

TABLE I.

Substance	$\beta$	$\delta$	$\gamma'$	$\gamma$	$\Gamma/\Gamma'$	$R_x = \Gamma D^{-\delta} B^{\delta-1}$	References*
$\alpha\text{-Fe}_2\text{O}_3$	$0.365 \pm 0.011$	$4.4 \pm 0.1$	$1.24 \pm 0.05$	$1.24 \pm 0.05$	$3.4 \pm 0.4$	$1.4 \pm 0.2$	[PW]
Similarity theory, $\varepsilon$ -expansion							
$n=1$	0.34	4.462	—	1.24	4.8	1.6	{ [12], [13] }
$n=2$	0.36	4.460	—	1.30	—	—	
$n=3$	0.38	4.458	—	1.34	—	1.33	

\*[PW]—present work.

$\beta, \delta, \gamma, \gamma', R_x = \Gamma D^{-\delta} B^{\delta-1}$  and  $\Gamma/\Gamma'$  depend mainly on the dimensionality  $n$  of the order parameter and on the dimensionality  $d$  of the lattice, should be the same for magnets belonging to one universality class  $(n, d)$ .<sup>12</sup> Hematite and iron borate are easy-plane antiferromagnets, their critical behavior is therefore expected to be close to that of magnets of class  $(2, 3)$  ( $XY$  model).

It is seen from the table that the values of  $\beta$ ,  $\delta$ , and  $R_x$  for hematite are in good agreement with the theoretical values for  $n=2$ , although the values of  $\gamma$  for both substances are somewhat lower than the theoretical one. On the other hand, the exponents  $\beta$ ,  $\gamma$ , and  $\gamma'$  for hematite agree within the limits of errors with the analogous exponents of iron borate. This also agrees with the assumptions that the two magnets belong to the same universality class.

The greatest difference between the critical parameters is observed for  $\delta$  and  $R_x$ , which deviate for  $\text{FeBO}_3$  towards the values predicted by the mean-field theory. This result can apparently be attributed to the fact that for hematite the interval of the reduced fields

$2 \cdot 10^{-3} \leq \tilde{H} \leq 8 \cdot 10^{-3}$  in which the values of  $\delta$  were determined lies somewhat closer to the critical point ( $T_N, 0$ ) than for iron borate, for which  $10^{-2} \leq \tilde{H} \leq 3 \cdot 10^{-2}$ .

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## Thermal resistance of metal-insulator boundary and nonlinear electric resistivity of metal films at low temperatures

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The size effect in heat transfer and in the nonlinear electric resistivity of current-heated metallic plates (films) is discussed under conditions when the electronic contribution to the formation of the spectral distribution of the nonequilibrium ballistic phonons emitted by the metal plays the decisive role.

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### 1. INTRODUCTION

Experiments on the propagation of heat pulses in solids at high temperatures (see, e.g., Ref. 1) have increased recently the interest in the thermal resistance of a metal-insulator (M-I) interface. For an experimental study of this resistance (as well as to obtain the thermal pulses themselves), extensive use is made of electric current to heat thin metallic films deposited on bulky single-crystal insulator substrates.

In these experiments one measures usually the magnitude and spectral intensity of the heat flux  $Q$  emitted by the film as a function of the electron temperature  $T_e$  and of the substrate temperature  $T$  (the latter can be regarded as equal to the temperature of the helium bath, inasmuch as the propagation of the phonons emitted by the film is ballistic). The electron temperature  $T_e$  is not measured directly but is calculated from the change of the film resistivity, using the known temperature dependence of the resistivity in the equilibrium case.<sup>2</sup>

The relation  $Q = Q(T_e, T)$  obtained in such experiments is usually interpreted in terms of a theory proposed by Little<sup>3</sup> for the temperature discontinuity at the boundary between two solids. If  $T_e \ll \Theta_d$  ( $\Theta_d$  is the Debye temperature of the metal), then Little's result can be expressed in the form  $Q = A(T_e^4 - T^4)$ , where  $A$  is determined only by the acoustic characteristics of the metal and of the insulator, the procedure accepted in the theory of acoustic mismatch. If  $\theta$  is the angle of the phonon incidence on the interface, then  $A$  is proportional to the interface transparency  $\alpha(\theta)$  averaged over the angles, a quantity representing the probability of penetration of the phonon in question through the M-I interface. In this approach no contribution is made by the electrons to the thermal resistance of the M-I interface.

However, as will be shown in the present article (preliminary results were reported in Ref. 4), the acoustic-mismatch theory is valid only for sufficiently bulky metallic samples. Of primary interest to us is the inverse limiting case of thin metallic plates (films—the pertinent estimates are presented later on—the role of the electrons in the formation of the temperature discontinuity on the M-I interface becomes decisive.<sup>1</sup>

To explain the physical cause of such an increase of the electron contribution with decreasing thickness of the metallic layer (I<sub>1</sub>—M—I<sub>2</sub> sandwich configuration, see Fig. 1), we consider qualitatively the mechanism of heat transfer through an M—I interface. We recall first that in sufficiently pure metals the main heat carriers are electrons and not phonons. However, the heat transfer through the M—I interface is effected only by phonons in proportion to its acoustic transparency. On the metal side of the interface there is located therefore a transition layer in which the thermal energy transported by the electrons is transformed into a phonon flux. The thickness of this layer is of the order of the mean free path of the thermal phonons relative to scattering by electrons,  $l_{pe}(T_e)$  ( $l_{pe} \sim \hbar v_F/kT_e \sim 10^{-3}$ — $10^{-4}$  cm at helium temperatures). It is intuitively obvious that if  $d \gg l_{pe}$  ( $d$  is the thickness of the metal plate), then the detailed structure of this transition layer is inessential for the calculation of the thermal resistance of the M—I interface; this is equivalent to the usual approach in which the electron contribution is neglected.

