Tetrahedra equations and integrable systems in three-dimensional space

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A (2 + 1)-dimensional factorized theory is developed for the scattering on infinite one-dimensional objects—"straight-line strings." The conditions for factoring the S matrix of straight strings—the tetrahedra equations—are determined. The tetrahedra equations are also the conditions of Z-invariance of a statistical system on a three-dimensional lattice. An explicit solution is obtained for the tetrahedra equations in the special "static" limit. This solution makes it possible to construct a parametric family of operators (transfer matrices) that commute at different values of Θ.

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1. INTRODUCTION

In an examination of the problem of (1+1)-dimensional nonrelativistic particles with δ-function pair interaction, Yang has found that in the case of distinguishable particles the conditions for the self-consistency of the Bethe substitution are special functional relations that must be satisfied by the two-particle scattering amplitudes. Similar relations (factorization equations) were deduced subsequently in the study of (1+1)-dimensional relativistic factorized S matrices (see the review[2] and the references therein). In a formal construction of a factorized S matrices, these relations can be used as the equations that make it possible, jointly with the analyticity and unitarity conditions, to calculate explicitly all the elements of a two-particle S matrix.[1)] Some explicit solutions of the factorization equations can be found in the review[2] and in Refs. 4–6.

In the most general case of a relativistic factorized scattering theory, involving n different species of particles (numbered by the index $i=1, 2, \ldots, n$), the factorization equations take the form[7]

$$S_{i_1i_2}^{k_1k_2}(\theta)S_{k_1i_3}^{j_1k_3}(\theta+\theta')S_{k_2k_3}^{j_2j_3}(\theta')=S_{k_1k_2}^{j_1j_2}(\theta)S_{i_1k_3}^{k_1j_3}(\theta+\theta')S_{i_2i_3}^{k_2k_3}(\theta'), \quad (1.1)$$

where $S_{ij}^{kl}(\theta)$ is the two-particle S matrix; i and j (k and l) designate the species of the initial (final) particles, and θ is the difference between the rapidities of the colliding particles. Summation from 1 to n over all the repeated indices k is implied in (1.1). The functional relations with the structure (1.1) will be called the triangles equations.

Baxter[8,9] has formulated in a number of paper a statistical eight-vertex lattice model and constructed its exact solution in the case of a rectangular lattice. Baxter has also shown[10] that the remarkable properties of the eight-vertex model manifest themselves most clearly if it is formulated on a general irregular lattice made up by intersection of a large number L of arbitrarily directed straight lines on a plane (the axes of the lattice). For the model with the irregular lattice to remain exactly solvable, the vertex statistical weights must be assigned a special dependence on the vertex angle (i.e., the angle of intersection of the two axes in the given vertex). The Baxter model with irregular lattice has a remarkable symmetry, which Baxter called Z-invariance. The partition function of the model is not altered by arbitrary parallel shifts of any of the lattice axes. We note that such shifts, generally speaking, alter significantly the coordination structure of the lattice. In fact, the Z-invariance of the Baxter model is connected with the fact that the corresponding vertex weights satisfy the triangles equations (1.1), where S_{ij}^{kl} are the vertex weights (the indices i, j, k, and l denote the states of the four edges joining in a given vertex, and in the Baxter model they take on two values, $i=\pm 1$), and θ is the vertex angle. Actually, any nontrivial solution of the triangles equation (1.1) makes it possible to construct a lattice statistical model that possesses Z-invariance.[7,11]

All Z-invariant lattice systems are apparently exactly solvable. At any rate, the transfer matrix of any Z-invariant model placed on a regular lattice defines a fully integrable quantum systems that can be investigated by the inverse-problem quantum method recently proposed by Faddeev, Sklyanin, and Takhtadzhyan (Ref. 12).[2)]

It can thus be said in a certain sense that the existence of a large class of two-dimensional exactly solvable problems is due to the presence of nontrivial solutions of the triangles equations (1.1). The meaning of the triangles equations for factorized S matrices and for Z-invariant lattice systems is briefly discussed in Sec. 2.

The present paper is an attempt to generalize some of the constructions mentioned above to include the case of three-dimensional space. A natural three-dimensional analog of the irregular Baxter lattice[10] is a lattice made up of an aggregate of intersecting planes in three-dimensional space. It is possible to formulate on this lattice a rather general class of statistical systems by attaching fluctuating variables (colors) to the lattice faces and by specifying vertex statistical weights for all the lattice vertices. We shall define a three-dimensional statistical system as Z-invariant if its partition function remains unchanged for all parallel shifts of the planes making up the lattice. The Z-invariance requirement leads to special functional equations that

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must be satisfied by the vertex weights; we shall call them the tetrahedra equations. The tetrahedra equations are the three-dimensional analogs of the triangles equations (1.1).

At the same time, the tetrahedra equations can be interpreted as the facterization conditions for a specific (2+1)-dimensional scattering theory. In this theory the scattering objects are not particles, but one-dimensional formations of the type of infinite linear domain walls; we shall call them "straight strings." While such a scattering theory is unusual, it appears to be the closest (2+1)-dimensional analog of the (1+1)-dimensional purely elastic S matrices. Just as in the (1+1)-dimensional case, scattering leaves unchanged all the kinematic state parameters (the directions and velocities of all the "straight strings"), and a "multistring" S matrix can be factorized into S matrices of elementary ("three-string") collisions. The factorization relations—the tetrahedron equations—impose limitations on the form of these "three-string" amplitudes.

Even in the simplest cases, the number of independent functional relations contained in the tetrahedra equations is very large and, at any rate is many times more than the number of unknown functions. Our principal hypothesis is that in spite of this circumstance the tetrahedra equations admit of nontrivial solutions. Unfortunately, we did not succeed in obtaining even one complete solution: the main difficulty in the investigations of the tetrahedra equations is the presence of a complicated algebraic relation between the variables in the equations. It is possible, however, to consider a special limiting case ("the static limit") in which the tetrahedron equations, remaining nontrivial, become greatly simplified. In the present paper we formulate a concrete model of a factorized scattering theory of straight strings (or equivalently, a statistical system on a three-dimensional irregular lattice) and obtain an explicit solution of the corresponding tetrahedra equations in the static limit. Although the presence of solutions in the static limit is only a necessary and by for not sufficient condition for the existence of the solutions of complete tetrahedra equations, we regard the result as a significant confirmation of our hypothesis.

The plan of the article is the following. Section 2 is devoted to (1+1)-dimensional factorized S matrices and to two-dimensional Z-invariant lattice systems, and is introductory in character. In Sec. 3 is formulated a (2+1)-dimensional theory of scattering of straight string, and the thetrahedra equations are derived as the condition for factorization of a multistring S matrix. In Sec. 4 the tetrahedra equations are obtained as the conditions of Z-invariance of statistical systems on a three-dimensional irregular lattice. The static limit of the tetrahedron equations is formulated in Sec. 5. In Sec. 6 we obtain an explicit solution of the static tetrahedra equations for a concrete model. It is shown in Sec. 7 that this solution defines on a two-dimensional lattice a parametric family of transfer matrices which commute at all values of the parameter.

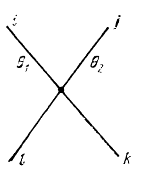


FIG. 1. Diagram representing the two-particle matrix $S_{ij}^{kl}(\theta)$; $\theta=\theta_1-\theta_2$.

2. FACTORIZED SCATTERING THEORIES, TRIANGLES EQUATIONS, AND Z-INVARIANT LATTICE SYSTEMS IN TWO-DIMENSIONAL SPACE

We present in this section a brief survey of the (1+1)-dimensional factorized scattering theory and of Z-invariant statistical systems on a flat lattice, since many details of these two-dimensional constructions will serve as the source for analogies in the treatment of the three-dimensional case.

Factorized S matrices (see, e.g., Ref. 2 and the references therein) arise, for example, in the study of quantum scattering in models of the (1+1)-dimensional field theory with soliton behavior, such as the sine-Gordon model (see Refs. 14 and 12). These S matrices contain only purely elastic channels, in which the following are conserved: a) the total number of particles, and b) the set of individual momenta of all the particles. If we assume for the sake of argument that the scattering theory contains n species of particles A_i ($i=1, 2, \ldots, n$) having identical masses m, and if we describe the asymptotic states of these particles by the values of their rapidities θ_n, defined by the formulas

$$p_a^0 = m \operatorname{ch} \theta_a, \quad p_a^1 = m \operatorname{sh} \theta_a, \tag{2.1}$$

where p_a^μ is the two-momentum of the a-th particle, then the restrictions a) and b) above are equivalent to the existence of a finite expansion.

$$|A_{i_1}(\theta_1), A_{i_2}(\theta_2), \ldots, A_{i_L}(\theta_L)\rangle_{out} = \sum_{j_1,\ldots j_L} S_{i_1, i_2, \ldots, i_L}^{j_1, j_2, \ldots, j_L}(\theta_1, \theta_2, \ldots, \theta_L) |A_{j_1}(\theta_1), \ldots, A_{j_L}(\theta_L)\rangle_{in}, \tag{2.2}$$

where

$$|A_{i_1}(\theta_1), A_{i_2}(\theta_2), \ldots, A_{i_L}(\theta_L)\rangle_{in\ (out)}$$

denotes the in (out) state of L particles $A_{i_1}, A_{i_2}, \ldots, A_{iL}$, having respective rapidities $\theta_1, \theta_2, \ldots, \theta_L$.

