty$. In particular, for the function

$$R_\infty(\rho,\rho_0)=H(0,\rho,\rho_0)-g(\rho-\rho_0),$$

which describes the back-reflected field, we get the equation

$$\int d\rho' d\rho''\hat{b}(\rho',\rho'')\frac{\delta R_\infty(\rho,\rho_0)}{\delta W(\rho',\rho'')}+i[(k^2+\Delta_\rho)^{1/2}+(k^2+\Delta_{\rho_0})^{1/2}]R_\infty(\rho,\rho_0)$$
$$=k^2\int d\rho_1[g(\rho-\rho_1)+R_\infty(\rho,\rho_1)]\varepsilon(I(\rho_1))[g(\rho_1-\rho_0)+R_\infty(\rho_1,\rho_0)], \tag{32}$$

where

$$I(\rho)=\left|\int d\rho_0[g(\rho-\rho_0)+R_\infty(\rho,\rho_0)]f(\rho_0)\right|^2.$$

In the important case in which $\varepsilon=\beta I$ and the intensity W of the incident field is sufficiently small, we can seek a solution of Eq. (32) in the form of a functional Taylor series in W. In first approximation,

$$R_\infty(\rho,\rho_0)=\int d\rho' d\rho'' W(\rho',\rho'')Q(\rho',\rho'',\rho,\rho_0),$$

and we obtain for the function Q the following operator equation:

$$[(k^2+\Delta_\rho)^{1/2}+(k^2+\Delta_{\rho_0})^{1/2}+(k^2+\Delta_{\rho'})^{1/2}-(k^{*2}+\Delta_{\rho''})^{1/2}]Q(\rho',\rho'',\rho,\rho_0)$$
$$=-ik^2\beta\int d\rho_1 g(\rho_1-\rho')g^*(\rho_1-\rho'')g(\rho_1-\rho_0)g(\rho-\rho_1), \tag{33}$$

which is easily solved with the help of the Fourier representation in the variables ρ.

We now consider the conditions under which the equation of nonlinear quasi-optics (3) is obtained from the equations thus found. The wave field inside the medium is described by the equation (26) for the Green's function, the coefficients of which and the initial condition are determined by the back-scattered field in the plane of the source $H(L,\rho,\rho_0)$. The function $H(L,\rho,\rho_0)$ satisfies the closed equation (30). The back scattering is due to the field ε; therefore, if we neglect the integral term in the operator $\hat{a}_\rho$, then the solution of the remaining equation has the form $H(L,\rho,\rho_0)=g(\rho-\rho_0)$, which corresponds to the presence of only the incident wave in the plane $x=L$.

Equation (26) then preserves its form and the operators are determined by the equalities

$$\hat{a}_{\rho_0}G(x,\rho;L,\rho_0)=i(k^2+\Delta_{\rho_0})^{1/2}G(x,\rho;L,\rho_0)$$
$$-k^2\int d\rho_1 G(x,\rho;L,\rho_1)\varepsilon(L,\rho_1;\bar{I}(\rho_1))g(\rho_1-\rho_0),$$
$$\hat{b}(\rho',\rho'')=W(\rho',\rho'')(\hat{a}_{\rho'}+\hat{a}_{\rho''}^*), \tag{34}$$

where

$$\bar{I}(\rho)=\int d\rho_1 d\rho_2 g(\rho-\rho_1)g^*(\rho-\rho_2)W(\rho_1,\rho_2).$$

As the initial condition (26) we use the equality

$$G(x,\rho;L,\rho_0)|_{L=x}=g(\rho-\rho_0). \tag{35}$$

It is obvious that the linear equation (26) is now equivalent to the integral equation

$$G(x,\rho;L,\rho_0)=g(L-x,\rho-\rho_0)-k^2\int_x^L dx_1\int d\rho_1 g(x-x_1,\rho-\rho_1)$$
$$\times\varepsilon(x_1,\rho_1;I(x_1,\rho_1))G(x_1,\rho_1;L,\rho_0), \tag{36}$$

which differs from the initial equation (21) only in the lower limit of integration. Consequently, the wave field (22) will satisfy the integral equation

$$U(x,\rho)=U_0(x,\rho)-k^2\int_x^L dx_1\int d\rho_1 g(x-x_1,\rho-\rho_1)\varepsilon(x_1,\rho_1;I(x_1,\rho_1))U(x_1,\rho_1), \tag{36'}$$

which describes the propagation of the wave forward and admits, generally speaking, of scattering at large angles (smaller, however, than $\pi/2$). The parabolic equation of quasi-optics (3) corresponds to the Fresnel expansion of the Green's function $g(x,\rho)$ in Eq. (36').