If, on the other hand,  $d \ll l_{pe}(T_e)$ , and  $\alpha \sim 1$ , then most phonons emitted by the metal electrons manage to leave the film without being absorbed in its interior even after several successive reflections from its

boundaries. Thus, in contrast to the preceding case, the spectral distribution of the phonons emitted by the film contains more information on the electron-phonon interaction (EPI) in the metal than on the transparency  $\alpha$  of the M—I boundary.

A consistent formulation of these qualitative considerations from the kinetic point of view makes it possible to describe quantitatively the size effect in the heat transfer from thin metal films under conditions of ballistic propagation of the phonons emitted by the film. This calculation, which takes into account the finite transparency of the M—I boundary in the spirit of the acoustic-mismatch theory,<sup>2</sup> is given in Sec. 1. As expected, the phonon distribution function in the interior of the metal, obtained as a result of this calculation, turns out to be essentially inhomogeneous in the  $l_{pe}$  scale and anisotropic, so that the phonon contribution  $R_{ph}^T(\mathcal{P})$ , to the sample resistance ( $\mathcal{P}$  is the dissipated power) calls for a special calculation (Sec. 3). In the concluding Sec. 4 we discuss questions connected with the organization of the experiment and formulate the conclusions.

## 2. ROLE OF CONDUCTION ELECTRONS IN THE THERMAL RESISTANCE OF THE METAL-DIELECTRIC INTERFACE

We consider a metallic plate (film) of thickness  $d$ , whose two faces are in contact with bulky insulators I<sub>1</sub> and I<sub>2</sub> with known temperatures  $T_1$  and  $T_2$  (see Fig. 1). Let a direct current  $j$  flow through the sample, such that the power dissipated in a unit volume is  $\mathcal{P} = j^2/\sigma$ , where  $\sigma$  is the conductivity of the metal. It is required to find the heat fluxes  $Q_1$  and  $Q_2$  if the transparencies  $\alpha_1$  and  $\alpha_2$  of the interfaces to phonons are known.

A consistent theoretical calculation of the heat released from the plate at a practical arbitrary relation between its thickness  $d$ , the quantity  $l_{pe}(T_e)$ , and the transparency parameters is possible because several physical circumstances simplify the problem in this situation.

One of them is connected with the possibility of neglecting the inhomogeneity of the electron distribution function over the film thickness if the parameter  $I$  is small ( $\kappa_p$  and  $\kappa_e$  are the phonon and electron thermal-conductivity coefficients). If the characteristic energies of the electrons are such that  $\kappa_e$  is formed over the length  $l_i$  of the elastic collisions of the electrons with the impurities or with the faces of the crystallites of the film, while  $\kappa_p$  is determined by the scattering of the phonons by the electrons, then the inequality given above is equivalent to the condition  $(l_D/l_{pe})^2 \ll 1$ , and the requirement  $T \ll \Theta_D$  is usually satisfied in the experiment with a large margin. Here  $l_D \sim (l_e l_i)^{1/2}$  is the diffusion length and  $l_e \sim l_{pe}(T)$  ( $\Theta_D/T$ )<sup>2</sup> is the mean free path of an electron of energy  $\varepsilon \sim kT$  with respect to scattering by phonons ( $k$  is Boltzmann's constant).

Another important simplifying circumstance is the possibility of introducing an electron temperature if

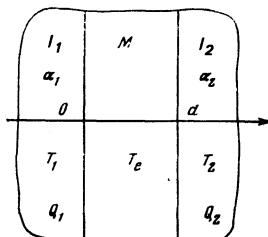


FIG. 1.

$T_e \gtrsim \Theta_D^2/\varepsilon_F$ , when a temperature Fermi distribution can no longer be established on account of direct inter-electron collisions. This value of  $T_e$  can be justified both in the limit<sup>6</sup>  $d \ll l_{pe}$  and in the limit  $d \gg l_{pe}$ , because of the effective interelectron collisions via the phonons. Therefore the statement that  $T$  can be introduced in the case  $d \gg l_{pe}$  can be regarded as a sufficiently reliable interpolation of the aforementioned limiting cases, where it can be rigorously proved.

These two circumstances allow us to reduce the posed problem to a solution of the kinetic equation for the phonon distribution function, and then determine  $T_e$  as a function of  $Q$  and  $T$  from the heat-balance equation.

Finally, one more important simplification, which is particularly significant in the good-transparency ( $\alpha \sim 1$ ) case of importance to us, is connected with the possibility of formulating simple boundary conditions for the phonon distribution function. As will be shown below, the possibility of ballistic propagation of the phonons emitted by the film not only simplifies greatly the allowance for the heat dissipation by the sample, but also creates conditions for a realization of the size effect.

Proceeding to the quantitative formulation of the problem, we direct the  $z$  axis perpendicular to the interfaces and assume that the problem is spatially homogeneous in the  $xy$  plane. In accord with the statements made above, we write down the kinetic equation for the phonon distribution function  $N(\mathbf{q}, z)$  ( $\mathbf{q}$  is the phonon momentum) in the form

$$s_z \partial N / \partial z = vN. \quad (1)$$

Here  $s_z$  is the projection of the phonon velocity on the  $z$  axis, and  $vN$  is the electron-phonon collision integral, which can be reduced in our case, when the electrons have an equilibrium Fermi distribution function with temperature  $T_e$ , to the form  $\nu_{pe} [n(T_e) - n(z)]$ , where  $n(T_e)$  is an equilibrium Bose function with temperature  $T_e$ , while  $\nu_{pe}$  has the meaning of the frequency of the collisions of a phonon of energy  $\hbar\omega$  with the electrons. In the simplest model considered by us, that of electrons with quadratic and isotropic dispersion and of the Debye model for the phonons,  $\nu_{pe}$  is determined by the magnitude of the EPI and is proportional to  $\omega (\nu_{pe} \sim \omega s_z/v_F)$ , ( $v_F$  is the Fermi velocity). More accurately,

$$\nu_{pe} = \mu^2 m^2 \omega / 2\pi \hbar^3 \rho s^2. \quad (2)$$

Here  $\mu$  is the EPI constant ( $\mu \sim \varepsilon_F$ ),  $m$  is the electron effective mass,  $\rho$  is the density of the substance, and  $s$  is the speed of longitudinal sound.

