Penetration of a magnetic field in a thin hollow superconducting cylinder

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The experimental results of Deaver and Fairbank [Phys. Rev. Lett. 7, 43 (1961)], and Doll and Näbauer [*ibid*., p. 51], Little and Parks [*ibid*. 9, 9 (1962) and Phys. Rev. **A133**, 97 (1964)] on the penetration of a longitudinal magnetic field in a thin hollow superconducting cylinder are discussed on the basis of the Ginzburg-Landau theory. It is shown that an important role is played here by the screening factor $\mu \approx r_1 d/\delta_L{}^2(T)$ (r_1 is the inside radius of the hollow cylinder, d is the wall thickness, and $\delta_L(T)$ is the London penetration depth). It is shown that in the case $\mu < 1$ the states of the cylinder are single-valued functions of the external field (there is no hysteresis). If $\mu > 1$ ambiguity appears (on account of states with frozen-in flux) and hysteresis is possible. It is shown that as $T \to T_c$ the hysteresis reported by Deaver and Fairbank and by Doll and Näbauer should vanish. On the other hand in the case considered by Little and Parks hysteresis may appear with increasing distance from T_c. The effect of the sample resistance oscillations observed by Little and Parks in magnetic field near T_c is discussed, as is also the parabolic growth of the resistance with the field.

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1. The question of penetration of a magnetic field in a hollow superconducting cylinder was discussed within the framework of the Ginzburg–Landau theory[1] in a number of theoretical papers.[2-4] It was observed that in view of the doubly connected geometry the magnetic field penetrates into the cavity in individual batches containing an integer number of flux quanta.[5] It was noted in addition that hysteresis states are present and give rise to "superheating" and "supercooling" fields in the magnetic field.[6,3] We recall in this connection the experiments of Kontarev,[7] who observed analogous effects in thick-wall cylinders.

We point out that the screening properties of a hollow cylinder differ greatly from those of a flat film in a magnetic field. Thus, the behavior of a flat film is determined by the ratio $\nu = d^2/\delta_L^2(T)$, where d is the film thickness and $\delta_L(T)$ is the London penetration depth. In the case of a thick-wall cylinder with wall width $d \ll \delta_L(T)$, as shown for example in Ref. 5, the equations contain a screening factor $\mu \approx r_1 d/\delta_L^2(T)$ (r_1 is the inside radius of the cylinder). This factor is much larger in the case $r_1 \gg d$ than the factor ν of a flat film of the same thickness d. As a result, the characteristic dimensions and the field values corresponding to first- and second-order transitions differ substantially in a cylinder[3] and in a flat film.[6,8]

The penetration of a magnetic field in a thin-wall cylinder was studied also in a number of experiments. Thus, Deaver and Fairbank[9] and Doll and Näbauer[10] observed, far from T_c, a jumplike penetration of the magnetic field into the cylinder, and measured the flux quantum $\Phi_0 = hc/2e \approx 2\cdot 10^{-7}$ G-cm². Little and Parks,[11,12] using thinner cylinders near T_c, observed an oscillatory dependence of the cylinder resistance R in an external field H_0 at a low measuring current. This effect, also due to successive penetration of individual flux quanta into the cylinder, was attributed in Refs. 11–15 to an oscillatory dependence of the film critical temperature on the magnetic field. Besides the resistance oscillations, Little and Parks observed, with increasing field, an additional parabolic background proportional to H_0^2 on the $R(H_0)$ plot, of unexplained origin. The experimental data were discussed in Refs. 11–15 on the basis of the Ginzburg–Landau theory, but the screening factor μ mentioned above was not taken into account in due manner. Under the experimental conditions of Refs. 9–12 this factor can be large; this leads, as will be shown below, to substantial peculiarities in the manner in which the field penetrates the cavity of the cylinder.

We present here formulas that describe the penetration of a field into a thin-wall cylinder, with screening taken into account, and discuss on the basis of these formulas the experimental results of Deaver,[9] Fairbank,[10] Doll,[11] and Näbauer.[12] It will be shown that the jumplike penetration of the field observed by Deaver *et al.*[9,10] changes when T_c is approached because of the change in the screening factor μ, and becomes smoother, while the hysteresis vanishes and the state of the cylinder becomes a single-valued function of the external field. Under the conditions of the experiments of Little and Parks,[11,12] however, with increasing distance from T_c and with increasing μ, ambiguity appears in the states and hysteresis becomes possible. In addition, it will be shown that allowance for the terms of order of d^2 in the formulas leads to the appearance, in the free energy of the cylinder, of a term proportional to the square H_0^2 of the external field, a term that must be taken into account when the aforementioned parabolic effect is discussed.[11,12]

2. According to the Ginzburg-Landau theory,[1] the free energy of a superconductor can be expressed in the form

$$F_s = F_{n0} + \int \left\{ \frac{H^2}{8\pi} + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4 + \frac{1}{2m}\cdot\left| i\hbar\nabla\Psi + \frac{2e}{c}\mathbf{A}\Psi \right|^2 \right\} dv, \qquad (1)$$

where F_{n0} is the free energy of the metal in the normal state in the absence of a field, Ψ is the wave function (order parameter) of the superconductor, $\alpha < 0$ and