Equation (36') can be rewritten in the form of an operator equation. Differentiating it with respect to x and using (24) and (31′), we obtain

$$\partial U(x,\rho)/\partial x=-i(k^2+\Delta_\rho)^{1/2}U(x,\rho)$$
$$-{}^1/_2 ik^2(k^2+\Delta_\rho)^{-1/2}\varepsilon(x,\rho;I(x,\rho))U(x,\rho),\quad U(0,\rho)=U_0(\rho).$$

4. We studied above the scalar wave equation. We now consider the incidence of a plane electromagnetic wave on the layer $L_0<x<L$ of a nonlinear medium. Let (x,y) be the plane of incidence of the wave, and p and q the coordinates of the wave vector along x and y, so that $k^2=p^2+q^2$. In this case, the problem is described by the Maxwell equations for the electric field **E** and magnetic field **H**. It is well known (see, for example, Ref. 14) that it suffices to consider only two polarizations, when the field **E** of the wave is perpendicular to the plane of incidence or parallel to it.

In the first case, the electric field $\mathbf{E}=[0,0,\nu u(x)e^{iqy}]$, while the quantity $u(x)$ satisfies the wave equation

$$d^2u/dx^2+p^2u(x)+k^2\varepsilon(x,w|u(x)|^2)u(x)=0 \tag{37}$$

with the condition of continuity of $u(x)$ and $du(x)/dx$ on the boundary. This equation is identical with Eq. (4); therefore, we can immediately use the results obtained above.

In the second case, in which the electric field of the wave is parallel to the plane of incidence, it is more convenient to study the equation for the magnetic field $\mathbf{H}=[0,0,\nu\varepsilon_0^{1/2}u(x)e^{iqy}]$:

$$\frac{d^2u}{dx^2}-\frac{\varepsilon'(x)}{1+\varepsilon(x)}\frac{du}{dx}+p^2u(x)+k^2\varepsilon(x)u(x)=0, \tag{38}$$

where $\varepsilon'(x)=d\varepsilon/dx$ and v is the amplitude of the electric field in the plane of incidence. Then the quantities $u(x)$ and $[1+\varepsilon(x)]^{-1}du/dx$ are continuous on the boundary. Since, in the given case,

$$E_x(x,y)=\frac{i\omega}{ck^2(1+\varepsilon)}\frac{\partial H_z}{\partial y}$$
$$E_y(x,y)=-\frac{i\omega}{ck^2(1+\varepsilon)}\frac{\partial H_z}{\partial x}, \tag{39}$$

it follows that

$$\varepsilon_x=\varepsilon(x,|E|^2)=\varepsilon\left(x,\frac{w}{|k|^2|1+\varepsilon_x|^2}\left[|qu(x)|^2+\left|\frac{\partial u(x)}{\partial x}\right|^2\right]\right). \tag{40}$$

Here, as before, $w=|v|^2$. By virtue of the boundary conditions on the boundary $x=L$,

$$u(x)|_{x=L}=u_L(w),\quad \left.\frac{\partial u(x)}{\partial x}\right|_{x=L}=ip(1+\varepsilon_L)[u_L(w)-2]. \tag{41}$$

Setting $x=L$ in (40) and substituting the values (41), we obtain a transcendental equation for the quantity ε_L:

$$\varepsilon_L=\varepsilon\left(L,\frac{w}{|k|^2|1+\varepsilon_L|^2}[|qu_L|^2+|p|^2|1+\varepsilon_L|^2|u_L-2|^2]\right). \tag{42}$$

We proceed next as in the second section. Equation (38) is equivalent to the integral

$$u(x;L,w)=g(x;L,w)-$$
$$-\frac{ip}{2}\int_{L_0}^{L}dx'g(x;x',w)\left(1-\frac{q^2}{p^2}\frac{1}{1+\varepsilon_{x'}}\right)\varepsilon_{x'}u(x';L,w), \tag{43}$$

where the function $g(x;x',w)$ is defined by the equality

$$g(x;x',w)=\exp\left[ip\,\mathrm{sgn}(x-x')\int_{x'}^{x}d\xi(1+\varepsilon_\xi)\right]. \tag{44}$$