The general solution of Eq. (1) is of the form

$$N(z) = C \exp(-z/l_z) + n(T_e), \quad (3)$$

where  $C$  is an arbitrary constant and  $l_z \equiv s_z/\nu_{pe}$ . We introduce now the two functions  $N \geq (q, z) \equiv N(z, q, q_z \geq 0)$ , and denote  $|l_z| = l$ , where  $l$  depends on the angle  $\theta$  between the direction of the vector  $q$  and the  $z$  axis. We

then have for  $N \geq$  the relations

$$N \geq (z) = C \geq \exp(-z/l) + n(T_e), \quad (4)$$

where the coefficients  $C \geq$  must be determined with account taken of the two boundary conditions on  $N(z)$  at  $z = 0$  and  $z = d$ , respectively. These boundary conditions can be easily written down in the case of interest to us, that of ballistic propagation of the phonons emitted by the metal, with account taken of the finite transparency of the M-I interface within the framework of the acoustic-mismatch theory.

If  $\beta_1$  and  $\beta_2$  are the coefficients of phonon reflection from boundaries 1 and 2 ( $\beta_i = 1 - \alpha_i$ ,  $i = 1, 2$ , where  $\alpha_i(\theta)$  is the transparency coefficient), then the necessary boundary conditions take the form

$$\begin{aligned} N > (0) &= \alpha_1 n(T_1) + \beta_1 N < (0), \\ N < (d) &= \alpha_2 n(T_2) + \beta_2 N > (d). \end{aligned} \quad (5)$$

In the acoustic-mismatch theory the interface transparency coefficient  $\alpha$  is connected with the angle  $\theta_1$  of phonon incidence on the interface, the refraction angle  $\theta_2$ , and the acoustic impedances of the adjacent media by the known relations (see, e.g., Ref. 2)

$$\alpha = \alpha_1(\theta_1) = \alpha_2(\theta_2) = \frac{4(\rho_2 s_2 / \rho_1 s_1)(\cos \theta_2 / \cos \theta_1)}{(\rho_2 s_2 / \rho_1 s_1 + \cos \theta_2 / \cos \theta_1)^2}, \quad (6)$$

where  $\theta_1$  and  $\theta_2$  and a given boundary are connected by the relation  $s_2 \sin \theta_1 = s_1 \sin \theta_2$ . (The subscripts 1 and 2 now pertain to one of the M-I interfaces).

Combining relations (4) and (5); we obtain the following expression for  $C^>$ :

$$C^> = [\alpha_1 n(T_1) + \beta_1 \alpha_2 n(T_2) x + n(T_e) (\alpha_1 - \beta_1 \alpha_2 x)] / (1 - \beta_1 \beta_2 x^2). \quad (7)$$

The expression for  $C^<$  differs from (7) in that the subscripts 1 and 2 are interchanged, and by an additional factor  $x \equiv \exp(-d/l)$ .

Using the expressions for  $C \geq$ , we can easily write down the solution of the kinetic equation (3) and an expression for the heat flux  $Q_1$  (the analogous formula for  $Q_2$  differs from  $Q_1$  only in that the subscripts 1 and 2 are interchanged and the sign is reversed) takes the form

$$Q_1 = \sum_{\alpha_i > 0} \alpha_i \gamma \{ (1 - \beta_2 x^2) [n(T_1) - n(T_e)] - \alpha_2 x [n(T_2) - n(T_e)] \}, \quad (8)$$

$$\gamma = \hbar \omega s_1 / (1 - \beta_1 \beta_2 x^2).$$

The quantity  $T$  is determined from the heat balance equation for the electrons  $Q = Q_1 - Q_2$ , where  $Q = \mathcal{P}d$  is the total flux density of the heat passing through the boundary of the metal plate.

We turn now to a question of importance for experiments with heat pulses,<sup>1</sup> that of the heat radiated from a current-heated metallic plate into a medium with temperature  $T$ . The sought connection between  $Q$  and  $T_e$ , which follows from (8) at  $T_1 = T_2 = T$ , can be represented in compact form by introducing the effective resultant transparency  $\tilde{\alpha}(q, d)$  of the boundaries, so

that

$$Q = \sum_{q>0} \hbar \omega_q s_z \tilde{\alpha}(q, d) [n(T_e) - n(T)], \quad (9)$$

$$\tilde{\alpha} = (1-x)[\alpha_1(1+\beta_2x) + \alpha_2(1+\beta_1x)]/(1-\beta_1\beta_2x^2). \quad (10)$$

We note that in contrast to the "bare" transparencies  $\alpha_i(\theta)$ , which depend only on the angle of incidence of the phonon on the interface in accord with Eq. (6), the effective transparency has not only an additional angular dependence connected with the quantity  $x$ , but also a dependence on the phonon frequency.

Changing in (10) from summation to integration, it is convenient to represent the heat flux  $Q$  in the form

$$Q = \frac{2\pi s^2}{(2\pi\hbar)^3} \int_0^\infty q^3 \chi_Q(q) [n_{T_e}(q) - n_T(q)] dq, \quad (11)$$

where we have introduced the effective transparency  $\chi_Q(q)$  averaged over the angles  $\theta$  and defined by the relation

$$\chi_Q(q) = \int_0^1 u \tilde{\alpha}(u, q) du. \quad (12)$$

Here  $u = \cos\theta$ ,  $n_T(q) = n(sq/kT)$ , and  $\tilde{\alpha}(u, d)$  is given by (10).

