[8]V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. 65, 1251 (1973) [Sov. Phys. JETP 38, 620 (1973)].
[9]A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Field-Theoretical Methods in Statistical Physics), Fizmatgiz, 1962 [Pergamon, 1965]
[10]Yu. A. Bychkov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Zh. Eksp. Teor. Fiz. 50, 738 (1966) [Sov. Phys. JETP 23, 489 (1966)].
[11]L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Nauka, 1976, Sec. 24 [Pergamon].

Translated by J. G. Adashko

Solitons and nonlinear resonance in two-dimensional lattices

L. A. Ostrovskiĭ, V. V. Papko, and Yu. A. Stepanyants

Institute of Applied Physics, USSR Academy of Sciences
(Submitted 18 August 1979)
Zh. Eksp. Teor. Fiz. 78, 831–840 (February 1980)

The behavior of solitary waves in two-dimensional lattices of nonlinear oscillators is investigated theoretically and experimentally. Solitons with an anisotropic relation between the amplitude and the duration are found. This anisotropy is preserved in the continual approximation. The proper "soliton modes" are studied in bounded lattices (resonators); it is shown that such modes are possible only for two configurations of the boundaries (rectangle and equilateral right triangle). The resonant excitation of soliton modes by a harmonic source (parametric generation of solitons) is considered. Experimental results of excitation of soliton modes in lattices of nonlinear electric oscillators are reported.

PACS numbers: 05.50. + q

INTRODUCTION

The role of solitary nonlinear waves—solitons—in lattice dynamics was recognized to be important quite long ago, starting with the attempts at interpretation of the known "paradox" of Fermi, Pasta, and Ulam,[1] which is connected with the anomalously slow stochastization in a system of nonlinear oscillators (we recall that the very term "soliton" was first introduced precisely in connection with this problem[2]). The properties of solitons in one-dimensional lattices (chains of coupled oscillators) were investigated in sufficient detail, and for particular forms of the interaction potential between the oscillators there are known solutions.[3] It was shown in addition that solitons can exist in multilayer semiconductor structures (superlattices),[4] as well as in multiband systems.[5] The role of soliton ensembles as collective excitations in lattices was investigated in a number of studies; conclusions were drawn that the solitons can contribute to the energy transport process that determines the thermal conductivity of crystals.[6] However, almost all the investigations were limited to one-dimensional processes. It is clear that both the properties of individual solitons and their collective behavior can be substantially different in two-dimensional and three-dimensional systems (see Ref. 6).

We consider in this paper solitons in two-dimensional lattices and investigate the resonant phenomena connected with the formation of "soliton modes" in bounded resonators of varying configurations. The theoretical analysis is based on the equations of a rectangular lattice of electromagnetic oscillators, for which the experimental results reported below were obtained. The conclusions, however, are more general in character and apply with practically no change, for example, to a corresponding system of mechanical oscillators.

THEORY

1. We consider a two-dimensional rectangular lattice consisting of identical elements and describable by the following nonlinear differential-difference equation:

$$L\frac{d^2Q_{m,n}}{dt^2}=(u_{m-1,n}-2u_{m,n}+u_{m+1,n})+(u_{m,n-1}-2u_{m,n}+u_{m,n+1}),\qquad(1)$$

where $Q_{m,n}$ is a specified nonlinear function of $u_{m,n}$, and the subscripts are the coordinates of a given lattice site in its two dimensions.

Equation (1) corresponds directly to the oscillations of a lattice of electromagnetic oscillators, which was used in the experiment described below (Fig. 1), where L denotes the inductance of the element, $u_{m,n}$ the vol-

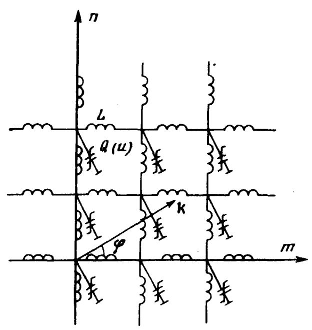

FIG. 1. Diagram of electric lattice used in the experiment.