Relation (2.2) serves simultaneously as a definition of the L-particle S matrix

$$S_{i_1, \ldots, i_L}^{j_1, \ldots, j_L}(\theta_1, \ldots, \theta_L). \tag{2.2'}$$

Owing to relativistic invariance, the elements of the L-particle S matrix are functions of only the rapidity differences

$$\theta_{ab} = \theta_a - \theta_b \ (\theta_a > \theta_b); \quad a, b = 1, 2, \ldots, L.$$

The two-particle S matrix $S_{ij}^{kl}(\theta)$, defined by the expansion (2.2) with $L=2$, is the principal object in the construction of the factorized scattering theory. It can be shown[2] that its elements are meromorphic functions of θ (of the rapidity difference of the colliding particles), and are real at pure imaginary θ; they satisfy the two-particle unitarity conditions

$$S_{ij}{}^{nm}(\theta) S_{mn}{}^{kl}(-\theta) = \delta_i{}^k \delta_j{}^l \tag{2.3}$$

and the crossing symmetry relations

$$S_{ij}{}^{kl}(\theta) = S_{jk}{}^{li}(i\pi - \theta) \tag{2.4}$$

(we assume that all the particles A_i are real: $CA_i = A_i$, where C is the charge conjugation). If we confine ourselves to P- and T-invariant scattering theories, then the matrix $S_{ij}^{kl}(\theta)$ should satisfy in addition the relations

$$S_{ij}^{kl}(\theta)=S_{kl}^{ij}(\theta)=S_{ji}^{lk}(\theta). \tag{2.5}$$

It will be convenient hereafter to represent the matrix $S_{ij}^{kl}(\theta)$ by the diagram shown in Fig. 1. To each of the two intersecting lines (which can be arbitrarily regarded as space-time particle trajectories) is assigned one of the rapidities θ_1 and θ_2 ($\theta_1 > \theta_2$), and the outer ends of the lines are marked by the indices i, j and k, l, which designate the species of the initial and final particles.

For a purely elastic S matrix, an arbitrary L-particle element is factorized into $L(L-1)/2$ two-particle S matrices (see Ref. 2). Concretely speaking, to construct the L-particle S matrix (2.2′) (we assume for the sake of argument that $\theta_1 > \theta_2 > \ldots > \theta_L$) it is necessary to consider a diagram consisting of L intersecting lines l_a:

$$x^0 \operatorname{sh} \theta_a - x^1 \operatorname{ch} \theta_a = \xi_a, \tag{2.6}$$

where $x^\mu = (x^0, x^1)$ are the coordinates of the Minkowski plane, while the parameters ξ_a are arbitrary (an example of such a diagram for the case $L = 4$ is shown in Fig. 2). The described diagram has, generally speaking, $L(L-1)/2$ vertices—points of pairwise intersection of the lines (it is assumed that no three lines ever intersect in a single point). We set the intersection points of lines l_a and l_b in correspondence with two-particles S matrices $S(\theta_{ab})$, where $\theta_{ab} = \theta_a - \theta_b$; $a > b$. Next, we assign to the upper (lower) "external" ends of the lines l_a the indices i_a (j_a), which number the species of the initial (final) particles, and label $L(L-2)$ "inner" segments of the lines l_a with summation indices k_s; $s = 1, 2, \ldots, L(L-2)$. The L-particle S matrix (2.2′) is equal to the product of two-particle S matrices over all the vertices of the diagram, and summation from 1 to n must be carried out over all the "inner" indices k_s.

For a prescribed two-particle S matrix $S_{ij}^{kl}(\theta)$, the described procedure is not unambiguous. The point is that by specifying the rapidities $\theta_1, \theta_2, \ldots, \theta_L$ we do not define uniquely the structure of the diagram; for given directions of the lines l_a it is possible to have essentially different diagrams that differ by parallel shifts of one or several lines. Different diagrams, obviously, correspond to different formal expressions for the L-particle S matrix in terms of the two-particle matrices. For the entire construction of the factorized S matrix to be self-consistent it is necessary that the L-particle S matrices constructed from such different diagrams actually coincide. The physical reasons for this requirement are discussed in Ref. 2. It is easily understood that to satisfy this requirement it suffices to have equality of the triangular diagrams shown in Fig. 3 for all values of the outer indices $i_1, i_2, i_3, j_1, j_2, j_3$ and for all values of the rapidities θ_1, θ_2, and θ_3. The equality of the triangles on Fig. 3 is expressed in essence by the functional equation (1.1), which must be satisfied by the two-particle S matrices of any factorized scattering theory. We shall call Eqs. (1.1) the factorization relations or the triangles equations.

At present, an appreciable number of explicit examples of factorized S matrices is known. Most were obtained by direct solution of the triangles equations (1.1) and of the unitarity and analyticity conditions (2.3) and (2.4). Examples can be found in a number of papers.[2,4-6]

Judging from the very method by which it is constructed from two-particle S matrices, the matrix element (2.2′) of a multiparticle S matrix can be regarded as a partition function of a certain lattice statistical system connected with a definite irregular lattice. More accurately speaking it is necessary to consider the analytic continuation of the L-particle S matrix to pure imaginary values of all the rapidities

$$\theta_a = i\alpha_a. \tag{2.7}$$

In this case all the elements of the matrices $S_{ij}^{kl}(\theta_{ab})$, and hence of the matrix (2.2′), will be real. We choose real variables α_a such that

$$\pi \geqslant \alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_L \geqslant 0. \tag{2.8}$$

Following this continuation, the diagram representing the L-particle S matrix (see Fig. 2) can be treated as a certain irregular lattice on a Euclidean plane (x^1, x^2) consisting of L intersecting lines l_a specified by the equations

$$x^2 \sin \alpha_a - x^1 \cos \alpha_a = \xi_a. \tag{2.9}$$

The variables $\alpha_{ab} = i\theta_{ab}$ are the geometric intersection points of the lines l_a and l_b. We shall call the points of intersection of the lines l_a and l_b the vertices of the lattice V_{ab}, and the quantities $\alpha_{ab} (a > b)$ the vertex angles. The segments of the lines l_a that join two "neighboring" vertices are the edges (bonds) of the lattice. The irregular lattice described above will be designated $\mathscr{L}(\{\alpha_a\}, \{\xi_a\})$.

Statistical models on the lattice $\mathscr{L}(\{\alpha_a\}, \{\xi_a\})$, which are connected with factorized S matrices (we shall call them S-models) are formulated in analogy with the Bax-

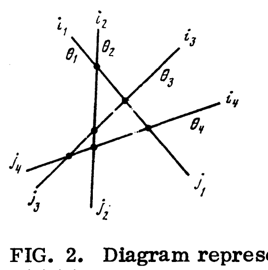


FIG. 2. Diagram representing the four-particle S matrix $S_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4}(\theta_1, \theta_2, \theta_3, \theta_4)$. The intersections of the straight lines correspond to two-particle S matrices.

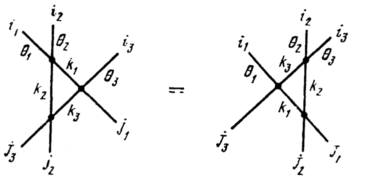


FIG. 3. "Factorization relation," viz., the equality of triangle diagrams that differ from each other by parallel shift of any of the straight lines l.

ter eight-vertex model[8-10]: the fluctuation variables $i=1,2,\ldots,n$ ("spins" or "colors") are set in correspondence with the edges of the lattice. Each configuration of colors i, j, k, and l on the four edges meeting in a given vertex V_{ab} (see Fig. 1) is assigned a statistical weight $S_{ij}^{kl}(i\alpha_{ab})$, which is a function (common to all vertices) of the vertex angle α_{ab}.

Thus, the two-particle S matrix plays the role of the matrix of the vertex weights. The partition function is defined as the sum over all the color configurations of all the edges of the lattice $\mathscr{L}(\{\alpha_a\},\{\xi_a\})$, and each configuration is taken with a weight equal to the product of the vertex weights over all the vertices. Owing to the symmetry property (2.4), it does not matter which of the angles, α_{ab} or $\pi-\alpha_{ab}$, is regarded as the vertex angle at a given vertex V_{ab}.

To determine completely the partition function of an S model on a lattice $\mathscr{L}(\{\alpha_a\},\{\xi_a\})$ it is necessary to specify the boundary conditions. The simplest possibility corresponds to the following definition of the partition function:

$$Z(\{\alpha_a\})=S_{i_1,\ldots,i_L}^{i_1,\ldots,i_L}(i\alpha_1,\ldots,i\alpha_L). \tag{2.10}$$

Summation over all the indices $i_a, a=1,\ldots,L$, is implied here.

In the formulation of the S model on the lattice $\mathscr{L}(\{\alpha_a\},\{\xi_a\})$ it is understood that the number L is chosen to be large enough to permit the system to exhibit statistical properties.

The partition function of any S model has a remarkable symmetry: for a lattice $\mathscr{L}(\{\alpha_a\},\{\xi_a\})$ it does not depend on the choice of the parameters ξ_a, although the coordination structure of the lattice can change substantially when these parameters are changed. The invariance of the partition function under arbitrary parallel shifts of the lattice axes l_a is ensured by the triangle equations (1.1) for the matrix of the vertex weights S_{ij}^{kl}. This symmetry was first observed by Baxter[10] for the exactly solvable eight-vertex model on the lattice $\mathscr{L}(\{\alpha_a\},\{\xi_a\})$. Baxter called it Z-invariance, a designation we shall also use. Thus, all the S models are Z-invariant.

Many regular lattices are particular cases of the lattice $\mathscr{L}(\{\alpha_a\},\{\xi_a\})$. For example, a regular lattice of parallelograms with N columns, M rows, and vertex angle α [we designate such a lattice by $\mathscr{L}_{NM}(\alpha)$] is obtained by putting

$$\alpha_1=\alpha_2=\ldots=\alpha_N,\quad \alpha_{N+1}=\alpha_{N+2}=\ldots=\alpha_{N+M}=\alpha_1-\alpha,\quad N+M=L$$

and by choosing ξ_a in suitable fashion.[3)]

Baxter[10] has shown that the partition function of an exactly solvable eight-vertex model on a lattice

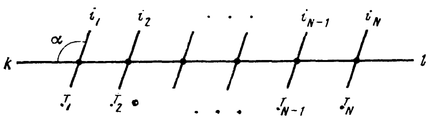


FIG. 4. Diagram showing the operator matrix $T_k^l(\alpha)$. The ends of the long horizontal line are assigned the matrix indices k and l, while the ends of the transverses correspond to the operator symbols of this matrix.