Expressions (11), (12), and (10) contain the connection between the experimentally observed function  $T_e(Q)$  (see articles 12 and 13 in Ref. 1, as well as the more detailed exposition in Ref. 2) and the film thickness and the transparency parameters. In the general case, however, since (10) is so complicated, the dependence of the "form factor"  $\chi_Q(q)$  on the phonon momentum  $q = \hbar\omega/s$  can be calculated out for  $\tilde{\alpha}$  only by numerical computer integration. We consider therefore two limiting cases of physical interest, when the expression for  $\chi_Q(q)$  can be written in explicit form.

We determine first the parameter  $\varepsilon$ , which depends on the electron temperature, using the formula

$$\varepsilon = 2d\beta_1\beta_2/l_{pe}(1-\beta_1\beta_2), \quad (13)$$

where

$$l_{pe} = \hbar v_F/kT_e.$$

The formulas corresponding to the usual interpretations,<sup>1,2</sup> in which the electron contribution to the thermal resistance of the boundary is ignored, are obtained from (11) and (12) at  $\varepsilon \gg 1$ . In this limit we have in fact  $\tilde{\alpha} = \alpha_1 + \alpha_2 \equiv \tilde{\alpha}$  and  $\chi_Q = \tilde{\alpha}$ , and at  $T_e \ll \theta_D$  Eq. (11) leads to Little's well known result

$$Q = \mathcal{P}d = \frac{\pi^2 \tilde{\alpha}}{120s^2\hbar^3} (T_e^5 - T^5). \quad (14)$$

We note that if  $\mathcal{P}$  is constant we have  $T_e \sim d^{1/4}$ , i.e., it increases, albeit weakly, with increasing film thickness  $d$ . The thermal regime of the film corresponds in this case to the usual "Joule" heating. In particular, the spectrum of the phonons emitted by the film is at equil-

ilibrium with a temperature  $T_e$ , so that the maximum of its spectral intensity corresponds to an energy  $\hbar\omega_m \sim 2.8kT_e$ .

We consider now the opposite limiting case, when primarily  $d \ll l_{pe}(T_e)$ , but still  $d > \lambda$ , where  $\lambda = \lambda(T_e)$  is the wavelength of the thermal phonon in the metal, so that we can disregard the deformation of the phonon spectrum of the film (we recall that in the "pure" limit considered by us we have  $\lambda/l_{pe} \sim s/v_F \ll 1$ , so that the double inequality indicated above can be satisfied). Now if in addition  $\varepsilon \ll 1$  then we easily find from (10) that  $\tilde{\alpha} \approx 2d/l$  and  $\chi_Q(q) = 2d/s\tau_{pe}$ . From (11) we obtain in this limit

$$\mathcal{P} = \frac{2\mu^2 m^2 D_5}{(2\pi\hbar)^3 \hbar^3 \rho s^4} (T_e^5 - T^5), \quad (15)$$

where

$$D_5 = \int_0^\infty \frac{z^4 dz}{e^z - 1} \approx 24.88. \quad (16)$$

It is seen from (15) that at constant  $\mathcal{P}$  the value of  $T_e$  does not depend on the transparency parameters  $\alpha$  and on the film thickness  $d$ , and is determined only by the intensity of the EPL.

In this limit ( $\varepsilon \ll 1$ ), which we shall call henceforth the electronic superheating regime, most phonons emitted by the electrons manage to leave the film without being reabsorbed in it, and the electrons and the lattice can be described with the aid of two different temperatures,  $T_e$  and  $T$ .<sup>7</sup> We note that even though Eq. (15) can in fact be written down directly, by using the heat-balance condition (as was done first in Ref. 7), its present derivation in the framework of the kinetic approach determines the limits of its applicability under conditions of stationary heating with an electric current. Namely, the electronic superheating regime is realized in the problem of interest to us in "pure form" only if the parameter  $\varepsilon$  (13) satisfies the inequality  $\varepsilon \ll 1$ .

We note incidentally that Eq. (11) describes also all the "intermediate" possibilities that are realized between the limiting situations of pure electronic superheating ( $\varepsilon \ll 1$ ) and "Joule" heating ( $\varepsilon \gg 1$ ) with increasing dissipated power  $\mathcal{P}$ , and consequently with increasing parameter  $\varepsilon$ . It is curious to note also that in the electronic superheating regime the emission of phonons from the film is in disequilibrium, and the maximum of its spectral intensity corresponds to an energy  $\hbar\omega_m \approx 3.9kT_e$ .

As to the "Joule" heating, Eq. (14) notwithstanding, it can still not be assumed, as is customary,<sup>3</sup> that the phonons in the film are at equilibrium with a temperature  $T_e$ , since such a distribution function causes the electron-phonon collision integral to vanish, and hence also the power  $\mathcal{P}$  transferred to the phonons. As will be shown by an analysis of the expression for  $N(q, z)$ , the phonon distribution function in this regime is essentially inhomogeneous over the scale of  $l_{pe}(T_e)$ . Therefore in the case of "Joule" heating one can only state that the distribution function  $N$  of the phonons emitted

on the film boundary  $z=d$  correspond to equilibrium phonons with temperature  $T_e$ .