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$\beta>0$ are temperature-dependent coefficients, and $\mathbf{A}$ is the vector potential of the magnetic field. As shown in Ref. 16 (see also Refs. 2 and 6), for a cylinder placed in an external magnetic field $\mathbf{H}_0$, the thermodynamic potential

$$\Phi_s = F_s - \frac{\mathbf{H}_0}{4\pi}\int_0^{r_2} \mathbf{H}\,dv + \frac{H_1^2}{8\pi}V_1, \tag{2}$$

is a minimum ($V_1=\pi r_1^2 l_z$ is the volume of the cylinder cavity, r_1 and r_2 are its inside and outside radii, and $\mathbf{H}_1$ is the field inside the cavity. We shall assume that the fields $\mathbf{H}_0$ and $\mathbf{H}_1$ are directed along the cylinder axis z. Since the problem is homogeneous along the z axis, a unity integration length ($l_z=1$) is implied in (1), (2), and everywhere below.

Using Maxwell's equations and integrating in (2) by parts, in analogy with the procedure used in Ref. 6 for the case of a solid cylinder, we arrive at the following expression for the thermodynamic potential of a hollow cylinder:

$$f=\frac{\Phi_s-\Phi_{n0}}{V_s H_{cm}^2/8\pi}=\psi^4-2\psi^2-\frac{4\pi}{V_s H_{cm}^2}\mathbf{M}\mathbf{H}_0+n\frac{\Phi_0}{8\pi}\frac{H_1-H_0}{V_s H_{cm}^2/8\pi}, \tag{3}$$

where Φ_s is defined in (2)

$$\Phi_{n0}=F_{n0}-VH_0^2/8\pi,\quad V=\pi r_2^2 l_z;$$

$V_s=\pi(r_2^2-r_1^2)l_z$ is the volume occupied by the superconductor, and $\mathbf{M}$ is the magnetic moment of the cylinder per unit length. It is taken into account in (3) that in the case of a cylinder $\Psi=|\Psi|e^{in\theta}$, where θ is the azimuthal angle in a cylindrical coordinate frame and n is an arbitrary integer that determines the number of flux quanta "frozen" into the cylinder. Equation (3) differs from that obtained by Ginzburg[6] in the last term, which is due to the presence of the cavity.

In Eq. (3) we have changed over, as is customary,[1,15] to reduced variables, introducing the quantities

$$H_{cm}^2=4\pi|\alpha|^2/\beta,\quad \Psi_0^2=|\alpha|/\beta,\quad \psi^2=\Psi^2/\Psi_0^2, \tag{4}$$

with

$$\delta_L^2(T)=mc^2\beta/16\pi e^2|\alpha|,\quad \xi^2(T)=\hbar^2/2m|\alpha|, \tag{5}$$

where $\xi(T)$ is the temperature-dependent coherence length of the superconductor; the parameter $\varkappa$ of the Ginzburg-Landau theory is equal to the ratio

$$\varkappa=\delta_L(T)/\xi(T). \tag{6}$$

For $\xi(T)$ we shall use also a relation that follows from the microscopic theory[15]:

$$\xi^2(T)=\frac{0.55\xi_0^2}{1-T/T_c}, \tag{7}$$

where ξ_0 is the pair-correlation length in the superconductor.

We shall consider the case of sufficiently thin cylinders, with a thickness defined by the condition

$$\xi_0<d<\delta_L(T),\ \xi(T). \tag{8}$$

The condition $\xi_0<\delta_L(T)$ actually defines the region of applicability of the Ginzburg-Landau theory. For type-I superconductors ($\varkappa<1/\sqrt{2}$) this region turns out to be quite narrow[17]: $T_c-T\ll\varkappa^2 T_c$.

By virtue of the conditions (8) we can regard the order parameter $|\psi|$ as a constant independent of the coordinates. Under these conditions the solution of the electrodynamic problem for the cylinder[2-4] leads to the following expression for the internal field H_1 and for the magnetic moment (we use the notation of Ref. 3):

$$H_1=\left[\frac{n\Phi_0}{\pi r_1^2}+\frac{2H_0}{\xi_1^2[K_0(\xi_1)I_0(\xi_2)-I_0(\xi_1)K_0(\xi_2)]}\right]\times\left[1+\frac{2}{\xi_1}\frac{K_0(\xi_2)I_1(\xi_1)+I_2(\xi_2)K_1(\xi_1)}{K_0(\xi_1)I_0(\xi_2)-I_0(\xi_1)K_0(\xi_2)}\right]^{-1}$$

$$M=-\frac{1}{4}\frac{r_2^2[aI_2(\xi_2)-bK_2(\xi_2)]-r_1^2[aI_2(\xi_1)-bK_2(\xi_1)]}{K_0(\xi_1)I_0(\xi_2)-I_0(\xi_1)K_0(\xi_2)}, \tag{9}$$

$$a=H_0K_0(\xi_1)-H_1K_0(\xi_2),\quad b=H_0I_0(\xi_1)-H_1I_0(\xi_2),$$

where $\xi_1=r_1/\delta$, $\xi_2=r_2/\delta$, $\delta=\delta_L(T)/|\psi|$, and K_n and I_n are Bessel functions of imaginary argument. Equations (9) are valid for arbitrary dimensions of the cylinder.