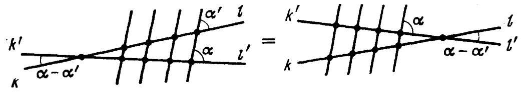


FIG. 5. Graphic representation of Eq. (2.14). Diagram b is obtained from a by successively shifting the transverses from right to left.

$\mathscr{L}(\{\alpha_a\},\{\xi_a\})$ in the thermodynamic limit ($L\to\infty$) is simply expressed in terms of the partition function of an eight-vertex model on a lattice $\mathscr{L}_{NM}(\alpha)(N,M\to\infty)$: the proof is entirely based on the Z-invariance property and can therefore be easily generalized to include the case of an arbitrary S model.[7]

The S models defined on a lattice $\mathscr{L}_{NM}(\alpha)$ are fully integrable and can be investigated by the inverse-problem quantum method.[12,13] In particular, any S model on a lattice $\mathscr{L}_{NM}(\alpha)$ is connected with a parametric family of transfer matrices $T(\alpha)$, which commute at different values of α:

$$T(\alpha)T(\alpha')=T(\alpha')T(\alpha). \tag{2.11}$$

The commutativity of (2.11) follows almost trivially from the relations (1.1).[4)] To verify this, we consider the operator matrix

$$T_{k\{i\}}^{l\{j\}}(\alpha)=S_{ki_1}^{k_1j_1}(i\alpha)S_{k_1i_2}^{k_2j_2}(i\alpha)\ldots S_{k_{N-1}i_N}^{lj_N}(i\alpha), \tag{2.12}$$

where $\{i\}=\{i_1,\ldots,i_N\}$ and $\{j\}=\{j_1,\ldots,j_N\}$ are the operator indices. This matrix has exactly the meaning of the global monodromy matrix, which plays the principal role in the quantum inverse-problem method.[12,13] It is convenient to represent it by the diagram shown in Fig. 4, where the graphic designation of Fig. 1 is used for the vertex-weight matrix $S_{ij}^{kl}(i\alpha)$. It is easily understood that the trace $T_k^l(\alpha)$ of the matrix coincides with the transfer matrix of the S model on the lattice $\mathscr{L}_{NM}(\alpha)$ with the periodic boundary conditions:

$$T(\alpha)=T_k^k(\alpha). \tag{2.13}$$

From (1.1) follows directly the relation

$$S_{kk'}^{nn'}(i\alpha-i\alpha')T_n{}^l(\alpha')T_{n'}^{l'}(\alpha)=T_{k'}^{n'}(\alpha)T_k{}^n(\alpha')S_{nn'}^{ll'}(i\alpha-i\alpha') \tag{2.14}$$

(we have left out the operator symbols of the matrices T_k^l), which is illustrated in Fig. 5. If we multiply both halves of (2.14) by $S_{ll'}^{kk'}(i\alpha'-i\alpha)$ and sum over k, k', l, and l', we obtain (2.11) by using the condition (2.3).

3. FACTORIZED THEORY OF SCATTERING OF STRAIGHT STRINGS IN (2+1)-DIMENSIONAL SPACE-TIME

In a space-time of dimensionality greater than two there can exist no nontrivial scattering theory that ensures conservation of all the kinematic parameters (individual momenta) in the collisions. This is true when it comes to the theory of particle scattering. With an aim of aconstructing a (2+1)-dimensional theory that is as close as possible an analog of the two-dimensional factorized scattering theory (see Sec. 1), we consider formally the scattering of one-dimensional objects—"straight strings." As the prototypes of the "straight strings" we can take infinite straight walls that arise in a number of models of (2+1)-dimensional classical field theory, for example in the (2+1)-dimensional Higgs model or in the (2+1)-dimensional since-Gordon model. We shall have in mind quantum objects of this type. The analysis is formal, i.e., without reference to any concrete model of field theory; in particular, many of the assumptions that we intend to use are in all probability not valid for the models mentioned above.

Assume that there exist quantum states corresponding to uniform free motion of an infinite straight string, and characterized by a constant momentum density along the string. The kinematics of such a state is described by a velocity 2-vector **v**, and the direction of this vector determines the orientation of the string on the plane (the vector **v** is normal to the direction of the string), while the length $v=|\mathbf{v}|$ is equal to the velocity (see Fig. 6). For a relativistically covariant description it is most convenient to introduce a unit 3-dimensional vector

$$n_\mu=(n_0,\mathbf{n}),\quad n_\mu n^\mu=n_0^2-\mathbf{n}^2=-1,\qquad n_0=v/(1-v^2)^{1/2},\quad \mathbf{n}=\mathbf{v}/v(1-v^2)^{1/2}. \tag{3.1}$$

Let, in addition, the "single-string" state be characterized by a certain quantum number (color), which can assume one of n values: $i=1,2,\ldots n$. We can imagine the string to be painted a certain color i over its entire length.

We assume next that there exist stationary states corresponding to two arbitrarily directed crossing strings moving uniformly with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ [in analogy with (3.1), we can introduce two 3-dimensional vectors n_1^μ and n_2^μ]. We assume that the two parts of this string, into which it is divided by the point of intersection with the other string, can be painted different colors. Accordingly, the two-string state is characterized by four colors (i_1,i_2,i_3,i_4), as shown in Fig. 7. It is important that all the characteristics of the two-string state (direction and velocity of each string, the presence of the intersection point, the coloring of each string element) remain unchanged in the course of motion, and therefore this state is more readily the state of free motion rather than a scattering state.

The situation is change if we consider a state with three moving straight strings s_1, s_2, and s_3 (here and elsewhere, when dealing with multistring states, we use the symbol s_a for the strings; $a=1,2,\ldots,L$, where L is the number of strings), described by the respective unit vectors n_1^μ, n_2^μ, and n_3^μ. As $t\to-\infty$, the three points of the pairwise intersection of the strings are well separated in space, and the color state of each of the strings $s_a, a=1,2,3$, is described by three colors (i_a,k_a,j_a), as shown in Fig. 8a. The entire state is

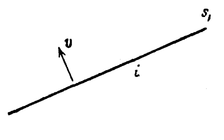


FIG. 6. Infinite straight string uniformly moving perpendicular to its length. The subscript $i=1,2,\ldots,n$ denotes the color of the string.

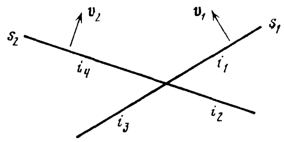


FIG. 7. Two infinite intersecting strings moving with velocities v_1 and v_2. The intersection point divides each string into two elements; the indices i_1, i_2, i_3, and i_4 denote the colors of these elements.

characterized consequently by nine colors. The string velocities are so directed that the size of the triangle in Fig. 8a decreases in the course of time. The instant of vanishing of this triangle will be called the collision of the three strings $s_a, a=1,2,3$.

Our most important assumption, analogous to the assumption that no particles are produced in collisions in the (1+1)-dimensional factorized scattering theory, is that after the collision there are produced only states of three "diverging" strings, having the same velocities and directions as the initial "converging" strings. The string that diverge after the collisions, which we shall likewise designate $s_a, a=1,2,3$, can differ from the initial ones only in the colors of the "inner" segments of each of the strings.

The possible state produced as a result of collision from the state shown in Fig. 8a, is shown in Fig. 8b, where in the general case $k_a'\neq k_a$. The states corresponding as $t\to-\infty$ $(t\to+\infty)$ to the three converging (diverging) strings s_1, s_2, and s_3 shown in Fig. 8a (Fig. 8b) will be called the *in* (*out*) states of the three-string scattering.

The gist of our assumption concerning the "pure elastic" character of the scattering of straight strings can be expressed by the formula

$$|(n_1^\mu;\, i_1, k_1', j_1)(n_2^\mu;\, i_2, k_2', j_2)(n_3^\mu;\, i_3, k_3', j_3)\rangle_{out} = \sum_{k_1,k_2,k_3} S^{i_1k_1j_1k_1'}_{\substack{i_2k_2j_2k_2'\\ i_3k_3j_3k_3'}}(n_1^\mu, n_2^\mu, n_3^\mu)\,|(n_1^\mu;\, i_1, k_1, j_1) \times(n_2^\mu;\, i_2, k_2, j_2)(n_3^\mu;\, i_3, k_3, j_3)\rangle_{in}. \tag{3.2}$$

This formula serves also as a definition of the three-string S matrix

$$S^{i_1k_1j_1k_1'}_{\substack{i_2k_2j_2k_2'\\ i_3k_3j_3k_3'}}(z^{(23)}, z^{(13)}, z^{(12)}), \tag{3.3}$$

which plays the principal role in our analysis. In (3.3) it is taken into account explicitly that owing to the relativistic invariance the elements of the three-string S matrix can depend only on the scalar products of the vectors n_a^μ:

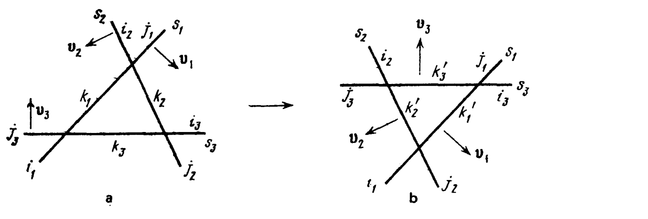


FIG. 8. Initial (a) and final (b) states of the scattering of three straight strings s_1, s_2, and s_3.