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tage, and $Q_{m,n}$ the charge of the nonlinear capacitor. However, the theory developed below holds without substantial changes also for other simple models of nonlinear lattices. For example, transverse oscillations in a mechanical model of a lattice of atoms with mass μ, coupled by elastic forces, are described by the equation

$$\mu\frac{d^2z_{m,n}}{dt^2}=f(z_{m,n}-z_{m-1,n})-f(z_{m+1,n}-z_{m,n})+f(z_{m,n}-z_{m,n-1})-f(z_{m,n+1}-z_{m,n}), \tag{2}$$

where $z_{m,n}$ is the deviation from the equilibrium position, and f is the force (generally speaking, nonlinear) that the nearest neighbors exert on the atom (for the linear case such models are considered in detail, for example, in Ref. 7). For one-dimensional chains it is frequently possible to establish a correspondence between the nonlinear functions in formulas (1) and (2), whereby these equations become identical.[3] In two-dimensional problems the situation is more complicated, but at moderate nonlinearities and in the continual approximation (see below) the principal results for these equations turn out to be analogous.

It is well known that in the linear approximation, when $Q_{m,n}=C_0u_{m,n}$, where C_0 is a constant, Eq. (1) has a solution in the form of plane harmonic waves $u_{m,n} \sim \exp\{i(\omega t-k_m m-k_n n)\}$. The wave frequency ω is connected with the projections k_m and k_n of the wave vector on the lattice axes by the following equation:

$$\omega=2v_0\left[\sin^2\frac{k_m}{2}+\sin^2\frac{k_n}{2}\right]^{1/2}, \quad v_0=\frac{1}{(LC_0)^{1/2}}. \tag{3}$$

In some cases one can obtain also exact particular solutions for nonlinear problems. In fact, Eq. (1) reduces to one-dimensional in two obvious cases: a) when u depends only on m or n; b) when u depends on $m\pm n$, i.e., for plane waves propagating respectively along the lattice axes and in the diagonal direction. In particular, if $Q=Q_0\ln(1+u/U_0)$, where Q_0 and U_0 are constants, the propagation of such waves is described by the well-known Toda equation,[3] for which exact solutions were obtained. The latter include plane solitions of the form

$$u=V_c\,\mathrm{sch}^2 S(\Omega t-kr), \quad V_c=U_0\,\mathrm{sh}^2 k, \quad \Omega=(U_0/LQ_0)^{1/2}\,\mathrm{sh}\,k, \tag{4}$$

where $r=m$ or n and $S=1$ in case (a) and $r=(m\pm n)/\sqrt{2}$, $S=\sqrt{2}$ in case (b). The anisotropy of the lattice manifests itself here in the fact that solitons propagating along the diagonal to its axis have a duration shorter by a factor $\sqrt{2}$ than solitons traveling with the same amplitude along the axes.

To consider waves propagating in an arbitrary direction, we make the two customary simplifying assumptions: 1) the amplitude of the waves is relatively small, so that $Q(u)\approx C_0u-\alpha u^2$, and 2) the spatial scale of the motions is much larger than the distance between the neighboring cells.

This enables us to replace the differences in (1) by the first terms of the corresponding Taylor series and to proceed to an analysis of a continual system. Under these assumptions we obtain from (1)

$$\frac{1}{v_0^2}\left(\frac{\partial^2u}{\partial t^2}-\frac{\alpha}{C_0}\frac{\partial^2u^2}{\partial t^2}\right)=\frac{\partial^2u}{\partial m^2}+\frac{\partial^2u}{\partial n^2}+\frac{1}{12}\left(\frac{\partial^4u}{\partial m^4}+\frac{\partial^4u}{\partial n^4}\right). \tag{5}$$

We consider next a plane wave traveling at an angle φ to the lattice axis (see Fig. 1), and change over from m and n to new coordinates x and y, which are connected with the front of the wave:

$$m=x\cos\varphi-y\sin\varphi, \quad n=x\sin\varphi+y\cos\varphi, \quad \varphi=\text{const}.$$

In these coordinates, the wave propagates along the x axis, and its front is parallel to the y axis. Equation (5) for this wave is of the form

$$\frac{1}{v_0^2}\left(\frac{\partial^2u}{\partial t^2}-\frac{\alpha}{C_0}\frac{\partial^2u^2}{\partial t^2}\right)=\frac{\partial^2u}{\partial x^2}+\beta(\varphi)\frac{\partial^4u}{\partial x^4}, \tag{6}$$

where $\beta(\varphi)=(2-\sin^2 2\varphi)/24$.