3. In the case of a thin wall cylinder ($d\ll\delta$), formulas (9) take a simpler form. Expanding the Bessel functions in powers of the small parameter $d/\delta\ll1$ at $d/r_1\ll1$ and retaining terms of order d^3 inclusive, we obtain for the thermodynamic potential (3) the expression

$$f=\psi^4-2\psi^2+\frac{2A(\Phi/\Phi_0^*-n)^2\psi^2}{1+\mu\psi^2/2}+C\left(\frac{\Phi}{\Phi_0^*}\right)^2\psi^2, \tag{10}$$

where

$$A=\frac{\xi^2(T)}{r_1^2}\left(1+\frac{1}{2}\frac{d}{r_1}\right),\quad C=\frac{2}{3}\frac{d^2}{r_1^2}\frac{\xi^2(T)}{r_1^2}\frac{1}{1+d/r_1},$$

$$\mu=\frac{r_1 d}{\delta_L^2(T)}\left(1+\frac{3}{2}\frac{d}{r_1}\right), \tag{11}$$

$$\Phi_0^*=\Phi_0\Big/\left(1+\frac{d}{r_1}\right),\quad \Phi_0=\frac{hc}{2e}=2\cdot10^{-7}\ \text{G-cm}^2,\quad \Phi=H_0\pi r_1^2.$$

The magnetic field H_1 in the cavity takes accordingly the form

$$H_1=H_0+\frac{\mu\psi^2/2}{1+\mu\psi^2/2}\frac{\Phi_0^*}{\pi r_1^2}\left(n-\frac{\Phi}{\Phi_0^*}\right). \tag{12}$$

Equations (10)–(12) are our fundamental formulas and will serve as the basis of the exposition that follows. We note first that expression (10) for the thermodynamic potential goes over into the expression used in Refs. 13–15 if one neglects the small terms $d/r_1\ll1$ and puts $\mu=C=0$. The resultant expression, cited by Tinkham[13,14] and by de Gennes,[15] was obtained by them within the framework of the Ginzburg-Landau equations under the assumption that the screening properties of a thin cylindrical film are the same as those of a flat film in a magnetic field parallel to their surface. In fact, as seen from (10) [as well as from the exact formula (3), cf. Refs. 2–4], a screening factor μ that can in general by large[1)] appears in a cylindrical film. The same factor enters in (12) and leads to a strong screening of the internal field ($H_1 \approx 0$ at $n=0$ and $\mu \gg 1$; cf. Ref. 3).

The presence of the last term proportional to $C\Phi^2$ in (10) is also understandable from simple physical considerations. In fact, with increasing field H_0, the internal field becomes comparable with the external field, $H_1 \approx H_0$, i.e., $n \approx \Phi/\Phi_0^*$ [see (12)]. Under these conditions the term with the coefficient A in (10) becomes small, and a cylindrical film in a magnetic field having the same value on the inside and outside behaves like a flat film in a field H_0 parallel to its surface. We note that the last term of (10) coincides exactly (at $r_1 \gg \delta_L$) with the corresponding expression for the thermodynamic potential of a flat film.[6,8] Thus, this term ensures the transition to the limit of a flat film in a strong field. The presence of the term $\sim CH_0^2$ that increases quadratically with the field in (10) causes the superconductivity of the film to become suppressed, sooner or later, by the external field. We shall see below that this term makes a substantial contribution to the parabolic effect observed in Refs. 11 and 12.

The number n in (10) is a function of the state of the system and can be determined from the requirement that the stable state of the system correspond to an absolute minimum of the thermodynamic potential f in the given external field H_0. With increasing field, transitions $n \to n+1$ should take place, and therefore the factor $\varphi_n = |\Phi/\Phi_0^* - n|$ in (10) is a bounded ($0 \leqslant \varphi_n \leqslant 1/2$) periodic function of the field. Thus, the periodic dependence is described by the terms $\sim(\Phi/\Phi_0^* - n)^2$ in (10).

To clarify the foregoing, we present several diagrams plotted with the aid of Eqs. (10)–(12) for the geometric parameters r_1 and d corresponding to the experimental conditions of Refs. 9–12. Figure 1 shows plots of the thermodynamic potential f [Eq. (10)] as functions of ψ^2 and H_0 ($\Phi = \pi r_1^2 H_0$). It is clear that the possible state of the system corresponds to the minimum of the thermodynamic potential, i.e.,

$$\partial f/\partial\psi = 0 \tag{13}$$

($\partial^2 f/\partial\psi^2 > 0$). Equation (13) enables us to obtain the value of $\psi_0(H_0)$ at the point of the minimum of f. This equation, according to (10), reduces to an equation cubic in ψ^2 and can be solved in principle analytically with the aid of the Cardan formulas. This solution, however, is quite unwieldy, and for the sake of clarity we present the values of $\psi_0(H_0)$ and $H_1(H_0)$ obtained from (10) (13) by numerical means (see Figs. 2–4). The figures show also the difference $\Delta = f_{\min} - f_{\max}$ between the thermodynamic potentials at the extremal points corresponding to the minimum and maximum of f. An examination of the curves in Figs. 2–4 makes obvious both the periodic field dependence contained in (10) and the important role of the parabolic term $\sim C\Phi^2$. In the absence of this term the curves would be strictly periodic at all values of the field.