$$z^{(ab)}=z^{(ba)}=n_a{}^{\mu}n_{b\mu}. \tag{3.4}$$

By definition, the three-string S matrix has the symmetry

$$\mathrm{S}\,{}^{\substack{i_1k_1j_1k_1'}}_{\substack{i_2k_2j_2k_2' \\ i_3k_3j_3k_3'}}(z^{(23)},\ z^{(13)},\ z^{(12)})=\mathrm{S}\,{}^{\substack{i_2k_2j_2k_2'}}_{\substack{i_3k_3j_3k_3' \\ i_1k_1j_1k_1'}}(z^{(13)},\ z^{(12)},\ z^{(23)}). \tag{3.5}$$

We assume also T-invariance of the theory of the scattering of straight string, which can be expressed by the relation

$$\mathrm{S}\,{}^{\substack{i_1k_1j_1k_1'}}_{\substack{i_2k_2j_2k_2' \\ i_3k_3j_3k_3'}}(z^{(23)},\ z^{(13)},\ z^{(12)})=\mathrm{S}\,{}^{\substack{j_1k_1'i_1k_1}}_{\substack{j_2k_2'i_2k_2 \\ j_3k_3'i_3k_3}}(z^{(23)},\ z^{(13)},\ z^{(12)}). \tag{3.6}$$

The three-string S matrix must obviously satisfy the unitarity relation

$$\sum_{k_1'k_2'k_3'}\left[\mathrm{S}\,{}^{\substack{i_1k_1j_1k_1'}}_{\substack{i_2k_2j_2k_2' \\ i_3k_3j_3k_3'}}(z^{(23)},\ z^{(13)},\ z^{(12)})\right]\left[\mathrm{S}\,{}^{\substack{i_1l_1j_1k_1'}}_{\substack{i_2l_2j_2k_2' \\ i_3l_3j_3k_3'}}(z^{(23)},\ z^{(13)},\ z^{(12)})\right]^{*}=\delta_{l_1}{}^{k_1}\delta_{l_2}{}^{k_2}\delta_{l_3}{}^{k_3}, \tag{3.7}$$

where the asterisk denotes complex conjugation. Since the analytic properties of the amplitudes (3.3) are at present not quite clear, we do not know how to present in the general case (3.7) in the form of an analytic relation [similar to (2.3)] that is valid for all complex z.

We turn now to multistring scattering processes. We consider states containing L straight strings $s_a=(s_1, s_2, \ldots, s_L)$, whose motion is described respectively by the unit vectors $n_a^{\mu}=\{n_1^{\mu}, \ldots, n_L^{\mu}\}$. As $t\to\infty$ these states describe an aggregate L of diverging strings that intersect in a definite manner. The term "diverging strings" means here a string arrangement such that the given kinematics (set of velocities) excludes the possibility of any three-string collisions in the future. Any string s_a is divided by the points of its intersection with the other strings into L elements: to outer segments and L-2 inner ones; altogether we have thus L^2 segments.

L-string *out* states are defined as those which have at $t\to\infty$ definite coloring of all L^2 segments. The *in* states are similarly defined. Then the assumption that L-string scattering is "pure elastic" means that any *out* state of the strings $s_a=\{s_1, \ldots, s_L\}$, described by the vectors $n_a^{\mu}=\{n_1^{\mu}, \ldots, n_L^{\mu}\}$, can be expanded into a finite superposition of *in* states, each of which is described by the same set of vectors $\{n_a^{\mu}\}$ and has the same coloring of all the outer segments of the strings s_a as the *out* states; the inner segments of the strings, however, can change color as a result of the scattering. The coefficients of this expansion constitute the elements of the L-string S matrix.

In analogy with the $(1+1)$-dimensional purely elastic particle-scattering theory (see Sec. 2), where the L-particle S matrix is expressed in terms of a product of $L(L-1)/2$ two-particle S matrices, we shall assume that the L-string purely elastic S matrix is a product of $L(L-1)(L-2)/6$ three-string S matrices, in accordance with the representation of L-string scattering as a sequence of $L(L-1)(L-2)/6$ three-string collisions. It is important that the sequence in which these three-string collisions take place depends not only on the directions and velocities of all the strings s_a, but also on their initial arrangement (asymptotic coordinates). Depending on the asymptotic coordinates of the strings, different sequences of three-string collisions are possible, corresponding obviously to different formal expressions for the L-string S matrix in terms of the three-string scattering amplitude. For the entire factorized theory of scattering of straight strings to be self-consistent it is necessary that the L-string S matrices corresponding to different sequences of the three-string collisions actually coincide; this requirement is perfectly analogus to the factorization conditions for $(1+1)$-dimensional S matrices (see Sec. 2).

To clarify the foregoing and derive the string analog of Eq. (1.1), we consider the scattering of four strings, s_1, s_2, s_3, and s_4, characterized respectively by the vectors n_1^{μ}, n_2^{μ}, n_3^{μ}, and n_4^{μ}. The initial state of these strings is shown schematically in Fig. 9a. It is seen from this figure that there exist two alternative sequences of three-string collisions that make up the entire considered process. One of them begins with collision of strings s_1, s_2, and s_3 (turning-over of triangle Ia on Fig. 9a), and the other begins with collision of strings s_2, s_3, and s_4 (turning-over of triangle Ib in Fig. 9a).

We consider in greater detail the first possibility. After collision of the strings s_1, s_2, and s_3, corresponding to the factor

$$\mathrm{S}\,{}^{\substack{j_1l_1k_1l_1''}}_{\substack{i_3k_3l_3k_3'' \\ k_2l_2j_2l_2''}}(z^{(23)},\ z^{(12)},\ z^{(13)}) \tag{3.8}$$

in the four-string S matrix, the state shown in Fig. 9b is produced. This is followed by collision of strings s_1, s_2, and s_4 (turning-over of triangle IIa on Fig. 9b), the result of which is the state shown in Fig. 9c. The succeeding events evolve as shown in Figs. 9d and 9e. The state of Fig. 9e, which is the result of four three-string collisions corresponds to diverging strings and is the final state of the scattering. The four-string S matrix describing the process Fig. 9a $\to$ Fig. 9e and corresponding to this sequence of three-string collisions, is given by the left-hand side of Eq. (3.9) below.

As an alternate, the process of scattering of the state

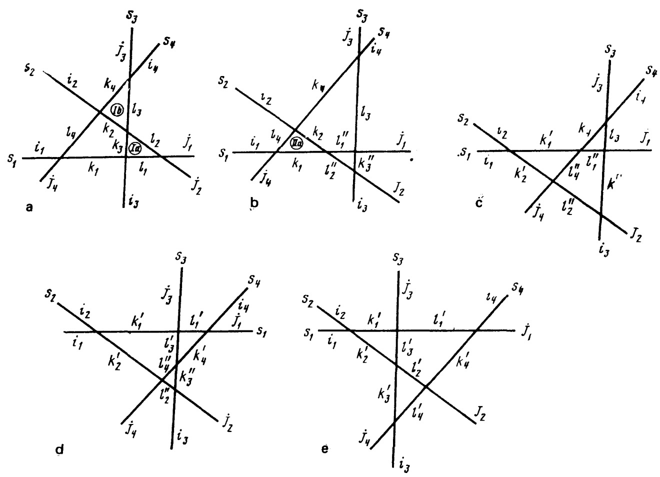


FIG. 9. One of the two possible sequences of configurations of four strings s_1, s_2, s_3, and s_4 constituting the four-string scattering process.

of Fig. 9a could start with turning over of triangle Ib. Then, four successive three-string collisions likewise result in the final stat of Fig. 9e, but the intermediate states differ from those shown in Figs. 9b–d. The four-string S matrix corresponding to this sequence of collision is given by the right-hand side of (3.9).

The factorization condition for the S matrix of straight strings is the requirement that the four-string S matrices corresponding to these two sequences of three-string collisions be equal:

$$
\begin{aligned}
&\sum_{l_1'',l_2'',l_4'',k_3''} S^{j_1l_1k_1l_1''}_{\substack{i_3k_3l_3k_3''\\ k_2l_2j_2l_2''}}(z^{(23)},z^{(12)},z^{(13)})\, S^{l_1''k_1i_1k_1'}_{\substack{j_4l_4k_4l_4''\\ i_2k_2l_2''k_2'}}(z^{(24)},z^{(12)},z^{(14)})\\
&\times S^{j_1l_1''k_1'l_1'}_{\substack{l_4''k_4i_4k_4'\\ j_3l_3k_3''l_3'}}(z^{(34)},z^{(13)},z^{(14)})\, S^{j_2l_2''k_2'l_2'}_{\substack{j_4l_4''k_4'l_4'\\ l_3'k_3''i_3k_3'}}(z^{(34)},z^{(23)},z^{(24)})\\
&=\sum_{k_1'',k_2'',k_4'',l_4''} S^{l_2k_2i_2k_2''}_{\substack{l_4k_4i_4k_4''\\ j_3l_3k_3l_3''}}(z^{(34)},z^{(23)},z^{(24)})\, S^{l_1k_1i_1k_1''}_{\substack{j_4l_4k_4''l_4'\\ l_3''k_3i_3k_3'}}(z^{(34)},z^{(13)},z^{(14)})\\
&\times S^{j_1l_1k_1''l_1'}_{\substack{l_4'k_4''i_4k_4'\\ k_2''l_2j_2l_2'}}(z^{(24)},z^{(12)},z^{(14)})\, S^{l_1'k_1''i_1k_1'}_{\substack{k_3'l_3''j_3l_3'\\ i_2k_2''l_2'k_2'}}(z^{(23)},z^{(12)},z^{(13)}). \qquad (3.9)
\end{aligned}
$$

The six variables $z^{(12)}$, $z^{(13)}$, $z^{(14)}$, $z^{(23)}$, $z^{(24)}$, and $z^{(24)}$ are not independent, but are connected by the algebraic relation

$$
\begin{aligned}
&1-(z^{(12)})^2-(z^{(13)})^2-(z^{(14)})^2-(z^{(23)})^2-(z^{(24)})^2-(z^{(34)})^2\\
&+(z^{(12)}z^{(34)})^2+(z^{(13)}z^{(24)})^2+(z^{(14)}z^{(23)})^2+2z^{(12)}z^{(24)}z^{(14)}+2z^{(13)}z^{(34)}z^{(14)}\\
&+2z^{(12)}z^{(23)}z^{(13)}+2z^{(23)}z^{(34)}z^{(24)}-2z^{(12)}z^{(23)}z^{(34)}z^{(41)}\\
&-2z^{(12)}z^{(24)}z^{(43)}z^{(31)}-2z^{(13)}z^{(32)}z^{(24)}z^{(41)}=0. \qquad (3.10)
\end{aligned}
$$

Thus, relations (3.9) constitute the functional equations that must be satisfied by the three-string S matrix in any factorized theory of scattering of straight strings. These equations are a direct analog of the triangles equations (1.1). For reasons that will be made clear in the next section, we shall call (3.9) the tetrahedra equations. It can be shown that when the tetrahedra equations are satisfied an L-string S matrix with any $L \geq 4$ does not depend on the sequence of the three-string collisions that make up the entire L-string scattering process.