For a fixed angle φ, Eq. (6) is the well known Boussinesq equation,[1)] the solutions of which have by now been well investigated. In particular, one of them describes the soliton

$$u=V_c\,\mathrm{sch}^2\frac{x-v_ct}{\Lambda}, \tag{7}$$

where

$$v=\frac{v_0}{(1-2\alpha V_c/3C_0)^{1/2}}, \quad \Lambda=\left\{4\beta(\varphi)\left[\left(\frac{v_c}{v_0}\right)^2-1\right]^{-1}\right\}^{1/2}.$$

Thus, the dependence of the soliton velocity on its amplitude is the same for all directions but its characteristic duration $T_c=\Lambda/v_c$ depends on the orientation of the front. Just as above, at a specified amplitude of the soliton the duration T_c is minimal for diagonal directions ($\varphi=\pm\pi/4$) and is larger by a factor $\sqrt{2}$ in the case of propagation along the lattice axes.

2. We proceed to the question of the natural and forced oscillations in bounded nonlinear lattices with reflecting boundaries (resonators). We assume that in the continual limit we have on the lattice boundary $\partial u/\partial n=0$ (in the electric model this means that the conduction current is zero). In the linear approximation, oscillations in such a system can be described by a set of normal harmonic modes with corresponding natural frequencies ω_i ($i=1,2,\ldots$). The nonlinearity produces, generally speaking, a coupling between the modes, and in the case of weak nonlinearity the principal role is played by the resonant interactions. In this sense, systems with an equidistant spectrum of the natural frequencies ($\omega_i\approx i\omega_1$), for which all the harmonics of the fundamental oscillation are at resonance, are a special case, and lead to establishment of essentially unharmonic (pulsed) oscillations. Instead of analyzing an interaction of a large number of normal harmonic modes in a resonator, it is then more effective to use an approach based on consideration of "pulsed" modes, each of which is represented, generally speaking, by an unlimited spectrum of harmonic modes.

We consider first a linear two-dimensional resonator without dispersion, when $Q=C_0u$ in (6), and there is no dispersion term $\sim\partial^4u/\partial x^4$. Obviously, a plane wave of arbitrary profile is not distorted in such a medium. Assume that the wave is short compared with the dimensions of the resonator ("flat δ pulse") and undergoes repeated reflections from the boundaries. As a result, the configuration of its front will vary continuously, but it is possible that after a finite number of reflections the front will reoccupy its previous position (Fig. 2). In

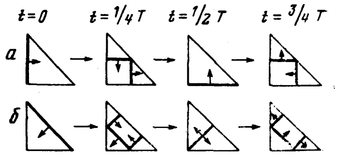

FIG. 2. Two possible types of pulsed modes excited in a nonlinear triangular resonator.

this case, obviously, natural pulsed oscillations are established in the resonator, in the form of a superposition of a finite number of flat δ pulses, that go over into one another by the reflection from the boundaries (they can be regarded as a set of an infinite number of harmonic modes with equidistant spectrum). Confining ourselves to resonators in the form of convex polygons, we can show[8] that in this class there exist only four types of resonators in which natural pulsed modes can be observed: rectangle, equilateral right triangle, equilateral triangle, and right triangle with angle $\pi/6$ at the vertex.