It is clear from Fig. 1b that the minima on the $f(\psi)$ curves at $\psi \neq 0$ vanish at the inflection point $\psi = \psi_{00}$, where

$$\partial^2 f/\partial\psi^2|_{\psi=\psi_{00}} = 0. \tag{14}$$

To find the simultaneous solution of Eqs. (13) and (14) it is necessary to consider two cases, $\mu < 1$ and $\mu > 1$. In the case $\mu < 1$ the solution of (13) and (14) yields $\psi_{00}^2 = 0$, $\phi = \phi_{cr}$, where $\phi = \Phi/\Phi_0^*$, while $\phi_{cr} = \Phi_{cr}/\Phi_0^*$ is the root of the equation

$$A(\phi - n)^2 = 1 - C\phi^2/2. \tag{15}$$

The superconductivity of a hollow cylinder thus vanishes gradually at $\mu < 1$ via a second-order phase transition ($\psi_{00} = 0$, see Fig. 1a) when the field reaches the value $\phi = \phi_{cr}$ from (15).

In the case $\mu > 1$, solution of Eqs. (13) and (14) yields

$$\psi_{00}^2 = {}^2/_3(1 - 1/\mu - {}^1/_2 C\phi_{cr}^2), \tag{16}$$

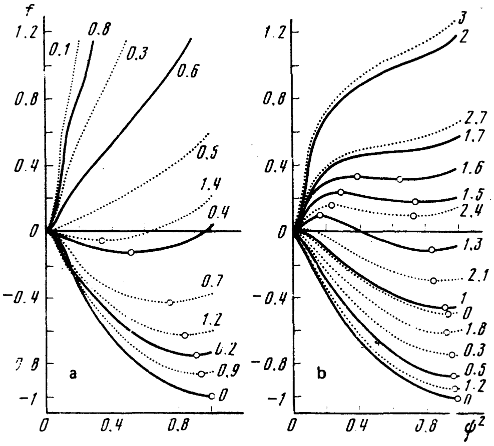

FIG. 1. Plots of $f(\psi^2)$ at various $\phi = \Phi/\Phi_0^*$ according to (10). The values of ϕ are marked on the curves. The solid lines correspond to the state $n=0$ and the dotted ones to $n=1$. The circles on the curves mark the extremum points of f (at $\psi^2 = \psi_0^2$). In Fig. b the minimum and the maximum coalesce at $\phi = 1.69$ at the inflection point $\psi_{00}^2 = 0.52$. The curves were plotted for a cylinder with the set of parameters $\mathscr{P}_{DF}$ (see footnote 2), a—at $t = 1 \times 10^{-4}$, b—at $t = 5 \times 10^{-4}$.

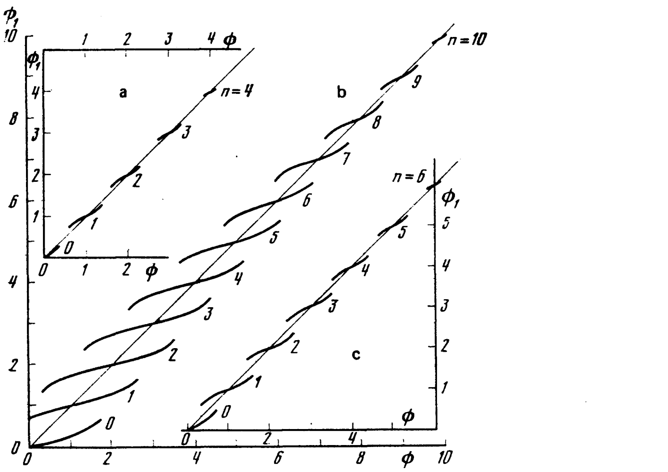

FIG. 2. Dependence of the internal field $\phi_1 = \Phi_1/\Phi_0^*$, $\Phi_1 = \pi r_1^2 H_1$) on the applied field ($\phi = \Phi/\Phi_0^*$, $\Phi = \pi r_1^2 H_0$) in states with different n for a cylinder with the parameters $\mathscr{P}_{DF}$: a—at $t = 1 \cdot 10^{-4}$ ($\mu = 1.2$ $n_{cr} = 4$), b—at $t = 5 \cdot 10^{-4}$ ($\mu = 5.6$ $n_{cr} = 10$), c—at $t = 2 \cdot 10^{-4}$ ($\mu = 2.2$ $n_{cr} = 6$).