The number of independent functional equations contained in (3.9) is in general quite large; at any rate, this number exceeds substantially the number of independent elements of the three-string S matrix (3.3). A most critical question in the entire proposed theory is the existence of solutions of the tetrahedra equations. Although at present we do not know of any explicit solution of these equations, we shall present in Secs. 6 and 8 our arguments in favor of their compatibility.

In concluding this section, we note the following circumstance. In $(1+1)$-dimensional space, the factorized scattering theory is a rather special particular case of the general relativistic particle-scattering theory, which admits, generally speaking, of all possible inelastic processes. In contrast, the $(2+1)$-dimensional theory of scattering of straight strings cannot be at all self-consistent if it is not factorized.

4. Z-INVARIANT STATISTICAL SYSTEMS ON A THREE-DIMENSIONAL IRREGULAR LATTICE

A $(1+1)$-dimensional factorized S matrix, at Euclidean values of the external momenta, can be regarded as a statistical system on a two-dimensional irregular lattice $\mathscr{L}(\{\alpha_a\},\{\alpha_a\})$. The $(2+1)$-dimensional theory of scattering of straight strings admits of an analogous interpretation.

We consider in three-dimensional Euclidean space an irregular lattice made up of intersections of L arbitrarily oriented planes, which we shall designate s_a, $a=1,2,\ldots,L$, i.e., in the same manner as used to designate the straight strings in the preceding section. The direction of each plane s_a can be described by a unit normal vector $n_a^\mu=\{n_a^1,n_a^2,n_a^3\}$; the plane s_a is thus given by the equation

$$x^\mu n_a^\mu = x^1 n_a^1 + x^2 n_a^2 + x^3 n_a^3 = \xi_a, \qquad (4.1)$$

where ξ_a are real numbers. The lattice structure is completely determined by the sets of vectors $\{n_a^\mu\}$ and of numbers $|\xi_a|$. We call this the $\mathscr{L}(\{n_a^\mu\},\{\xi_a\})$ lattice. We shall assume that no four of the planes s_a have a common point.

The planes s_a divide the three-dimensional space into polyhedra—the lattice cells. Pairs of planes, s_a and s_b, intersect in straight lines l_{ab}. Each plane s_a is broken up by the lines of intersection with the other planes into polygons, which we shall call the faces of the lattice. A distinction can be made between "outer" and "inner" faces: the "outer" faces are not compact (in particular, they are infinite in area), and the compact faces are called "inner." The points of intersections of triads of planes s_a, s_b, s_c are called the lattice vertices (and are designated V_{abc}), while the segments of the lines l_{ab} between "neighboring" vertices are the bonds (edges) of the lattice $\mathscr{L}(\{n_a^\mu\},\{\xi_a\})$. Each vertex is the junction of six bonds, twelve faces, and eight cells.

The following formulation of a statistical system on a lattice $\mathscr{L}(\{n_a^\mu\},\{\xi_a\})$ is a direct generalization of the formulation of the S models on the two-dimensional lattice $\mathscr{L}(\{\alpha_a\},\{\xi_a\})$ (see Sec. 2). We associate the faces of the lattice $\mathscr{L}(\{n_a^\mu\},\{\xi_a\})$ with the summation variables or colors, which take on n values: $i_p=1,2,\ldots,n$. The subscript p numbers here the faces. For each vertex V_{abc} we specify a set of n^{12} statistical weights (vertex weights) corresponding to all possible i colors on the 12 faces meeting in the given vertex. The set of vertex weights for a given vertex weights for a given vertex can be treated as a matrix with 12 indices—the vertex-weight matrix. We assume that the vertex-weight matrix is a function (the same for all vertices) of the relative orientation of the three planes intersecting at the given vertex. The relative orientations of the planes s_a, s_b, and s_c can be described by three variables:

$$z^{(ab)}=n_a^\mu n_b^\mu,\quad z^{(ac)}=n_a^\mu n_c^\mu,\quad z^{(bc)}=n_b^\mu n_c^\mu; \qquad (a)$$

it is convenient three angles defined by

$$z^{(ab)}=\cos\theta^{(ab)},\quad z^{(ac)}=\cos\theta^{(ac)},\quad z^{(bc)}=\cos\theta^{(bc)}. \qquad (4.2)$$

Thus, the elements of the matrix of the vertex weights are functions (which we assume to be analytic) of the three angles θ.

To introduce a concrete notation for the vertex-weight matrix we consider the vertex produced by intersection of the three planes s_1, s_2, and s_3 (see Fig. 10a). Let the faces adjacent to this vertex have 12 colors: i_a, k_a, j_a and k_a'; $a=1,2,3$, as marked in Fig. 1b. This figure shows separately the three planes (s_1, s_2, s_3) as seen by the observer. We designate the vertex-weight element corresponding to this disposition of the colors on the faces in the following manner:

$$S^{\,i_1k_1j_1k_1'}_{\,i_2k_2j_2k_2' \atop i_3k_3j_3k_3'}(\theta^{(23)},\theta^{(13)},\ \theta^{(12)}). \tag{4.3}$$

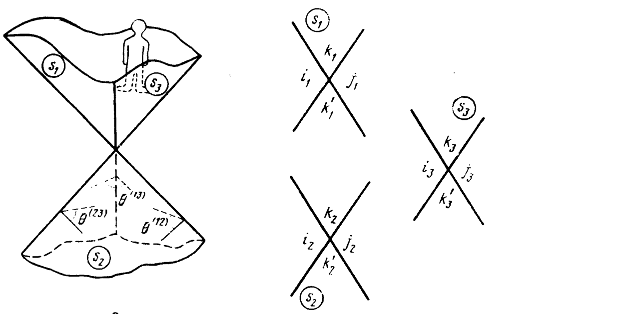
The angles $\theta^{(12)}$, $\theta^{(13)}$, and $\theta^{(23)}$ are shown in Fig. 10.

We stipulate that the vertex-weight matrix satisfy relations (3.5) and (3.6) and, in addition, that it have the "crossing symmetry":

$$S^{\,i_1k_1j_1k_1'}_{\,i_2k_2j_2k_2' \atop i_3k_3j_3k_3'}(\theta^{(23)},\ \theta^{(13)},\ \theta^{(12)})=S^{\,j_1k_1'i_1k_1}_{\,k_2'j_2k_2i_2 \atop k_3i_3k_3'j_3}(\theta^{(23)},\ \pi-\theta^{(13)},\ \pi-\theta^{(12)}). \tag{4.4}$$

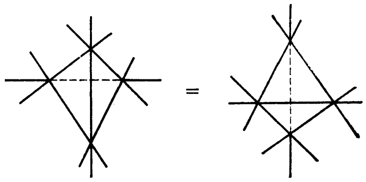
The partition function Z of such a system on the lattice $\mathscr{L}(\{n_a\},\{\xi_a\}$ is defined as the sum over all the possible colors of all the faces of the lattice, with each color configuration taken with a weight equal to the product of the vertex weights over all the lattice vertices. At an arbitrary choice of the matrix (4.3), the partition function depends both on the set of directions $\{n_a^\mu\}$ and on the parameters $\{\xi_a\}$. In analogy with the two-dimensional case, however, we can stipulate Z-invariance of the statistical system. In this case this stipulation is formulated as the requirement that the partition function Z be independent of the choice of the parameters ξ_a. In other words, the partition function must no be changed by a parallel transfer of any of the planes s_a making up the lattice.

The Z-invariance requirement is equivalent to definite functional equations for the vertex-weight matrix (4.3) (the tetrahedra equations), which can be obtained in the following manner. We consider four planes (s_1, s_2, s_3, and s_4) with normals n_1^μ, n_2^μ, n_3^μ, and n_4^μ, which form a tetrahedron in three-dimensional space (see Fig. 11a). Each of the planes $s_a, a=1,2,3,4$ is divided by the lines of intersection with the other plane into seven faces, six outer and one inner. We have thus altogether 24 outer and 4 inner faces (the latter are, in fact, the faces of the tetrahedron). We assign to the four vertices of the tetrahedron vertex-weight



a b

FIG. 10 a) Intersection of three planes s_1, s_2, and s_3 (the vertex of the lattice). b) The planes s_1, s_2, and s_3 as seen by an observer on Fig. a. The indices i_a, k_a, j_a, k_a'; $a=1,2,3$ denote the colors of the faces. The indicated coloring of the cases corresponds to the vertex weight (4.3).



a b

FIG. 11. Tetrahedra equation. The vertices of the tetrahedra correspond to the vertex-weight matrices (4.3). The colors of all the "outer" faces are fixed and the same in both sides of the equation; summation is carried out over all the colors of the inner faces.

matrices (4.3). Next, we fix the colors $i_1,\ldots,i_{24}$ of all the outer faces and sum the product of these four matrices over all possible color combinations of the four inner faces. We call the resultant quantity the statistical weight of the tetrahedron. The aggergate of these statistical weights can be regarded as a matrix with 24 indices $i_1,\ldots,i_{24}$, whose elements depend on the relative orientations of the planes $s_a, a=1,2,3,4$ (i.e., are functions of the angles $\theta^{(ab)}$).