We turn now to the nonlinear problem described by Eq. (6) with the same boundary condition ($\partial u/\partial n = 0$). It is natural to assume that in the case of small nonlinearity and weak dispersion there can exist in the resonators listed above free nonlinear oscillations in the form of plane solitons with the same front configuration as for the δ pulses in the linear system without dispersion. For this it is necessary, however, that the soliton remain stationary upon reflection. In the case of small nonlinearity, the reflection of the soliton proceeds in the same manner as in a linear medium (with conservation of the duration and waveform of the pulse), but owing to the anisotropy of the lattice, the duration of the reflected pulse may in general not correspond to the soliton, and then the subsequent propagation becomes nonstationary. Since the amplitude and duration of the soliton remain unchanged upon reflection, and consequently its spatial length does not change, it follows that in order for the pulse to remain stationary it is necessary that the orientation of the front on the reflected soliton preserve its symmetry with respect to the principal axis of the lattice [i.e., φ in (6) should change by $\varphi \pm \pi/2$]; then the incidence and reflection angles are equal. It follows therefore that the boundaries must be oriented either along the principal axes of the rectangular lattice, or make an angle $\pi/4$ with them. This in turn means that out of the four indicated resonator configurations only two—rectangle and equilateral right triangle—admit of the existence of stationary "soliton" modes. On the other hand, in each of these resonators the structure of the pulsed mode is not the only one. For example, in a rectangle with natural frequencies ω_{ij} ($i,j = 1, 2, \ldots$) different spatial structures correspond to equidistant sets of harmonic modes with arbitrarily specified ratio $i/j = \text{const}$.

For a triangular resonator, only two pulsed modes are possible, and are shown in Fig. 2. These modes consist of plane fronts propagating either along the normal to one of the sides of the triangle (Fig. 2a) or at an angle $\pi/4$ to them (Fig. 2b). The natural frequencies of these modes are chosen in the linear approximation from among the natural frequencies of a square resonator with side equal, in case a, to the hypothenuse, and in case b to the side of the triangular resonator, and in this case the fundamental frequency ω_a is smaller by a factor $\sqrt{2}$ than the fundamental frequency ω_b.

Thus, the nonlinear resonators of the two considered types turn out to be singled out in that "soliton modes" can exist in them in the form of solitons with plane fronts propagating at definite angles to the boundaries. Of course, the amplitudes of these waves must be high enough to make the length of the soliton much shorter than the dimension of the resonator.[2)] Since the soliton velocity depends on its amplitude, a nonlinear frequency shift of the oscillations ω_s must take place with change of the oscillation amplitude. It is easily seen that the relative frequency shift is equal to

$$\frac{\omega_s - \omega_l}{\omega_l} = \frac{\alpha V_s}{3C_0} + \frac{\beta(\varphi) k_l^2}{2}, \tag{8}$$

where ω_l is the natural frequency in the linear approximation (with account taken of dispersion), and k_l is the corresponding wave number.

It must be emphasized at the same time that in view of the reversibility of the interaction of the solitons, different modes, when simultaneously excited, do not exert on the average any influence on one another (a nonlinear frequency shift due to their frequency interaction appears only in the next order in the amplitude). At the same time, for modes with close frequencies, a substantial periodic exchange of energy is possible, with a period determined by the frequency difference.

3. We examine briefly the excitation of a resonator by a harmonic external source with frequency close to one of the natural modes. One can expect the synchronous upward frequency conversion to produce in the resonator forced oscillations corresponding to the indicated soliton modes.

To describe such oscillations we can use an approach developed earlier for one-dimensional systems—the so-called parametric pulse generators (PPG). A PPG constitutes a one-dimensional resonator, in which the energy of the harmonic excitation (pump) is converted into the soliton energy. The theory of such system is developed in Ref. 9 and is based on consideration of solitons propagating in a media whose parameter (in this case the distributed capacitance $C = dQ/du$) change because they are modulated by the harmonic "pump" field $u_p = U_p \cos(\omega_p t - k_p x)$, so that $C - C_0 \sim u_p(x - v_p t)$, where v_p is the propagation velocity of the pump wave in the system. It is important that account is taken here of only the pump-field component that moves together with the soliton, since the opposing component, when averaged over the period, does not interact with the soliton. The co-moving wave, on the other hand, delivers to the soliton a power $(E/C)\partial C/\partial t$, where E is the soliton energy. The behavior of the soliton in the field of the parameter wave is characterized by the difference between the velocities of the pump and the soliton (which depends on the amplitude $A = \alpha V_c/3C_0$), and also by its phase $\psi = \omega_p T_0$, which is determined by the interval T_0 to the nearest, from the side of negative t, maximum of the field. The equations for A and ψ become similar in form to the equations for the particles in a cyclic accelerator[9,10]:

$$\frac{dA}{d\tau}=-\frac{2}{3}A(M\sin\psi+\nu),\quad \frac{d\psi}{d\tau}=A-M\cos\psi-\Delta, \tag{9}$$

where $\tau=\omega_p t$, $\Delta=v_p/v_0-1$, and ν is the loss coefficient [see Ref. (9)]. This theory can be applied without substantial changes to two-dimensional resonators. In fact, the field of the resonator mode which corresponds to the pump wave, can be represented as a superposition of plane waves having the same front orientation as for the pulsed modes shown in Fig. 2:

$$u(\zeta,\eta,t)=U_p[\cos(k\zeta-\omega_p t)+\cos(k\eta-\omega_p t)+\cos(k\zeta+\omega_p t)+\cos(k\eta+\omega_p t)], \tag{10}$$

where for the case shown in Fig. 2a we have $\zeta=n$, and $\eta=m$, while for Fig. 2b $\zeta=(n-m)/\sqrt{2}$ and $\eta=(n+m)/\sqrt{2}$; U_p is the pump amplitude.

A plane soliton propagating in the corresponding direction interacts with only one of the traveling harmonic components of the pump field, so that it is possible to describe the amplification of the soliton by using the one-dimensional equation (9). In Ref. 9 was presented a detailed analysis of Eqs. (9), on the basis of which an investigation was made of the soliton generation regimes and their stability. As a result it was established analytically and confirmed experimentally that in PPG there can exist several stable equilibrium states corresponding to stationary solitons moving in synchronism with the pump. We note that the dependence of the natural frequencies of the soliton modes on the amplitude (8) leads, just as for the usual nonlinear oscillator to a jumplike dependence of the amplitude of the oscillations on the excitation frequency (the hard generation regime). Such a nonlinear resonance, however, is more complicated because of the multimode character of the system. In this case this means that different numbers of solitons in the resonator can exist simultaneously. Examples of such resonance curves are given in the experimental part of the paper (see Fig. 6 below). In addition, in the regime of generation of soliton modes the nonlinear resonance is more noticeably pronounced, since according to (6) the nonlinear frequency shift is proportional to the first power and not to the square of the oscillation amplitude.

The specifics of the two-dimensional resonator consists, in particular, in the existence of several "soliton" modes that differ in the orientation of the wavefronts relative to the boundaries (see Fig. 2). These modes can be excited simultaneously, if several independent harmonic sources act on the resonator. Although, as already noted, soliton modes do not interact energy-wise with one another, the resultant oscillations can differ substantially from a simple superposition of individual modes, since every one of them is now in a pump field with a complicated space-time structure. In the given-pump-wave approximation, the excited oscillations are described by a system of pairs of equations of the type (9), which are not connected with one another, and whose number is equal to the number of harmonic sources, while the dimensionless pump amplitude M is a complicated quasiperiodic function. In the simplest case the influence of this pump on each soliton mode reduces to quasiperiodic changes of the amplitude and phase of the solitons in the vicinities of the stationary state. Thus, in the case of a two-frequency pump wave, the parameters of the solitons are modulated with a period $T=2\pi(\omega_1-\omega_2)^{-1}$. Approximately half of this time the soliton wave is additionally amplified by the "foreign" pump wave, and during the other half it is weakened, so that one should expect appreciable deviations of the amplitude and phase of the soliton from the stationary state at $T\gg T_{1,2}$. On the other hand if $T\approx T_{1,2}$, then the crossover interactions are inessential and the pulsed oscillations should be close to superpositions of individual soliton modes.

EXPERIMENT

The experiments were performed in a two-dimensional electromagnetic resonator in the form of a rectangle or an equilateral right triangle, made up of oscillations shown in Fig. 1 with inductance $L=40\ \mu$H. The nonlinear element was the capacitance of a p-n junction of semiconductor diodes blocked by a dc bias voltage $E_b=-0.8$ V; such a capacitance can be approximated by the formula $C(u)=C_0-2\alpha u$, where $C_0=442$ pF and $\alpha=66$ pF/V. The resonators were excited in one corner link by a source of harmonic oscillations through a resistor $R=24$ kΩ. The orientation of the lattice (the axes m and n) coincided in all the cases with the directions of the sides at the vertex of the right angle (Fig. 2). The size of the rectangular resonator was 20–30 links, while the size of the side of the triangle was 20 links.