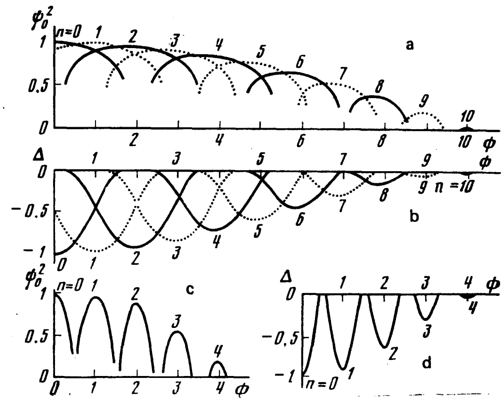

FIG. 3. Dependence of ψ_0^2 and Δ on ϕ in states with different n for a cylinder with the parameters $\mathscr{P}_{\rm DF}$. The curves of Figs. a and b—at $t=5\times10^{-4}$, c and d—at $t=1\times10^{-4}$.

where $\phi_{\rm cr}$ is the root of the equation

$$\mu A(\phi-n)^2=(^2/_3+{}^1/_3\mu-{}^1/_6\mu C\phi^2)^3. \tag{17}$$

Thus, if $\mu>1$ we have $\psi_{00}^2>1$ (at $1/\mu<1-1/2C\phi_{\rm cr}^2$) and the superconductivity of the hollow cylinder (in the n state) vanishes via a first-order phase transition (jumpwise at finite ψ_{00}, see Fig. 1*b*) when the field reaches the value $\phi=\phi_{\rm cr}$ from (17). In essence, the values of ψ_{00} and $\phi_{\rm cr}$ from (16) and (17) determine the points of the maximum "superheating" and "supercooling"[18] in a magnetic field [the extreme points on the $\phi_1(\phi)$ curves in Figs. 2 and 4].

In the case $\mu=1$ and $\phi_{\rm cr}=1$ the roots of Eqs. (15) and (17) coalesce. Obviously, the value $\mu\approx r_1 d/\delta_L^2=1$ separates the regions of the first and second order transitions (Ref. 2).[2)]

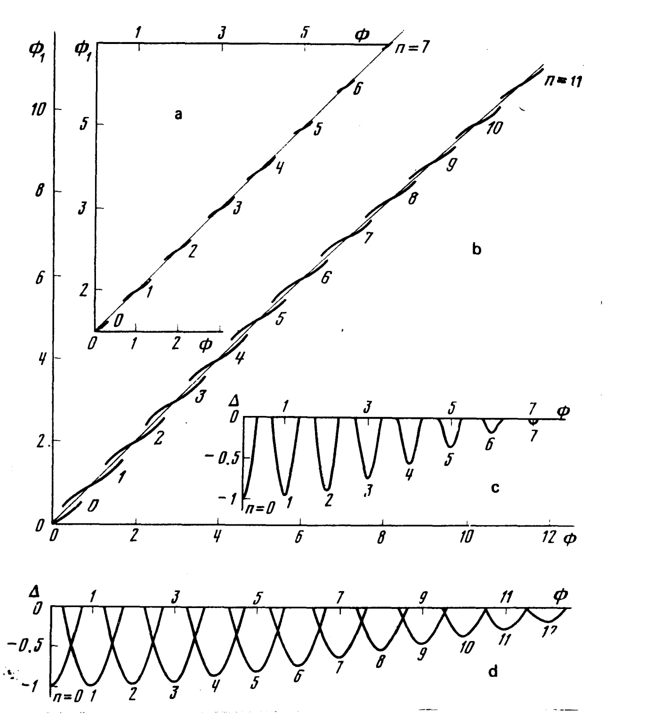

FIG. 4. Plots of ϕ_1 and Δ against ϕ in states with different n for cylinders with the parameters $\mathscr{P}_{\rm LP}$ (see footnote 1). Curves of Figs. a and b—at $t=1-T/T_c=5\times10^{-3}$ ($\mu=0.32, n_{\rm cr}=7$), c and d—at $t=2\times10^{-2}$ ($\mu=1.3, n_{\rm cr}=15$).

We determine also the slopes of the $\phi_1(\phi)$ curves on Figs. 2 and 4 at the points $\phi=n$ (the points of intersection with the straight line $\phi_1=\phi$). From Eqs. (13) and (12) we obtain at $\phi=n+\xi(\xi\ll1)$

$$\psi_0^2=1-{}^1/_2Cn^2-Cn\xi,\quad \phi_1=n+\xi\,\mathrm{tg}\,\alpha, \tag{18}$$

where $\tan\alpha\equiv d\phi_1/d\xi$ is equal to

$$\mathrm{tg}\,\alpha=1-\frac{{}^1/_2\mu(1-{}^1/_2Cn^2)}{1+{}^1/_2\mu(1-{}^1/_2Cn^2)}. \tag{19}$$

From (19) and from Figs. 2 and 4 it is clear that the slopes of the $\phi_1(\phi)$ curves depend on the dimensions of the cylinder and on the temperature [via $\mu(T)$ and $C(T)$] and on the number n. At $\mu\gg1$ we have $\tan\alpha\approx0$, i.e., for cylinders with a large screening factor μ the curves $\phi_1(\phi)$ are almost horizontal (as was the case in the experiments described in Refs. 9 and 10). The $\phi_1(\phi)$ curves then overlap, i.e., in a specified external field the system has several possible states with different values of n. This means that hysteresis is possible under these conditions. When the dimensions are decreased and as $T\to T_c(\mu\to0)$ the screening properties of the cylinder become weaker and $\tan\alpha\to1$, B, i.e., $\phi_1\approx\phi$. The hysteresis also vanishes in this case, since the states of the system become single-valued functions of the external field.