By parallel displacement of any of the planes s_a (say s_1) we can transform the tetrahedron shown in Fig. 11a into the turned-over tetrahedron shown in Fig. 11b. The Z-invariance requirement is satisfied if for any color combination of the 24 outer faces and for all orientations of the planes s_1, s_2, s_3, s_4 the statistical weights of the tetrahedra of Figs. 11a and 11b are equal. It is readily understood that the requirement that the statistical weights of the tetrahedra of Figs. 11a and 11b be equal coincides exactly with the conditions for factorization of the straight-string S matrix (3.9), if the vertex-weight matrix (4.3) is identified with the three-string matrix (3.3). There exists thus a formal connection between the factorized theory of scattering of straight strings and the Z-invariant statistical systems on a three-dimensional lattice, a connection fully analogous to that discussed in Sec. 2 for the two-dimensional case.

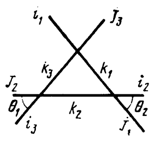
Z-invariant systems on a two-dimensional lattice are fully solvable. We propose that Z-invariant systems on the lattice $\mathscr{L}(\{n_a^\mu\},\{\xi_a\})$ (if they exist) are also exactly solvable.

5. STATIC LIMIT OF THE TETRAHEDRA EQUATIONS

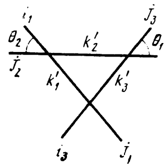
The algebraic equation (3.10) that connects the variables $z^{(ab)}$ greatly complicates the investigation of the tetrahedra equations (3.9). In this paper we study a certain limiting case of these equations—the static limit. In this and following sections the discussion is in terms of the theory of the scattering of straight strings (see Sec. 3).

Given the reference frame, the kinematics of an L-string state containing the strings $s_a, a=1,2,\ldots,L$, can be described by a set of velocity 2-vectors $\mathbf{v}_a, a=1,\ldots,L$ [see (3.1)]. We consider the L-string amplitudes and go formally to the limit $v_a\to 0, a=1,\ldots,L$, where $v_a=|\mathbf{v}_a|$. In this limiting case, which we call the static limit, the scattering amplitudes depend only on the directions of all the strings on the plane, i.e., actually on the set of angles between the directions of the various strings. For example, the elements of the three-string S matrix (3.3) in the static limit are functions of two independent angles.

We introduce a suitable notation. We agree to represent the asymptotic states of the scattering of the strings s_a by means of diagrams of the type shown in Figs. 8 and 9, on which are marked the colors of all the string segments and the angles between the string directions. We can then rewrite (3.2) in the static limit as follows:



$$= \sum_{k_1', k_2', k_3'} S\begin{smallmatrix} i_1k_1j_1k_1' \\ i_2k_2j_2k_2' \\ i_3k_3j_3k_3' \end{smallmatrix}(\theta_1, \theta_2)$$



(5.1)

The triangular diagram in the right (left) sides of the equations denotes the *in* (*out*) state, and the expansion coefficients

$$S\begin{smallmatrix} i_1k_1j_1k_1' \\ i_2k_2j_2k_2' \\ i_3k_3j_3k_3' \end{smallmatrix}(\theta_1, \theta_2) \tag{5.2}$$

denote a three-string S matrix in the static limit. This matrix has the symmetries

$$S\begin{smallmatrix} i_1k_1j_1k_1' \\ i_2k_2j_2k_2' \\ i_3k_3j_3k_3' \end{smallmatrix}(\theta_1, \theta_2) = S\begin{smallmatrix} j_1k_1'i_1k_1 \\ j_2k_2'i_2k_2 \\ j_3k_3'i_3k_3 \end{smallmatrix}(\theta_1, \theta_2) = S\begin{smallmatrix} i_2k_2j_2k_2' \\ i_3k_3j_3k_3' \\ i_1k_1j_1k_1' \end{smallmatrix}(\pi - \theta_1 - \theta_2, \theta_1) = S\begin{smallmatrix} j_3k_3i_3k_3' \\ j_2k_2i_2k_2' \\ j_1k_1i_1k_1' \end{smallmatrix}(\theta_2, \theta_1). \tag{5.3}$$

The last of these equalities expresses the P-invariance of the scattering theory.

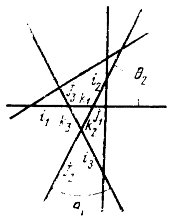
Using the analogy with particle scattering theory (Sec. 2), we make the assumption (which will be verified by the result of Sec. 6) that the elements of the static S matrix (5.2) are real in the "physical region"

$$0 \leqslant \theta_1 \leqslant \pi, \quad 0 \leqslant \theta_2 \leqslant \pi, \quad 0 \leqslant \pi - \theta_1 - \theta_2 \leqslant \pi. \tag{5.4}$$

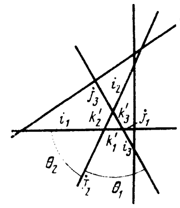
Then the unitarity condition (3.7) takes in the static limit the form

$$\sum_{k_1'k_2'k_3'} S\begin{smallmatrix} i_1k_1j_1k_1' \\ i_2k_2j_2k_2' \\ i_3k_3j_3k_3' \end{smallmatrix}(\theta_1, \theta_2)\, S\begin{smallmatrix} j_1k_1'i_1l_1 \\ j_2k_2'i_2l_2 \\ j_3k_3'i_3l_3 \end{smallmatrix}(\theta_1, \theta_2) = \delta_{k_1}^{l_1}\delta_{k_2}^{l_2}\delta_{k_3}^{l_3}. \tag{5.5}$$

Equation (5.5) allows us to forget that the diagrams in the right and left sides of (5.1) are states of different type (*in* and *out*, respectively), and regard (5.1) as a formal rule for transformation of the diagrams. This "diagram calculus" can be extended to include the case of arbitrary L-string diagrams. Any of the lines s_a constituting the L-string diagram can be parallel-shifted, and each passage of a given line through the intersection point to the other two lines (i.e., each turning-over of the triangle) corresponds to an expansion in accord with Eq. (5.1). For example, a five-string diagram is expanded into a superposition of diagrams:



$$= \sum_{k_1'k_2'k_3'} S\begin{smallmatrix} i_1k_1j_1k_1' \\ i_2k_2j_2k_2' \\ i_3k_3j_3k_3' \end{smallmatrix}(\theta_1, \theta_2)$$



(5,6)

We have left out the colors of some of the string segments; these colors are assumed fixed and equal in both sides of the equation.

For the described rule for transformation of L-string diagrams to be unique, it is necessary to stipulate that the coefficients of the expansion of any given diagram into diagrams obtained from the given one by a definite parallel shift of several lines s_a not depend on the sequence of these shifts. This requirement is equivalent to the following functional equation for the coefficients (5.2):

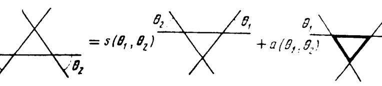
$$\sum_{l_1'', l_2'', l_4'', k_3''} S\begin{smallmatrix} i_3k_3l_3k_3'' \\ k_2l_2j_2l_2'' \\ j_1l_1k_1l_1'' \end{smallmatrix}(\theta_1, \theta_2)\, S\begin{smallmatrix} j_4l_4k_4l_4'' \\ i_2k_2l_2''k_2' \\ l_1k_1i_1k_1' \end{smallmatrix}(\theta_1, \theta_2 + \theta_3) \times S\begin{smallmatrix} l_4''k_4i_4k_4' \\ j_3l_3k_3''l_3' \\ j_1l_1''k_1'l_1' \end{smallmatrix}(\theta_1 + \theta_2, \theta_3)\, S\begin{smallmatrix} j_4l_4''k_4'l_4' \\ l_3'k_3''i_3k_3' \\ j_2l_2''k_2'l_2' \end{smallmatrix}(\theta_2, \theta_3) = \sum_{k_1'', k_2'', k_4'', l_3''} S\begin{smallmatrix} k_3'l_3''j_3l_3' \\ i_2k_2''l_2'k_2' \\ l_1'k_1''i_1k_1' \end{smallmatrix}(\theta_1, \theta_2)$$
$$\times S\begin{smallmatrix} l_4'k_4''i_4k_4' \\ k_2''l_2j_2l_2' \\ j_1l_1k_1''l_1' \end{smallmatrix}(\theta_1, \theta_2 + \theta_3)\, S\begin{smallmatrix} j_4l_4k_4''l_4' \\ l_3''k_3i_3k_3' \\ l_1k_1i_1k_1'' \end{smallmatrix}(\theta_1 + \theta_2, \theta_3)\, S\begin{smallmatrix} l_4k_4i_4k_4'' \\ j_3l_3k_3l_3'' \\ l_2k_2i_2k_2'' \end{smallmatrix}(\theta_2, \theta_3). \tag{5.7}$$

Relation (5.7) is the tetrahedra equation (3.9) in the static limit.

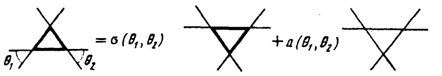
6. MODEL OF TWO-COLOR STRINGS. SOLUTION OF TETRAHEDRA EQUATIONS IN THE STATIC LIMIT

We consider now a concrete model of a (2+1)-dimensional theory of scattering of straight strings. Let the color index i take on two values, $i = 1, 2$. We shall say that the segments of the straight strings can be colored in two ways, black or white. We say that an L-string state is allowed if all the segments of its strings s_a, $a = 1, 2, \ldots, L$, are so colored that an even number of black segments meet at each point of intersection of two strings. Next, let all the elements of a three-string S matrix, which transform the allowed states into forbidden ones, be equal to zero. Then the scattering theory can be formulated in closed fashion in the sector of the allowed states.[5)] This scattering theory will be assumed invariant to P and T inversions, and also symmetrical with respect to replacement of all the black segment by white ones and vice versa (color symmetry).