We present some data concerning the investigation of the triangular resonator. At low excitation amplitudes (up to 30 mV), nearly linear harmonic oscillations were produced in the system, with resonances at the natural frequencies. The first four resonances were observed at the frequencies $f_1=178$ kHz, $f_2=$ kHz, $f_3=353$ kHz, $f_4=505$ kHz, with the frequencies f_1 and f_3 of the order of $2f_1$ corresponding to the modes of a quadratic resonator with a side equal to the hypothenuse of the triangle, and with f_2 and f_4 of the order of $2f_2$ corresponding to the modes of a quadratic resonator with a side equal to the side of the triangle. As already mentioned above, two series of natural frequencies, multiples of f_1 and f_2, are present in this case.

At relatively large amplitudes U_p of the external source (pump) acting near of the one of the resonant frequencies ($U_p\gtrsim 0.2$ V), the oscillations in the resonator acquired a pulsed character in accordance with the theory.

Typical oscillograms of the voltage oscillations across the capacitor of the resonator corner link located farthest from the source ($m=1$, $n=20$) are shown in Fig. 3 for the cases when the values of f_p corresponded to the fundamental frequencies of the two indicated sequences. Depending on the relative detuning $\Delta=(f_p-f_i)/f_i$ between the excitation frequency f_p and the natural frequency f_i of the linear resonator, the waveform of the oscillations can be substantially different. At large negative Δ, the oscillations consist of series

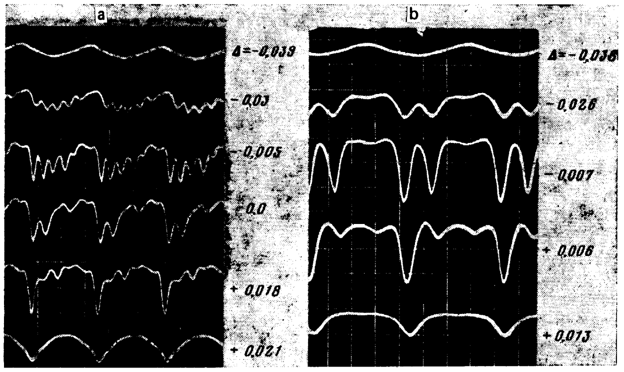

FIG. 3. Waveform of voltage oscillation at the cornerpoint of a triangular resonator: a) at $f_p = (1+\Delta)f_1$; b) at $f_p = (1+\Delta)f_2$. The values of Δ are indicated in the figure.

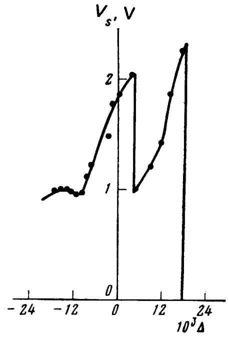

FIG. 5. Nonlinear resonance for soliton modes in a resonator near the frequency f_3.

of pulses in each period of the pump. The number of pulses in the series decreases with increasing f_p, and at small positive Δ there remains one pulse for each period. Measurements of the spatial distribution of the field in the resonator at different instants of time have made it possible to establish that the pulsed oscillations have plane fronts, whose configurations correspond fully through the pulsed modes graphically shown in Fig. 2 (Fig. 2a for $f_p \approx f_1$ and Fig. 2b for $f_p \approx f_2$). The profiles of the pulses agreed well with the solitons described by formula (7). Thus, in the case $f_p \approx f_1$ at a soliton amplitude $V_c = 0.87$ the soliton duration was 0.45 μsec, while formula (7) yields 0.46 μsec. At $f_p \approx f_2$ the pulse duration at the same amplitude was smaller by a factor 1.3 (instead of 1.4 according to the theory). The slight difference between theory and experiment can be attributed to the influence of the field of the pump, which produces a "pedestal" for the soliton and by the same token alters slightly its parameters. Judging from the measurements of the pulse duration, up to 12 harmonics of the fundamental frequency were effectively excited in the system.