To have a clear picture of the appearance of the size effect considered above, we deem it of interest to analyze in greater detail the expression for the phonon distribution function. For simplicity we confine ourselves to an analysis of the symmetrical case ( $\alpha_1 = \alpha_2 \equiv \alpha$ ,  $T_1 = T_2 = T$ ). Then, if we put

$$\kappa = \alpha / (1 - \beta x), \quad (17)$$

then expressions (4) for  $N \geq$  take the form

$$N^>(z) = \kappa e^{-z/l} n(T) + (1 - \kappa e^{-z/l}) n(T_e), \quad (18a)$$

$$N^<(z) = \kappa e^{(z-d)/l} n(T) + (1 - \kappa e^{(z-d)/l}) n(T_e). \quad (18b)$$

It is seen from (18) that the characteristic length of the spatial variation of  $N \geq$  is

$$l = |s_z| / v_{pe} = l(q, \theta). \quad (19)$$

The functions  $N \geq$  are "weighted" sums of two equilibrium Bose distribution functions:  $n(T)$  of the phonons from the thermostat and  $n(T_e)$  of the "hot" phonons emitted by the electrons of the metal. The relative weight of each of these terms is determined by the quantity  $\kappa$  ( $0 < \kappa < 1$ ), which has a simple physical meaning—it is the effective probability that a phonon incident from the metal on the M-I boundary will leave the film without colliding with electrons. It is easy to show that even in this case, when the bare transparency  $\alpha$  is small ( $\bar{\alpha} \ll 1$ ) the quantity  $\kappa$  can nevertheless be of the order of unity if  $\xi = d/l \ll \alpha$ . This means that if  $\xi \ll \alpha$  then practically all the phonons emitted by the electrons and satisfying this condition leave the film (without being reabsorbed in it) with a probability on the order of unity, despite of the possibility of a number of successive reflections, from the interfaces inside the film. In accord with this lucid treatment, it follows readily from the definition (17) that  $d\kappa/d\alpha > 0$  and  $d\kappa/d\xi < 0$ , so that the effective probability  $\kappa$  increases with increasing  $\alpha$  and decreases with increasing film thickness. In the limiting cases we have  $\kappa \rightarrow 1$  as  $d \rightarrow 0$  and  $\kappa \rightarrow \alpha$  as  $d \rightarrow \infty$ .

We turn now to an analysis of the function  $T_e(Q)$ , using expressions (9) and (10). It is seen from (9) that  $Q$  depends on  $T$  only via  $n(T_e)$ . It is easily seen then that  $dQ/dT_e > 0$  and  $d^2Q/dT_e^2 < 0$ , i.e., the function  $T_e(Q)$  is in the general case monotonic and "convex upward." An approximate form of these functions at various values of the transparency is shown in Fig. 1. The less sloping curves correspond to smaller transparency. The transition from the  $T^4$  to the  $T^5$  dependence takes place at  $\varepsilon \sim 1$ . An important feature of the curves of Fig. 2 is that the  $Q \sim T_e^5$  dependence, which corresponds to pure electronic heating, lies lower in the figure than the remaining curve (except for the sections where they practically coincide). This means that in the limit  $\varepsilon \ll 1$  the heat dissipation from the film is the most effective (minimal  $T_e$  at a given  $Q$ ), i.e., it can no longer be increased by improving the acoustic matching of the metal of the substrate.

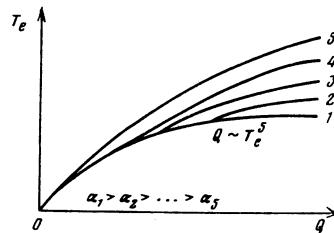


FIG. 2.

It should be noted that in a number of experiments<sup>2,8</sup> the matching of the film to the substrate turns out to be better than expected from the acoustic-mismatch theory (which in our case corresponds to the sections of curves with  $\varepsilon > 2$ ). To interpret these results, the authors of Refs. 2 and 8 use the so-called absolute black body (ABB) model,<sup>8</sup> which generally speaking is not at all theoretically valid in the case  $\alpha \neq 1$ . It follows from the foregoing that the actually possible regime is one in which, so long as  $\varepsilon \ll 1$ , the heat release is a maximum and does not depend on the transparency coefficients. In contrast to the ABB model, however, in the electronic superheating regime the heat release is determined only by the metal parameters that are responsible for the magnitude of the EPI.

We turn now to the expression for the effective transparency  $\tilde{\alpha}$  [Eq. (10)]. It can be shown that both derivatives  $d\alpha/d\alpha_1$  and  $d\tilde{\alpha}/d\alpha_2$  are larger than zero, so that with increasing "bare" transparencies  $\alpha_1$  and  $\alpha_2$ , other conditions being equal, the effective transparency increases. Accordingly,  $dQ/d\alpha_1 > 0$ , i.e., given  $\alpha_2$  and  $T_e$  the dissipated power increases with increasing transparency.

We mention also the dependence of the effective transparency  $\tilde{\alpha}$  on the film thickness  $d$  (other conditions being equal). It is easy to show that  $d\tilde{\alpha}/d\xi > 0$ , i.e.,  $\tilde{\alpha}$  decreases with decreasing film thickness. It might seem that this contradicts the statement that as  $d \rightarrow 0$  and at constant  $T_e$  an ever increasing fraction of the phonons emitted by the electrons leaves the metal unimpeded ( $\kappa \rightarrow 1$ ), and that at first glance this should involve an increase (and not a decrease) of the effective transparency  $\tilde{\alpha}$ . In fact, of course, there is no contradiction at all if it is recognized that as  $d \rightarrow 0$  the total number of phonons emitted by the electrons decreases with decreasing film thickness, something that should be treated at constant  $T_e$  as a decrease of  $\tilde{\alpha}$ .

### 3. CALCULATION OF THE ELECTRIC RESISTANCE

As shown in the preceding section, the phonon distribution function in a metal is inhomogeneous over the plate thickness and is anisotropic [owing to the dependence on  $l$ , Eq. (19)]. Therefore the calculation of the phonon contribution to the nonlinear electric resistance of the film, in contrast to the well-known case of electron scattering by equilibrium phonons, calls for a special analysis.