Putting $\xi=0$ in (18), we obtain the maximum value $n=n_{\rm cr}$ at which a superconducting transition is still possible ($\psi_0^2>0$):

$$n_{\rm cr}=[\sqrt{2/C}], \tag{20}$$

where the brackets $[a]$ denote the integer part of the number a. It is seen from (20) and from Figs. 2–4 that the number of oscillations of the curves $\phi_1(\phi)$ and $\Delta(\phi)$ is limited. With increase in temperature, $T\to T_c$, the number $n=n_{\rm cr}$ decreases in accord with (11) and (20).

In the case $\mu\ll1$ the cubic equation (13) becomes simpler and a more complete solution of the problem can be obtained:

$$\psi_0^2=\frac{1-A(\phi-n)^2-{}^1/_2C\phi^2}{1-\mu A(\phi-n)^2},\quad \psi_{00}=0, \tag{21}$$

$$\phi_1=\phi+\frac{{}^1/_2\mu\psi_0^2}{1+{}^1/_2\mu\psi_0^2}(n-\phi),\quad \Delta=f(\psi_0)=-\frac{[1-A(\phi-n)^2-{}^1/_2C\phi^2]^2}{1-\mu A(\phi-n)^2}.$$

Equations (21) describe analytically the behavior of the curves ψ^2, ϕ_1, and Δ given in Figs. 2–4 (at small values of μ).

4. In this section we discuss in greater detail the experiments of Little and Parks.[11,12] The effect of penetration of the field into the cylinder was recorded by them not directly (as in Refs. 9 and 10), but indirectly, by observing the change of the resistance R of the cylinder in the transition region near T_c in a magnetic field H_0 at a small measuring current.[3)] It was observed that $R(H_0)$ contained a component that oscillated with the field, against a background that increased with the field quadratically (and depended strongly on d, r_1, T, and the sample orientation). The number of oscillations of the resistance was limited, and with increasing field the resistance became a monotonic function of the field.

This behavior can be qualitatively explained on the basis of the thermodynamic analysis presented above. Indeed, it is intuitively clear that the appearance of a resistance in the transition region near T_c might be attributed to a fluctuation-induced onset of normal regions in the superconducting metal (cf. Refs. 20–25). However, estimates of the fluctuation resistance of the films near T_c, deduced from the formulas of Refs. 20–25, show that the temperature region in which these dynamic fluctuations are significant is very narrow ($T - T_c \lesssim 10^{-5}$ K). The smallness of this region is due to the following: 1) In our case we are considering films that are thicker by one or two orders than in Refs. 20–25. 2) The case of pure superconductors ($l \gg \xi_0$) is considered. 3) The case of type-I superconductors (relatively large ξ_0) is considered. It can therefore be assumed that the appearance of resistance in the transition region $T - T_c \sim 10^{-3} - 10^{-4}$ K is due not to dynamic fluctuations, but more readily to structural fluctuations, i.e., to inhomogeneities that broaden also the phase-transition curve.

As the measure of the proximity to the phase transition we can use the difference $\Delta = f_{\min} - f_{\max}$ between the neighboring minimum and maximum of the thermodynamic potential ($|\Delta|$ is the potential barrier that the system must overcome to go from the superconducting to the normal state). The increase of the resistance in the transition region can be interpreted here as an approach of the system to a purely normal state (in which case $|\Delta|$ decreases and the role of the structural fluctuations increases). Conversely, it can be assumed that the decrease of the resistance is due to the increased deviation of the system from the normal state ($|\Delta|$ increases and the role of the structural transitions is suppressed). The plots of $\Delta(H_0)$ ($\Phi = \pi r_1^2 H_0$) shown in Figs. 3 and 4 show the characteristic features of the regularities observed in Refs. 11 and 12, namely the presence of oscillations, the parabolic background, the existence of a limiting value $n_{\rm cr}$, and the dependence on the dimensions.

For a quantitative comparison of the theory with experiment, the intuitive arguments advanced above are of course insufficient. We need for this purpose a consistent theory in which the sample resistance can be directly expressed in terms of the free energy of the system, with account taken of the structural fluctuations. There is still no such theory for a cylindrical system. However, the qualitative arguments advanced are useful even in this case, so that greater attention must be paid to the role of the screening factor, and a number of effects that one can attempt to observe in experiment (for example, the specific dependences on the size and on the temperature, the interchange of the regions with and without hysteresis, and others).

As to the aforementioned parabolic effect, it is desirable here, too, to make a more detailed experimental investigation aimed at estimating the contribution made by the last term of (10) and the experimentally observed relations. It is probable that this term alone is incapable of describing the parabolic effect [in particular, no account is taken in the free energy (10) of the important component due to the inaccurate orientation of the cylinder in the magnetic field, cf. Ref. 15]. It is evident from the foregoing results, however, that the term $\sim C\Phi^2$ plays an essential role in (10) and must be taken into account.