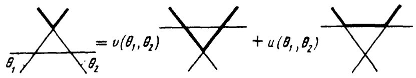
We consider in this section the described model of two-color strings in the static limit (see Sec. 5). All the independent nonzero elements of the three-string S matrix are defined in this model as the coefficients of the following expansions (in the formulas that follow, the black and white segments are represented by thick and thin lines, respectively):



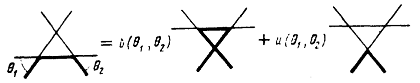
(6.1a)



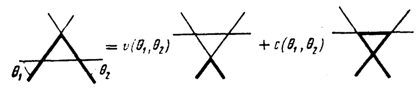
(6.1b)



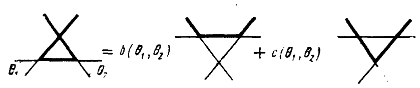
(6.1c)



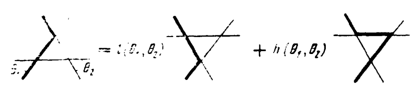
(6.1d)



(6.1e)

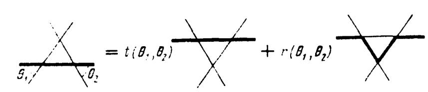


(6.1f)

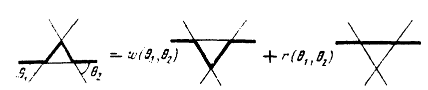


(6.1g)

(6.1h)



(6.1i)



(6.1j)

We seek for the described model the solution of the "static" tetrahedra equations; this solution has the property

$$a(\theta_1,\theta_2)=0,\quad r(\theta_1,\theta_2)=0. \tag{6.2}$$

In this case there are 11 different nonzero elements of the three-string S matrix. Obviously, the amplitudes s and σ must satisfy the symmetry relations

$$s(\theta_1,\theta_2)=s(\theta_2,\theta_1)=s(\theta_1,\pi-\theta_1-\theta_2),$$
$$\sigma(\theta_1,\theta_2)=\sigma(\theta_2,\theta_1)=\sigma(\theta_1,\pi-\theta_1-\theta_2), \tag{6.3}$$

and the eight amplitudes v, u, b, c, l, g, t, and ω must be symmetrical with respect to the interchange $\theta_1 \leftrightarrow \theta_2$.

The unitarity condition (5.5) imposes significant limitations on the form of the three-string amplitudes. For example, the unitarity conditions for channels (6.1g) and (6.1h) are of the form

$$l^2(\theta_1,\theta_2)+h^2(\theta_1,\theta_2)=g^2(\theta_1,\theta_2)+h(\theta_1,\theta_2)=1,$$
$$l(\theta_1,\theta_2)h(\theta_2,\theta_1)+h(\theta_1,\theta_2)g(\theta_1,\theta_2)=0, \tag{6.4}$$

consequently

$$g(\theta_1,\theta_2)=-\varepsilon l(\theta_1,\theta_2),\quad h(\theta_1,\theta_2)=\varepsilon h(\theta_2,\theta_1), \tag{6.5}$$

where ε is an arbitrary sign, $\varepsilon^2=1$, which is yet to be determined. Similarly, considering the unitarity conditions for the channels (6.1a) to (6.1f), (6.1i), and (6.1j), we get

$$s^2=\sigma^2=t^2=w^2=1,$$
$$b(\theta_1,\theta_2)=\varepsilon_1 v(\theta_1,\theta_2),\quad c(\theta_1,\theta_2)=-\varepsilon_1 u(\theta_1,\theta_2) \tag{6.6}$$

(here ε_1 is an undetermined sign, $\varepsilon_1^2=1$), and

$$l^2(\theta_1,\theta_2)+h^2(\theta_1,\theta_2)=1,\quad v^2(\theta_1,\theta_2)+u^2(\theta_1,\theta_2)=1. \tag{6.7}$$

We turn now to the tetrahedra equations. The concrete form of these equations for the model considered can be obtained both by writing out explicitly the general relations (5.7) and directly, by considering all possible "initial" states of four strings, expanding them in terms of the "final" states by two different methods (that differ in the sequence of the three-string collisions), as explained in Sec. 3, and equating the coefficients of these alternative expansions. The total number of the equations involved here is very large (several hundred). Six of them are shown in the Appendix by way of illustration.

We have written out 50 different equations (chosen randomly) and have verified that all are satisfied by the formulas

$$s(\theta_1,\theta_2)=\sigma(\theta_1,\theta_2)=t(\theta_1,\theta_2)=w(\theta_1,\theta_2)=1,$$
$$b(\theta_1,\theta_2)=l(\theta_1,\theta_2)=-g(\theta_1,\theta_2)=\varepsilon_1 v(\theta_1,\theta_2)=\left[\operatorname{tg}\frac{\theta_1}{2}\operatorname{tg}\frac{\theta_2}{2}\right]^{1/2},$$
$$h(\theta_1,\theta_2)=u(\theta_1,\theta_2)=-\varepsilon_1 c(\theta_1,\theta_2)=\varepsilon_2\left[\frac{\cos(\theta_1/2+\theta_2/2)}{\cos(\theta_1/2)\cos(\theta_2/2)}\right]^{1/2}, \tag{6.8}$$

where ε_1 and ε_2 are arbitrary signs, $\varepsilon_1^2=\varepsilon_2^2=1$.

7. TRANSFER MATRIX OF THREE-DIMENSIONAL LATTICE SYSTEM

It was explained in Sec. 4 that the solutions of the tetrahedra equations can be used to construct a lattice statistical system on any three-dimensional lattice $\mathscr{L}(\{n_a^\mu\},\{\xi_a\})$ that has Z-invariance. In this section we show that the static solution obtained in the preceding section makes it possible to construct a parametric family of transfer matrices $T(\theta)$ that commute at all values of the parameter θ.

We consider a planar lattice consisting of N arbitrarily directed intersecting straight lines s_a, $a=1,\ldots,N$ [i.e., in fact the lattice $\mathscr{L}(\{\alpha_a\},\{\xi_a\})$ defined in Sec. 2]. The detailed structure of this lattice is of no importance whatever in the reasoning that follows (this may be, e.g., a regular rectangular lattice). We shall therefore designate it simply by $\mathscr{L}$. An example of the lattice $\mathscr{L}$ is shown in Fig. 12. The convex dashed contour on this figure encloses all the intersection points of the lines s_a. The agregate of bonds contained within the contour will be called the interior of the lattice, and the bonds crossing the contour will be called the outer bonds.

Each inner bond of the lattice $\mathscr{L}$ should be colored in one of two ways, black or white. We introduce a certain space Φ of the color states of the lattice, spanning

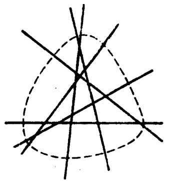


FIG. 12. Example of lattice $\mathscr{L}$.

all the possible color states of all the inner bonds (the allowed color states are defined at the beginning of Sec. 5), with all the outer bonds assumed to be white. We define a family of operators $T(\theta)$ that act in the space Φ in the following manner.

Let $\{i_r\}$ be a certain allowed color state of the inner bonds of the lattice $\mathscr{L}$ (the subscript r numbers the bonds). We introduce an auxiliary line s, directed at an angle θ to the "horizontal," and not crossing the inner part of the lattice, as shown in Fig. 13a. This line, as well as the entire outer part of the lattice $\mathscr{L}$ on Fig. 13a, is all colored white. Figure 13a can be regarded as a diagram corresponding to the $(N+1)$-string case in the static limit (see Secs. 5 and 6). By parallel shift of the line s we can move it over the entire inner part of the lattice $\mathscr{L}$. This procedure consists of successively transferring the line s through all the intersection points of the lines s_a contained inside the lattice $\mathscr{L}$. Each such elementary shift (turning-over of some triangle) corresponds to an expansion in accord with Eqs. (6.1) with coefficients (6.8). The result is an expansion of the diagram of Fig. 13a in terms of diagrams of the type of Fig. 13b. In the right side of this expansion there are terms of two types. We shall designate as "correct" all those expansion terms in which the entire line s is white after being transported through the lattice, as is also the entire outer part of the lattice; the remaining terms will be called "incorrect." The coefficients of the correct terms in the expansion of Fig. 13 are the matrix operators of the operator $T(\theta)$.

The operators $T(\theta)$ can be regarded as transfer matrices of certain three-dimensional lattice system of the type described in Sec. 4, with a special "white" boundary condition. In view of the equality $a(\theta_1, \theta_2) = 0$ [see (6.2)] a completely white lattice $\mathscr{L}$ is an exact eigenvector of the operators $T(\theta)$ for all θ.

We shall show that the operators $T(\theta)$ commute at different values of θ, i.e.,

$$T(\theta)T(\theta') = T(\theta')T(\theta). \tag{7.1}$$

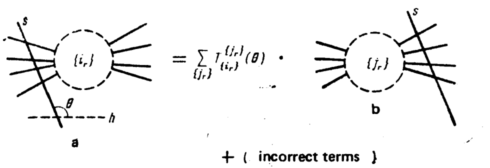


FIG. 13. Definition of operator $T(\theta)$. Only the outer part of the lattice $\mathscr{L}$ is shown explicitly. The coefficients $T\{^{j}_{i}\}(\theta)$ in the right-hand side of the equation are the matrix elements of this operator. The meaning of the term "incorrect terms" is explained in the text. The dashed line h is the "horizontal".