We note first that the quantity measured in the experiment<sup>9</sup> was the phonon contribution  $R_{ph}^T(E)$ , to the total resistance of the film, a contribution connected with the differential conductivity  $\sigma_{ph}^T(z)$  by the simple

relation

$$\frac{R_{ph}^T}{R_i} = -\rho_i \bar{\sigma}, \quad \bar{\sigma} = \frac{1}{d} \int_0^d \sigma_{ph}^T(z) dz, \quad (20)$$

where  $R_i$  and  $\rho_i$  are respectively the residual resistance and the resistivity of the sample ( $\bar{\sigma} \ll \rho_i$ ). We assume therefore that the dominant mechanism of momentum relaxation of the electrons is their elastic scattering (by impurities, crystallite boundaries, etc.). At  $T_e \ll \Theta_D$ , for sufficiently thin films, this is precisely the typical experimental situation.<sup>9</sup>

Using the noted smallness  $\rho_{ph} \ll \rho_i$ , we can obtain by standard methods the following expression for  $\sigma_{ph}^T(z)$ :

$$\sigma_{ph}^T(z) = -\left(\frac{e\tau}{mE}\right)^2 \sum_q \sum_p (qE)(pE) \frac{\partial f}{\partial \xi_p} B_{p,p+q}, \quad (21)$$

$$B_{p,p+q} = (1-f_{p+q}) W_q [N_q (\epsilon_{p+q} - \epsilon_p - \Omega) + (N_{-q} + 1) \delta(\epsilon_{p+q} - \epsilon_p + \Omega)] \\ + f_{p+q} W_q [(N_q + 1) \delta(\epsilon_{p+q} - \epsilon_p - \Omega) + N_{-q} \delta(\epsilon_{p+q} - \epsilon_p + \Omega)]. \quad (21a)$$

Here  $e$  and  $m$  are the charge and effective mass of the electron,  $E$  is the electric field intensity,  $f_p \equiv f(\epsilon_p/T_e)$  is the equilibrium Fermi distribution temperature of the electrons with temperature  $T_e$ ,  $\epsilon_p = p^2/2m$  is the energy of an electron with momentum  $p$ ,  $\xi_p = \epsilon_p - \epsilon_F$  is the electron energy reckoned from the Fermi level,  $\Omega = \hbar\omega$  is the energy of a phonon of frequency  $\omega$ ,  $W_q \equiv \pi \mu^2 \omega / ps^2$ , and  $N_q \equiv N(q, z)$  and is given by Eqs. (4) and (7). We note that in the general case  $\beta_1 \neq \beta_2$ , so that  $N_q \neq N_{-q}$ .

We choose the coordinate system such that the  $z$  axis, just as in Sec. 2, is perpendicular to the interfaces of the media, and the  $x$  axis is directed along the vector  $E$ . Let  $\theta_1$  and  $\varphi_1$  be respectively the polar and azimuthal angles of the vector  $q$ , and let  $\theta_2$  and  $\varphi_2$  be the analogous angles for the vector  $p$ . Then, changing in (21) for summation to integration, we obtain after a number of transformations

$$\sigma_{ph}^T(z) = -\left(\frac{e\tau}{m}\right)^2 \frac{mcN(0)}{(2\pi\hbar)^3} \int_0^d q^3 dq \int_{-\infty}^{\infty} d\xi \frac{df}{d\xi} \\ \times \int_{-1}^{+1} du \int_{-1}^{+1} dv \frac{uv+v}{[(1-u^2)(1-v^2)-(uv+v)^2]^{\nu_2}} [N_q + N_{-q} + f(\Omega+\xi) + f(\Omega-\xi)]. \quad (22)$$

The quantity  $c$  is defined here by the relation  $W_q \equiv cq$ ,  $v \equiv q/2p_F$ , where  $p_F$  is the Fermi momentum,  $u = \cos\theta_1$ , and  $v = \cos\theta_2$ . In the derivation of (22) we used the fact, demonstrated in Appendix I, that

$$I = \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \cos\varphi_1 \cos\varphi_2 \Phi \{ \cos(\varphi_1 - \varphi_2) \} = -2\pi \int_{-1}^{+1} \frac{x dx}{(1-x^2)^{\nu_2}}. \quad (23)$$

Here  $\Phi(x)$  is an arbitrary even function of  $x \equiv \cos(\varphi_1 - \varphi_2)$ . Then, choosing  $\Phi$  to be the  $\delta$ -functions contained in the expression for  $B_{p,p+q}$  (21a), we can obtain Eq. (22). We use next the fact that the angular dependence of  $N_q$  is connected only with the quantity  $u$ , and integrate with respect to  $v$ . As shown in Appendix II,

$$K = \int_{-1}^{+1} \frac{dv(uv+v)}{[(1-u^2)(1-v^2)-(uv+v)^2]^{\nu_2}} = -\pi v(1-u^2). \quad (24)$$

We recognize also that

$$\bar{N}_q(u) + \bar{N}_{-q}(u) = D(u) [n(T) - n(T_c)] + 2n(T_c), \quad (25)$$

where the bar denotes the spatial averaging (20), and

$$D(u) = l(u) \bar{\alpha}(u, d)/d \quad (26)$$

while  $l$  and  $\bar{\alpha}$  are given by (19) and (10).

It is convenient to write the final expression for  $\bar{\sigma}$  (20) in the form

$$\bar{\sigma} = \sigma_{ph}(T_c) + \frac{\pi c N(0) e^2 \tau^2}{(2\pi\hbar)^3 v_F m^2} \int_0^d q^4 \chi_R(q) [n_{T_e}(q) - n_T(q)], \quad (27)$$

where

$$\chi_R(q) = \int_0^1 du (1-u^2) D(u, q). \quad (28)$$

The quantity  $\bar{\sigma}$  is represented in (27) as a sum of two terms. The first does not depend explicitly on the transparency parameters and is the expression for the conductivity of electrons with temperature  $T_e$ , scattered by equilibrium phonons with the same temperature, so that

$$\sigma_{ph}(T_e) = -\rho_i^2 \sigma_{ph}(T_e) = \eta T_e^3, \quad \eta = \frac{20\pi D_b c N(0)}{3(2\pi\hbar)^3 n_e^2 e^2 v_F s^2}. \quad (29)$$