It is now easy to find the equations analogous to (13) and (16):

$$\frac{\partial u(x;L,w)}{\partial L}=a(L,w)u(x;L,w)+b(L,w)\frac{\partial u(x;L,w)}{\partial w},$$
$$a(L,w)=ip\left[1+\varepsilon_L-\frac{\varepsilon_L}{2}u_L(w)\left(1-\frac{q^2}{p^2}\frac{1}{1+\varepsilon_L}\right)\right], \tag{45}$$
$$b(L,w)=w[a(L,w)+a^*(L,w)],\quad u(x;L,w)|_{L=x}=u_x(w),$$
$$\frac{\partial u_L(w)}{\partial L}=ip(1+\varepsilon_L)[u_L(w)-2]+a(L,w)u_L(w)+b(L,w)\frac{\partial u_L(w)}{\partial w}, \tag{46}$$
$$u_{L_0}(w)=1.$$

The coefficient of reflection of the wave from the layer is determined by the equation $R_L(w)=u_L(w)-1$.

If the medium occupies the halfspace $x<0$ and ε_x does not contain an explicit dependence on x, then we can, with the help of the limiting transition $L_0\to-\infty$, obtain an equation similar to (18) for the reflection coefficient:

$$b(w)\frac{dR_\infty}{dw}+2ip(1+\varepsilon_\infty)R_\infty-\frac{ip}{2}(1+R_\infty)^2\left(1-\frac{q^2}{p^2}\frac{1}{1+\varepsilon_\infty}\right)\varepsilon_\infty=0 \tag{47}$$

with the initial condition $R_\infty=0$ at $w=0$, if $\varepsilon(0)=0$.

In the case in which $\varepsilon=\beta|E|^2$, we obtain from formula (40) a cubic equation for the quantity ε_L. At low intensity of the incident wave ($\beta w\ll 1$) $\varepsilon_L\approx\beta w$, and we can seek a solution of Eq. (47) in the form $R_\infty=\tilde{R}w$. We then get for the quantity $\tilde{R}$ the following equation:

$$\tilde{R}=\frac{\beta}{4}\left(1-\frac{q^2}{p^2}\right)\frac{ip}{ip-\gamma},\quad k=\varkappa+i\gamma, \tag{47'}$$

which is identical with (18′) in the limiting case of grazing incidence $p\to 0$, $q\to k$.

[1]S. A. Akhmanov and R. V. Khokhlov, Problemy nelineĭnoĭ optike (Problems of Nonlinear Optics) VINITI (All-Union Institute of Scientific and Technological Information Press), Moscow, 1964.
[2]N. Bloembergen, Nonlinear Optics (Benjamin, 1965).
[3]F. Zernike, and J. Midwinter, Applied Nonlinear Optics, Wiley, 1973.
[4]A. V. Gurevich and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. 45, 1242 (1963) [Sov. Phys. JETP 18, 855 (1964)].
[5]V. P. Silin, Zh. Eksp. Teor. Fiz. 53, 1662 (1967) [Sov. Phys. JETP 26, 955 (1968)].
[6]F. G. Bass and Yu. G. Gurevich, Goryachie élektrony i sil'nye élektromagnitnye volny v plazme poluprovodnikov i gazovogo razryada (Hot Electrons and Strong Electromagnetic Waves in a Plasma of Semiconductors and of a Gaseous Discharge) Nauka, 1975.
[7]A. E. Kaplan, Zh. Eksp. Teor. Fiz. 72, 1710 (1977) [Sov. Phys. JETP 45, 896 (1977)].
[8]V. E. Zakharov and A. B. Shabat Zh. Eksp. Teor. Fiz. 61, 118 (1971) [Sov. Phys. JETP 34, 62 (1972)].
[9]G. B. Witham, Linear and Nonlinear Waves, Wiley, 1974.
[10]J. Casti and R. Kalaba, Imbedding Methods in Applied Mathematics, Addison-Wesley, 1973.
[11]G. I. Babkin and V. I. Klyatskin, Izvestiya vuzov, Radiofizika 23, No. 6 (180).
[12]V. I. Klyatskin, Stokhasticheskie uravneniya i volny v sluchaĭno-neodnorodynykh sredakh (Stoachastic Equations and Waves in Randomly Inhomogeneous Media) Nauka, 1980.
[13]G. I. Babkin, V. I. Klyatskin and L. Ya. Lyubavin, Dokl. Akad. Nauk SSSR 250, 1112 (1980) [Sov. Phys. Doklady 25,98 (1980)].
[14]V. L. Ginzburg, Rasprostranenie elektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in Plasma) Nauka, 1967 [Pergamon, 1970].

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