Similar pulsed oscillations were observed when the resonator was excited at the frequencies f_3 or f_4, i.e., at the second harmonics of the fundamental frequencies. The period of the pulsed process could take on in this case two different values corresponding to the pump frequency or to half this frequency; in the latter case (frequency-division regime) the soliton was excited not in each period of the pump, but in every other period (Fig. 4).

Figure 5 shows the resonance curves characterizing the dependence of the soliton amplitude V_s on the pump frequency when the resonator is excited at frequencies close to the resonant frequency f_3. It can be seen that the resonance has two clearly pronounced peaks; the left-hand peak corresponds to oscillations with frequency f_p, and the right-hand side to frequency $f_p/2$. The $V_s(\Delta)$ curve has the asymmetric form typical of nonlinear systems; in particular, a change from one oscillation regime to another took place jumpwise following small changes of the frequency f_p near f_2, in full agreement with the theory.

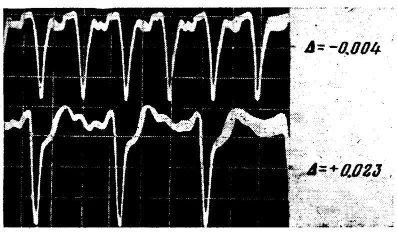

FIG. 4. Frequency division of pulsed oscillations in a resonator at small changes of the pump frequency near f_3.

Figure 6 shows the waveform of the characteristic pulsed oscillations in a rectangular resonator. Just as above, different pulsed modes are possible with corresponding equidistant spectra. Figure 6a shows oscillations produced at $f_p^{(1)} = 140$ kHz, and Fig. 6b—at $f_p^{(2)} = 251$ kHz, the fronts of the corresponding pulsed modes are shown on the left. Figure 6c demonstrates the form of the oscillations excited by two independent pump sources with frequencies $f_p^{(1)}$ and $f_p^{(2)}$, and in this case both pulsed modes are excited, and in view of the relative proximity of the difference $f_p^{(2)} - f_p^{(1)}$ to each of these frequencies, weak amplitude modulation was produced and the oscillations were close to a superposition of separately excited pulse sequences. The same figure shows the oscillations in a rectangular resonator when excited at $f_p^{(3)} = 345$ kHz (Fig. 6d) and $f_p^{(4)} = 434$ kHz (Fig. 6e). These frequencies correspond to more complicated pulsed modes (see Fig. 6d and 6e, left). In this case, joint excitation of the resonator at

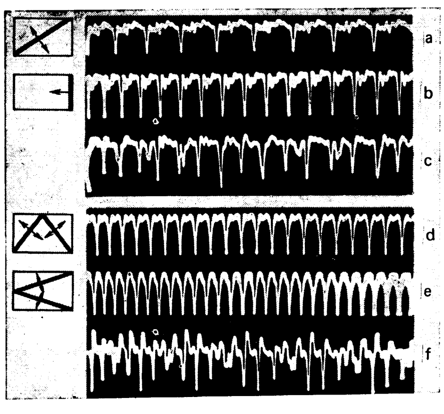

FIG. 6. Voltage oscillations at the corner points of a rectangular resonator, excited by a single source (a, b, d, e) or simultaneously by two (c, f) sources of sinusoidal oscillations.

both frequencies $f_p^{(3)}, f_p^{(4)}$ leads to a substantial modulation of the amplitude of the pulsed sequences with the difference frequency $f_p^{(4)} - f_p^{(3)} = 89$ kHz (Fig. 6f).

We note in conclusion that the foregoing analysis can be generalized to include three-dimensional lattices. In particular, in the case of a logarithmic nonlinearity it is possible to obtain the exact solution of three-dimensional equations similar to (1). These solutions correspond to a soliton whose front is oriented at various angles to all three axes of the lattice. Such a soliton is shorter by a factor $\sqrt{3}$ than the soliton propagating along one of the axes.