The second term of (27) contains the explicit dependence of  $\bar{\sigma}$  on the transparency parameter [via the quantity  $\chi_R(q)$ ]. It is important to note that the expression for the conductivity  $\bar{\sigma}$  as a function of  $T_e$  and of the transparency parameters [Eqs. (27) and (28)] is valid at any ratio of  $d$  and  $l_{pe}$ , and describes, just as in the case of (11) and (12), all the "intermediate" possibilities between the limiting cases of electronic superheating ( $\varepsilon \ll 1$ ) and "Joule" heating ( $\varepsilon \gg 1$ ). In the general case, however, the calculation of the "form factor"  $\chi_R(q)$ , even in the simplest case of bare transparencies  $\alpha$  independent of the phonon incidence angle, cannot be carried out analytically, and a computer must be used. Therefore, just as in Sec. 2, we consider the already mentioned two limiting cases, when  $\chi_R(q)$  can be calculated explicitly.

We consider first the case of electronic superheating ( $\varepsilon \ll 1$ ), it is easy to show that in this case  $\chi_R = 4/3$  and a simple comparison of (27) and (11) yields

$$\delta\rho_{ph}(E) = \rho_{ph}^T(E) - \rho_{ph}^T(E=0) = 4\mathcal{P}/3j_0^2, \quad (30)$$

where  $j_0 \equiv nes$  is a characteristic of the metal. Thus, in the electronic superheating regime ( $\varepsilon \ll 1$ ) the connection between the experimentally observed quantities  $\delta\rho_{ph}(E)$  and  $\mathcal{P}$  is linear, and the EPI constant  $\mu$  therefore drops out.

In the opposite limiting case of "Joule" heating ( $\varepsilon \ll 1$ ) the connection of  $\rho_{ph}^T(E)$  with  $\mathcal{P}$  is more complicated. Assuming for simplicity that  $\alpha$  is constant, we can easily show that  $\chi_R(q) = s\tau_{pe}\alpha/4d \sim q^{-1}$ . Then, using (27) and (11), we get

$$\rho_{ph}^T(E) = \rho_{ph}(T_e) - \mathcal{P}/4j_0^2, \quad (31)$$

where  $T_e$  is given by the Little formula (14), i.e.,  $T_e \sim \mathcal{P}^{1/4}$ . Inasmuch as  $\rho_{ph}(T_e) \sim T_e^5$ , by virtue of (29), we find that the first term of (31) is proportional to  $\mathcal{P}^{5/4}$ . The presence in (31) of a second term proportional to  $\mathcal{P} \sim T^4$  can be easily explained from the physical point of view. In fact, in the "Joule" heating regime, the phonon distribution function is at equilibrium over a large part of the film thickness, with an electron temperature  $T_e$ , so that the main contribution to the resistance [the first term of (31)] takes the form usual for the equilibrium case. However, even in the case  $\varepsilon \gg 1$ , at distances on the order of  $l(T_e)$ , as already noted, the phonon distribution function remains essentially inhomogeneous, and it is this which leads to the appearance of the second term in (31). The foregoing arguments show that the second term of (31) should be small compared with the first in proportion to the parameter  $\varepsilon^{-1} \ll 1$ . A more detailed estimate shows that asymptotically, at  $\varepsilon \gg 1$ ,

$$\frac{\mathcal{P}}{4j_0^2} = \frac{\pi^4}{200D_s\varepsilon} \rho_{ph}(T_e) \sim \frac{1}{50\varepsilon(T_e)} \rho_{ph}(T_e). \quad (32)$$

It should be noted, however, that it is obvious from physical considerations that the estimate (32) cannot be extrapolated to the case  $\varepsilon \sim 1$ . Indeed, in the interval of  $\mathcal{P}$  corresponding to  $\varepsilon(T_e) \sim 1$ , a change should take place from the relation  $\rho_{ph} \sim \mathcal{P}$  typical of  $\varepsilon \ll 1$  to the relation  $\rho_{ph} \sim \mathcal{P}^{5/4}$  corresponding to the limit  $\varepsilon \gg 1$ . Therefore in the case  $\varepsilon \sim 1$  both terms of (27) should be of the same order.

#### 4. CONCLUSION

It follows from the preceding analysis that in the electronic superheating regime ( $\varepsilon \ll 1$ ) both the integral  $Q(T_e)$  dependence and the spectral distribution of the phonons emitted by the film are determined only by the properties of the metal and are practically independent of the substrate characteristics. Thus, in contrast to the "Joule" heating regime ( $\varepsilon \gg 1$ ), in the case of electronic superheating there is no need to take explicitly into account the mechanisms of the heat removal from the sample, and the effects nonlinear in the electric field are not masked by the pure thermal effects. Therefore the results of the corresponding experiments in the  $\varepsilon \ll 1$  regime contain information on the EPI in the metal. In particular, experiments on the determination of the thermal resistance of an M-I interface make it possible to estimate a microscopic quantity such as  $l(T_e)$ .<sup>9</sup> Without dwelling in greater detail on these questions,<sup>2</sup> we make only a few remarks on the possibility of experimentally realizing the electronic superheating mechanism.

The point is that this realization entails not merely formal satisfaction of the inequality  $\varepsilon \ll 1$ . It must also be remembered that the criterion  $\varepsilon \ll 1$  was derived under the assumption that the phonons radiated by the film electrons propagate ballistically, i.e., that it is possible to neglect the flux of the so-called reverse phonons. In the case  $\alpha \ll 1$  this neglect is always theoretically justified. It follows from (13), however, that

the electronic superheating mechanism is easiest to realize precisely at  $\alpha \sim 1$ , and in this case the presence of even a small number of "reverse" phonons can greatly distort the experimental results even if the criterion  $\varepsilon \ll 1$  is formally satisfied.