5. In this section we deal with an interpretation of the resistance oscillation observed in Refs. 11 and 12 in terms of the oscillation of the effective transition temperature[4)] (see also Refs. 13–15). This interpretation follows, in particular, from the formulas (15) and (17) above. In fact, these formulas establish the presence of a critical value $\phi_{\rm cr}(T)$ at a specified measurement temperature T. At the point $\phi = \phi_{\rm cr}$ the minimum of the thermodynamic potential (at $\psi_0 \neq 0$) vanishes and the system should go over into the normal state ($\psi_0 = 0$). This transition can be effective at a given field ϕ by changing the sample temperature. Solving Eq. (15) for T [with account taken of (7) and (11)] we obtain at $\mu < 1$

$$\frac{T_c - T^*}{T} = 0{,}55\,\frac{\xi_0^2}{r_1^2}\left[a_0\left(\frac{\Phi}{\Phi_0^*} - n\right)^2 + c_0\left(\frac{\Phi}{\Phi_0^*}\right)^2\right],$$

$$a_0 = 1 + \frac{1}{2}\frac{d}{r_1}, \quad c_0 = \frac{1}{3}\frac{d^2}{r_1^2}\frac{1}{1 + d/r_1}. \tag{22}$$

Here T^* has the meaning of the effective transition temperature. (In Refs. 11 and 12, T^* was chosen to be the temperature of the steepest part of the function $R(T)$ in the transition region near T_c). Equation (22) goes over into the equation given in Refs. 13–15 if we replace in (22) the renormalized value Φ_0^* by Φ_0 and discard the last term in the square bracket. The shift of the transition temperature at $|\Phi/\Phi_0^* - n| = 1/2$ and $n \sim 1$ under the experimental conditions of Refs. 11 and 12 amounts according to (22) to $T_c - T^* \sim (1 - 5) \cdot 10^{-4}$ K.

Equation (22) contains a periodic dependence of $T^*(\Phi)$ with a period $\Phi_0^* = \Phi_0(1 + d/r_1)^{-1}$, that differs from the flux quantum[5)] Φ_0. Let us explain why our equations contain the renormalized value Φ_0^*. In experiment[9-15] the quantity studied is usually the periodicity with respect to the applied flux $\Phi = \sigma H_0$, where $\sigma = \pi r_1^2$. It is clear, however, that the size of the region in which a field is present in the cylinder exceeds πr_1^2 (owing to the penetration of the field into the superconductor). As shown by a rigorous calculation in accord with (3), the expression (10) contains in fact the effective field-penetration region $\sigma_{\rm eff} = \pi r_1 r_2 = \sigma(1 + d/r_1)$. The quantity $\phi = \Phi/\Phi^*$ in (22) could then be written in the form $\phi = \Phi^*/\Phi_0$, where $\Phi^* = \sigma_{\rm eff} H_0$. In this form it becomes obvious that allowance for the penetration effect renormalizes in fact not the flux quantum Φ_0 (which is a fundamental constant), but the area of the region in which a field is present. If the relations obtained in Refs. 11 and 12 are plotted not as functions of Φ but as functions of the effective flux Φ^*, then a strict periodicity should take place according to (22), with a period Φ_0.

In analogy with the derivation of (22), in the case $\mu > 1$ we get from (17) the formula

$$\frac{T_c - T^*}{T_c} = 0.55\varkappa^2\frac{\xi_0^2}{r_1 d}\left[a_1\left(\frac{\Phi}{\Phi_0^*} - n\right)^{2/3} + c_1\left(\frac{\Phi}{\Phi_0^*}\right)^2 - 2\right],$$

$$a_1 = \frac{3}{1 + {}^3/_2 d/r_1}\left[\frac{d^2}{\varkappa^2 r_1}\left(1 + 2\frac{d}{r_1}\right)\right]^{1/3}, \quad c_1 = \frac{1}{3}\frac{d^3}{\varkappa^2 r_1^3}\frac{1}{1 + d/r_1}. \tag{23}$$

An estimate for the shift $T - T^*$ at $|\phi - n| = 1/2$ and $n \sim 1$ yields according to (23) values close to the estimate obtained from (22).

We present also a formula obtained for $T_c - T^*$ obtained from the conditions $f = 0$ and $\partial f/\partial \psi = 0$ (see footnote 2). These conditions lead to the relation $4A\mu(\phi - n)^2 = [1 + \mu(1 - 1/2\, C\phi^2)]^2$, from which we get

$$\frac{T_c - T^*}{T_c} = 0.55\varkappa^2 \frac{\xi_0^2}{r_1 d} \frac{1}{1 + {}^3/_2 d/r_1} \left(a_3 \left| \frac{\Phi}{\Phi_0^*} - n \right| + c_1 \left(\frac{\Phi}{\Phi_0^*} \right)^2 - 1 \right),$$

$$a_3 = 2[(d/\varkappa^2 r_1)(1 + 2d/r_1)]^{1/2}. \tag{24}$$

The estimate of $T_c - T^*$ according to (24) differs little from that obtained from (22) and (23).