The singularities of the propagation and resonant excitation of solitons in anisotropic nonlinear lattices, which were considered above, can apparently play a substantial role in the analysis of various types of collective excitations in a solid. As already indicated, the importance of the soliton concept in the explanation of energy transport processes and transport of thermal excitations in crystals was recently made clear.[11,12] In recent papers by Bishop (see Ref. 13 and the bibliography therein) used the methods of statistical mechanics to analyze the contribution of the soliton component to the lattice oscillations. On the basis of the results of the present paper one can expect the anisotropy of the soliton parameters to influence the thermodynamic properties of nonlinear lattices. At the same time, crystals of a definite regular shape (for example, cubic) can be singled out under corresponding boundary conditions in that respect, that it is possible to excite effectively in them soliton modes of high intensity.

[1)]For a travelling plane wave this leads, as usual, to the Korteweg-de Vries equation.

[2)]A more general approach should include in the analysis, in place of solitons, periodic (conoidal) waves, this would make it possible to describe oscillations of any amplitude.

[1]E. Fermi, Collected Works (Russ. transl.), Vol. 2, Nauka, 1972.
[2]N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 240 (1965).
[3]M. Toda, Phys. Rep. **18**, 1 (1975).
[4]É. M. Epshtein, Fiz. Tverd. Tela (Leningrad) **19**, 3456 (1977) [Sov. Phys. Solid State **19**, 1618 (1977)].
[5]Yu. K. Bogatyrev, K. A. Gorshkov, and L. A. Ostrovskii, Radiotekh. Elektron. **21**, 458 (1976).
[6]N. J. Zabusky, Comp. Phys. Commun. **5**, 1 (1973).
[7]L. Brillouin and M. Parodi, Wave Propagation in Periodic Structures, Dover, 1953.
[8]V. V. Korneichik and A. P. Khapalyuk, Radiotekh. Élektron. **21**, 853 (1976).
[9]K. A. Gorshkov, L. A. Ostrovskiĭ, and V. V. Papko, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **16**, 1195 (1973).
[10]N. N. Filonenko, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **13**, 925 (1970).
[11]V. Narayanamurti and C. M. Varma, Phys. Rev. Lett. **25**, 1105 (1970).
[12]F. D. Tappert and C. M. Varma, Phys. Rev. Lett. **25**, 1108 (1970).
[13]A. R. Bishop, Solitons in Action, Acad. Press, New York, 1978.

Translated by J. G. Adashko

Phase transitions in superconducting compounds with superstructure *C*-15

V. A. Finkel' and E. A. Pushkarev

Khar'kov Physicotechnical Institute, Ukrainian Academy of Sciences

(Submitted 18 August 1979)
Zh. Eksp. Teor. Fiz. **78**, 842–846 (February 1980)

The crystal structure of the superconducting compounds ZrV_2 and HfV_2 with C-15 structure is investigated at 77–300 K, and the temperature dependence of their electric resistance is investigated at 61–300 K. At $T_m = 80$ K (for ZrV_2) and $T_m = 100$ K (HfV_2) the cubic lattice symmetry is lowered to a rhombohedral and tetragonal, respectively, an abrupt change takes place in the atomic volume, and maxima appear on the temperature dependence of the resistivity. Inflections of the temperature-dependence curves of the crystal lattice parameter and of the resistance are observed at ~ 157 K (ZrV_2) and ~ 120 K (HfV_2). It is suggested that the loss of stability of the crystal lattice is the result of two successive phase transitions, one of the second order and the other of first order.

PACS numbers: 61.60. + m, 64.70.Kb, 72.80.Ga

Much attention has been paid of late to the study of lattice instability of superconducting Laves phases with C-15 structure (in particular ZrV_2, HfV_2). Interest in the question is quite understandable: ZrV_2, HfV_2, the ternary compounds $Zr_xHf_{1-x}V_2, Hf_xTa_{1-x}V_2$, and others are unique examples of a very strong difference between the critical temperatures of the transition to the superconducting state ($T_c \sim 10$ K) and the temperatures at which the cubic crystal lattice of the type C-15 lose stability ($T_m \sim 100$ K). The existence of such a situation uncovers extensive prospects for experimental and theoretical study of the relation between the two phenom-

0038-5646/80/020422-03$02.40