Without entering into a theoretical discussion of the methods of decreasing the number of "reverse" phonons, we mention only that in addition to using single-crystal bulk substrates there are at least two other experimental possibilities of minimizing the number of these phonons. The first is to decrease (other conditions being equal) the widths of the investigated films, to make maximum possible use of the effect of "spreading" of the heat into the substrate. The second possibility is connected with using pulsed heating of the film in such a way that the characteristic time of return of the "reverse" phonons to the film are longer than the duration of the pulse (which must, however, not be long enough to establish a stationary state in the film). In this case the requirements on the width of the film can be greatly relaxed (compared with the case of stationary heating), owing to the lack of a characteristic thermal "background" for this heating.

A simple experimental criterion for determining that the film is actually in the electronic superheating regime is the absence of discontinuities of the observed physical quantities when the helium bath temperature goes through the  $\lambda$  point. In fact, since the physical characteristics of the metal electrons in this regime are no longer dependent on the bare transparency  $\alpha$ , it follows that further improvement due to cooling the film with superfluid helium,<sup>3</sup> can only strengthen the inequality  $\varepsilon \ll 1$ . Moreover, if the discontinuities exist nevertheless, their magnitude can be a measure of the "distance" from the regime of interest to us.

We list, finally, those of our results which are of greatest importance for experiments.

1. In the experimental situation of interest to us (see the Introduction), depending on the value of the parameter  $\varepsilon \sim 2d/\alpha l(T_e)$ , two substantially different heat-removal mechanisms are possible. If  $\varepsilon \gg 1$ , the usual "Joule" heating, which has been well investigated in a number of studies of the thermal resistance of an M-I boundary,<sup>1,2</sup> is realized. In the case  $\varepsilon \ll 1$  electronic superheating becomes possible, wherein the effects nonlinear in the electric field intensity are determined only by the properties of the metal and are no longer dependent on the acoustic transparency of the M-I interface.

2. Equations in closed form were obtained for the experimentally observed quantities  $Q(T_e)$  and  $R_{ph}(T_e)$  for arbitrary  $\varepsilon$  [Eqs. (11) and (27)]. These permit, in principle, comparison with experiment.

3. Even in the case of ideal acoustic matching of the metal and the substrate in sufficiently long films, electronic superheating cannot be eliminated in principle [and depends on the dissipated power  $\mathcal{P}$  in accord with Eq. (15)].

4. In the "Joule" heating regime  $\varepsilon \gg 1$ , the nonlinear

dependence of the electric resistance on the field strength does not reduce to a pure temperature dependence [with  $T_e$  determined from Eq. (11)], but contains a correction proportional to  $\epsilon^{-1}$  and due to the inhomogeneity of the phonon distribution function over the length  $l_p(T_e)$  [Eqs. (31) and (32)].

## APPENDIX I

Let us prove Eq. (23). To this end we change over in the integral I from the variables  $\varphi_1$  and  $\varphi_2$  to the new variables  $\chi = (\varphi_1 + \varphi_2)/2$  and  $\psi = \varphi_1 - \varphi_2$ . We can then show that

$$I = \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 Z(\varphi_1, \varphi_2) = \int_{-2\pi}^0 d\psi \int_{-\psi/2}^{2\pi+\psi/2} d\chi Z(\chi, \psi) + \int_0^{2\pi} d\psi \int_{\psi/2}^{2\pi-\psi/2} d\chi Z(\chi, \psi), \quad (1.1)$$

where

$$Z(\chi, \psi) = (\cos 2\chi + \cos \psi) \Phi \{ \cos \psi \}/2.$$

Integration with respect to  $\chi$  yields

$$I = \int_0^{2\pi} d\psi \Phi \{ \cos \psi \} [(2\pi - \psi) \cos \psi - \sin \psi]. \quad (1.2)$$

The change of variable  $\psi = \psi' + \pi$  reduces this integral to one with symmetric limits, after which (1.2) simplifies to

$$I = 2\pi \int_0^{2\pi} d\psi \cos \psi \Phi \{ -\cos \psi \}. \quad (1.3)$$

Finally, from (1.3) we readily obtain (23).

## APPENDIX II

We calculate the integral (24), where the integration limits  $x_1$  and  $x_2$  are determined by the roots of the trinomial

$$x = (1-u^2)(1-v^2) - (uv+v)^2 = av^2 + bv + c > 0;$$

where  $a = -1$ ,  $b = 2uv$  and  $c = 1 - u^2 - v^2$ , so that  $b^2 - 4ac \equiv 4(1-u^2)(1-v^2)$ . Then, breaking up the integral

$$K = \int_{x_1}^{\infty} \frac{dv (uv+v)}{\sqrt{x}}$$

into two parts and using the fact that

$$\int \frac{dv}{\sqrt{x}} = -\frac{1}{(-a)^{1/2}} \arcsin \frac{2av+b}{(b^2-4ac)^{1/2}}, \quad \int \frac{v dv}{\sqrt{x}} = \frac{\sqrt{x}}{a} - \frac{b}{2a} \int \frac{dv}{\sqrt{x}}$$

we readily obtain (24).

<sup>1</sup>) The question of the effect of the electrons on the size of the Kapitza jump between a metal and HeII was considered earlier in the known paper by Andreev.<sup>5</sup> The mechanism proposed there for this influence is most effective precisely in the case of strong acoustic mismatch between the media in contact ( $\alpha \lesssim 10^{-2}$  for the boundary between a metal and HeII). The mechanism whereby the electrons influence the temperature jump, which is discussed below, is most effective precisely in the opposite limiting case of high transparency ( $\alpha \sim 1$ ).

<sup>2</sup>) A generalization of some of the conclusions of this paper to include real spectra of electrons and phonons, and a discussion of the ensuing possibilities of extracting information on real EPI in a metal from experiments performed in the electronic superheating regime, will be reported in a separate paper.

<sup>3</sup>) It is assumed that liquid helium flows on one side of the metallic film.

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