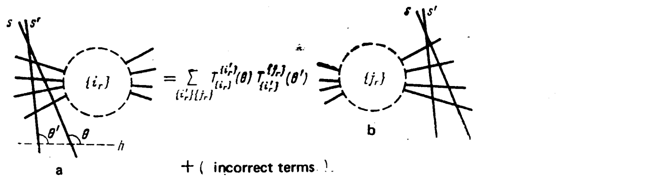


FIG. 14. Diagram equation resulting from successive shifting of the auxiliary lines s and s' over the inner part of the lattice $\mathscr{L}$; first to be shifted is s, followed by s'.

We introduce two auxiliary straight lines, s and s', which do not pass through the interior of the lattice and are directed respectively at angles θ and θ' to the horizontal, as shown in Fig. 14a. Both lines s and s', as well as the entire outer part of the lattice $\mathscr{L}$, are white. We transfer in succession (first s and then s') the two auxiliary line over the inner part of the lattice $\mathscr{L}$. The result of the transfer is shown in Fig. 13b. The coefficients of the correct terms in the right-hand side of the expansion on Fig. 14 coincide with the matrix elements of the product $T(\theta')T(\theta)$.

Since the coefficients in the expansions (6.1) satisfy the static tetrahedra equations, the result of any shift of the lines s and s' does not depend on the sequence in which the shift is made. In particular, the same expansion shown in Fig. 14 can be obtained by shifting the lines s and s' in the sequence shown in Fig. 15. The diagrams 15a and 15b are equal because $s(\theta_1, \theta_2) = 1$, $a(\theta_1, \theta_2) = 0$, and the outer part of the lattice is white. Similar arguments were used to obtain the equations in Figs. 15c and 15d. The coefficients of the exansion in Fig. 15d are the matrix elements of the operator $T(\theta)T(\theta')$. Equation (7.1) is therefore valid.

For a three-dimensional lattice system specified by the operator $T(\theta)$ and by some fixed θ, Eq. (7.1) means that there exists an infinite series of operators that commute with the transfer matrix and with one another. Of course, this still does not prove complete integrability of this system. The family of operators $T(\theta)$ can hardly contain the complete set of the integrals of motion. We propose that the static solution of the tetrahedra equations, obtained in Sec. 6, is the limiting case

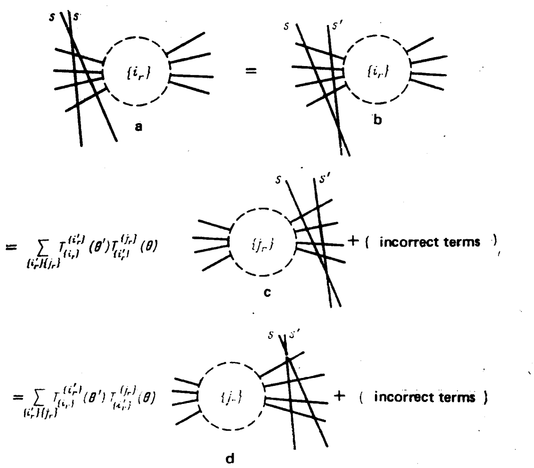


FIG. 15. Sequence of shifts of lines s and s', whereby s' is shifted over the inner part of the lattice ahead of s.

of a certain solution of the "complete" tetrahedra equations (3.8). If this is so, then there exists apparently a larger family of commuting operators $T(\theta, v)$ that depend, besides on the direction θ of the auxiliary line s, also on the rate v at which this line is shifter over the lattice $\mathscr{L}$. The family $T(\theta)$ determined by us is then the limiting case of $T(\theta) = T(\theta, v)\,|_{v\to 0}$.

I thank A. A. Belyavin, A. V. Mikhaĭlov, A. M. Polyakov, and V. A. Fateev for helpful discussions, and also Yu. G. Stroganov for pointing out an error in the initial draft of the article.

APPENDIX

In the cited equations we used the following abbreviations:

$$\theta_{12}=\theta_1+\theta_2, \quad \theta_{23}=\theta_2+\theta_3, \quad \theta_{123}=\theta_1+\theta_2+\theta_3;$$

$$\begin{aligned} &b(\pi-\theta_{12}, \theta_2)c(\pi-\theta_{123}, \theta_{23})l(\pi-\theta_{123}, \theta_3)c(\theta_3, \pi-\theta_{23}) \\ &+u(\pi-\theta_{12}, \theta_2)\sigma(\theta_1, \theta_{23})u(\theta_{12}, \theta_3)b(\theta_3, \pi-\theta_{23}) \\ &=b(\theta_2, \pi-\theta_{23})c(\theta_{12}, \pi-\theta_{123})l(\pi-\theta_{123}, \theta_1)c(\theta_1, \pi-\theta_{12}) \\ &+u(\theta_2, \pi-\theta_{23})\sigma(\theta_{12}, \theta_3)u(\theta_1, \theta_{23})b(\theta_1, \pi-\theta_{12}); \end{aligned} \tag{A.1}$$

$$\begin{aligned} &c(\theta_2, \pi-\theta_{12})b(\theta_{23}, \pi-\theta_{123})b(\theta_{12}, \pi-\theta_{123})c(\pi-\theta_{23}, \theta_2) \\ &=l(\theta_2, \theta_3)c(\theta_3, \theta_{12})c(\theta_{23}, \theta_1)l(\theta_1, \theta_2) \\ &+h(\theta_2, \theta_3)l(\theta_3, \pi-\theta_{123})l(\pi-\theta_{123}, \theta_1)h(\theta_1, \theta_2); \end{aligned} \tag{A.2}$$

$$\begin{aligned} &b(\theta_1, \pi-\theta_{12})u(\theta_{23}, \pi-\theta_{123})u(\theta_{12}, \pi-\theta_{123})c(\theta_3, \pi-\theta_{23}) \\ &+u(\theta_1, \pi-\theta_{12})s(\theta_1, \theta_{23})s(\theta_{12}, \theta_3)v(\theta_3, \pi-\theta_{23}) \\ &=s(\theta_2, \theta_3)v(\theta_3, \pi-\theta_{123})l(\theta_{23}, \pi-\theta_{123})u(\theta_1; \pi-\theta_{12}). \end{aligned} \tag{A.3}$$

$$\begin{aligned} &g(\theta_2, \pi-\theta_{12})t(\theta_1, \pi-\theta_{123})h(\theta_3, \pi-\theta_{123})b(\theta_2, \theta_3) \\ &+h(\theta_2, \pi-\theta_{12})h(\theta_{23}, \pi-\theta_{123})v(\theta_{12}, \theta_3)c(\theta_2, \theta_3) \\ &=v(\theta_3, \pi-\theta_{23})c(\theta_3, \pi-\theta_{123})v(\theta_1, \theta_{23})v(\theta_1, \pi-\theta_{12}), \\ &+u(\theta_3, \pi-\theta_{23})g(\theta_3, \theta_{12})c(\theta_1, \pi-\theta_{123})c(\theta_1, \pi-\theta_{12}); \end{aligned} \tag{A.4}$$

$$\begin{aligned} &g(\theta_2, \pi-\theta_{12})t(\theta_1, \pi-\theta_{123})l(\theta_3, \pi-\theta_{123})h(\theta_3, \pi-\theta_{23}) \\ &+h(\theta_2, \pi-\theta_{12})h(\theta_{23}, \pi-\theta_{123})u(\theta_{12}, \theta_3)g(\theta_3, \pi-\theta_{23}) \\ &=v(\theta_3, \pi-\theta_{23})c(\theta_3, \pi-\theta_{123})c(\theta_1, \theta_{23})c(\theta_1, \theta_2) \\ &+u(\theta_3, \pi-\theta_{23})g(\theta_3, \theta_{12})v(\theta_1, \pi-\theta_{123})v(\theta_1, \theta_2); \end{aligned} \tag{A.5}$$

$$\begin{aligned} &h(\theta_2, \pi-\theta_{12})l(\theta_{23}, \pi-\theta_{123})s(\theta_3, \theta_{12})t(\theta_2, \pi-\theta_{23}) \\ &=v(\theta_3, \pi-\theta_{23})v(\theta_3, \pi-\theta_{123})t(\theta_1, \theta_{23})h(\theta_2, \pi-\theta_{12}) \\ &+u(\theta_3, \pi-\theta_{23})h(\theta_3, \theta_{12})h(\theta_{23}, \pi-\theta_{123})l(\theta_2, \pi-\theta_{12}). \end{aligned} \tag{A.6}$$

1) This method was first proposed by Karowski, Thun, Truong, and Weisz.[3]

2) The triangles equations are in fact a component part of the quantum inverse-problem problem, since this equation is satisfied by the R matrix that defines the commutation relations between the elements of the global monodromy matrix (see Ref. 13).

3) Of course, the lattice $\mathscr{L}_{NM}(\alpha)$ does not differ in its coordinate structure from a rectangular lattice, and we speak of a lattice of parallelograms only to maintain the geometric meaning of the parameter α.

4) The idea of the derivation presented below stems from the papers of Baxter[8] and of Faddeev, Sklyanin, and Takhtadzhyan.[12]

5) In the "lattice" interpretation (see Sec. 4) of this model, the condition for allowed states corresponds to the fact that in the three-dimensional lattice $\mathscr{L}(\{n_a\}\{\xi_a\})$ it is permissible to color the faces black and white only in a way that the black faces form closed surfaces without edges.

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Translated by J. G. Adashko

Influence of spatial dispersion on the image forces and electron energy spectrum above the surface of liquid helium

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An analytic expression is obtained for the potential of the electrostatic image forces above the surface of liquid helium, with account taken of the spatial dispersion of its dielectric constant. The calculated frequencies of the transition between the surface electron levels agree well with the experimental data for He^3 and He^4.

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1. INTRODUCTION

The first to point out the possibility of the onset of localized electronic states over the surface of liquid helium under the influence of electrostatic image forces were Cole and Cohen[1,2] and Shikin.[3] The existence of such surface (two-dimensional) states was experimentally confirmed by Brown and Grimes[4,5] for He^4, and

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