We note that our analysis, based on Eq. (10), is valid under the condition $d/\delta \ll 1$, where $\delta = \delta_L/\psi$. Using Eqs. (5)–(7) and (22), this criterion can be reduced to the form

$$\frac{d}{2\varkappa r_1} \psi \left\{ a_0 + 2c_0 \left(\frac{\Phi}{\Phi_0^*} \right)^2 \right\}^{1/2} \ll 1, \tag{25}$$

which differs from the criterion $(d/2\varkappa r_1 \ll 1)$, given in Refs. 13–15 for the applicability of the theory. The presence of the factor ψ in (25) ($\psi \ll 1$ in the case of a second-order phase transition) extends greatly the region of applicability of the theory developed above. In the case of a first-order transition ($0 < \psi \leqslant 1$), to obtain a criterion similar to (25) it is necessary to use (23) in lieu of (22).

From (17) can obtain also a formula for the temperature T_1 at which a state with a frozen-in flux can exist inside the cylinder in the absence of an external field. Putting in (17) $\phi = 0$ and $n = 1$ (this corresponds in Fig. 3a to the outermost-left-hand point of the curve with $n = 1$ falling on the ordinate axis) we get

$$\frac{T_1}{T_c} = 1 - 0.55\varkappa^2 \frac{\xi_0^2}{r_1 d} (a_1 - 2), \tag{26}$$

where a_1 is defined in (23).

We note in conclusion that, as follows from the foregoing analysis, for type-*I* superconductors ($\varkappa < 1/\sqrt{2}$), under the conditions of the experiments in Refs. 9–12, a change in temperature can be accompanied by either a first-order or a second-order transition. For type-*II* superconductors, on the other hand ($\varkappa > 1/\sqrt{2}$), in view of the smallness of the correlation radius ξ_0 (Ref. 15), the parameter μ in the same temperature interval is much smaller than in the case of a type-*I* superconductor, so that only second-order phase transitions without hysteresis should be realized. This is seen, in particular, from the condition for the existence of a first-order phase transition in a hollow cylinder[6)]: $r_1 d/\delta_L^2 > 1$.[3] This criterion can be reduced with the aid of (5)–(7) and (22) to the form

$$\frac{d}{\varkappa^2 r_1} \left| n - \frac{\Phi}{\Phi_0^*} \right|^2 > 1. \tag{27}$$

Since $d/r_1 < 1$ and $|n - \Phi/\Phi_0^*| \leqslant 1/2$, the condition (27) can be satisfied only for type-*I* superconductors.

In conclusion we make the following remark: The foregoing picture of the behavior of a thin-wall cylinder in an external field is perfectly similar physically to that observed when a bulky superconducting ring with weak binding (a Josephson junction) is placed in an external field.[26,27] In the latter case the external field likewise penetrates inside the ring through the weak link in individual batches (flux quanta), and regimes with and without hysteresis are possible (for details see, e.g., Ref. 28). The role of the screening parameter μ in the case when rings with weak binding is played in this case by the self-induction coefficient l of the ring, which is equal (in dimensionless units, see Ref. 28) to the product of the width of the junction by the ring area. At $l < 1$ there is no hysteresis, and at $l > 1$ hysteresis appears, in full analogy with the situation in a thin-wall cylinder.

1) By way of example we consider below cylinders with two sets of parameters $\mathscr{P}_{DF}$ and $\mathscr{P}_{LP}$. The set $\mathscr{P}_{DF}$ corresponds to the conditions of the experiments[9,10] on a tin cylinder: $\xi_0 = 2\cdot 10^{-5}$ cm, $d = 1.4\cdot 10^{-4}$ cm, $r_1 = 7\cdot 10^{-4}$ cm, $\varkappa = 0.2$. The set $\mathscr{P}_{LP}$ corresponds to the conditions of the experiments of Refs. 11 and 12: $\xi_0 = 2\cdot 10^{-5}$ cm, $d = 0.7\cdot 10^{-5}$ cm, $r_1 = 0.8\cdot 10^{-4}$ cm. For a cylinder with the parameters $\mathscr{P}_{DF}$ we have $\mu = 5.6$ at $t = 1 - T/T_c = 5\times 10^{-4}$ and $\mu = 1.1$ at $t = 1\times 10^{-5}$; for a cylinder with the parameters $\mathscr{P}_{DF}$ we have $\mu = 1.3$ at $t = 2\times 10^{-2}$, i.e., the factor μ is important.

2) We note that the end of the superconducting state could be defined not by the conditions (13) and (14) but by the relations $f = 0$ and $\partial f/\partial\psi = 0$. The formulas obtained thereby would lead to qualitatively analogous conclusions (see the end of Sec. 5).

3) The influence of the measuring current in experiments of the type in Refs. 9 and 10 was discussed in a paper by Kolpazhiu and Shvets.[19]

4) We note that an analogous interpretation can be made in the case when the superconductivity of a flat film is destroyed by a magnetic field (cf. Refs. 2–4, where they usually begin with finding the critical field at a given temperature T).

5) We note that the ratio d/r_1 in the experiments of Refs. 11 and 12 was ~10 – 20%, and the measured period differed from the flux quantum Φ_0 by just 10 – 20%. It is possible that the observed deviation is due to inclusion of terms of the order of d/r_1 in Φ_0^*.

6) When account is taken of the parabolic term, this condition takes according to (16) and (17) the form $r_1 d/\delta_L^2 > 1 - \frac{1}{2}C\phi_{cr}^2$.

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