Low-angle multiple scattering by static inhomogeneities

S. V. Maleev and B. P. Toperverg

B. P. Konstantinov Leningrad Institute of Nuclear Physics, Academy of Sciences USSR
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The problem of the angular distribution of particles in multiple elastic scattering is discussed. It is shown that in the case of scattering by large-scale inhomogeneities possessing a sharp boundary the dependence of the halfwidth of the distribution on the sample thickness is determined by two parameters. Thus, two parameters of the scattering system, such as the mean size of the scatterers and their concentration, or the size and interaction energy, etc., can be determined in a single experiment. Multiple low-angle scattering by critical fluctuations is considered. The scattering intensity is found to depend on the angle in a power-law manner and to become isotropic rapidly with increase in the sample thickness.

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1. INTRODUCTION

As is well known, scattering is essentially single if the free path length in the medium is large in comparison with the thickness of the scattering sample. In this case, the most detailed information on the scattering system is obtained from the experimental data. In case of necessity it is not difficult to treat the contribution from multiple scattering as a small correction in the parameter L/l (l is the free path length, L is the sample thickness). However, in a number of cases, the ratio L/l cannot easily be made small, and then an attempt is made to satisfy the opposite condition: $L/l \gg 1$. To the present time, the question of multiple scattering of charged particles in a medium has been studied most completely, both experimentally and theoretically. An exposition of the theory of Moliere and Bethe that is usually applied in this case can be found in the book by Mott and Massey[1] (see also the review of Scott,[2] cited in "Review of Particle Properties,"[3] and the work of Case and Battle[4] and Highland.)[5]

From among the researches on the multiple scattering of neutral particles, we should note the experiments of Dexter and Beeman[6] on the multiple scattering of x-rays with wavelength of the order of 1 Å by powdered carbon, and the experiments of Shil'shteĭn *et al.*[7] on the multiple scattering of thermal neutrons in multi-domain ferromagnets. In both cases, the single-scattering angle turns out to be so small that it cannot be resolved, and the method of multiple scattering is used for the determination of the dimensions of the particles.

For the interpretation of the results of their experiments, the authors of Refs. 6 and 7 used the following simple idea: if the particle which is multiply scattered is deflected each time by a small angle of the order of some characteristic angle ϑ_0, then the probability of scattering through and angle $\theta \gg \vartheta_0$ (but $\theta \ll 1$) upon traversing a large path L ($L \gg l$) should be described by the diffusion formula

$$I(\theta, L)=I(0, L)\exp\{-\theta^2 L/\vartheta_0^2 l\}. \tag{1}$$

The halfwidth of the scattering curve $\bar{\theta}$, measured at half its height, is proportional here to the square root of the thickness of the sample:

$$\bar{\theta}(L^{1/2})=\vartheta_0(Ll^{-1}\ln 2)^{1/2}. \tag{2}$$

From the slope of this line, we can determine the angular diffusion coefficient $\vartheta_0^2 l$ and then compare this quantity with the theoretical results, obtained on the basis of one model or another.

On the graphs shown in Refs. 6 and 7, the experimental dependence of $\bar{\theta}(L^{1/2})$ actually seems to be linear at large values of $L^{1/2}$. However, the corresponding straight lines, extrapolated to small thicknesses, do not go through the origin, but intersect a certain positive segment on the abscissa (we call this phenomenon the "non-zero intercept" in what follows).

If we assume that the data given in these researches are well described by a straight line, then this means that at large L there should be a nontrivial correction to the diffusion law

$$\bar{\theta}^2(L)=\vartheta_0^2 Ll^{-1}\ln 2 \tag{3}$$

proprotional to $L^{1/2}$. Since the diffusion law (3) is asymptotic, there are also corrections to it, and their contributions to the dependence of the halfwidth of the scattering curve on $L^{1/2}$ vanish as $L\to\infty$. As will be seen below, the principal of these make a contribution of the order of ϑ_0^2 to $\bar{\theta}^2$.

In addition, there are other corrections, connected with the fact that the angular random walks in multiple scattering take place not on a plane but on a sphere. The chief of these corrections at $\bar{\theta}^2 \ll 1$ should obviously be of the order of $\bar{\theta}^4 \sim \vartheta_0^4 (L/l)^2$ and should increase in absolute value with increase in L.

It is clear from physical considerations that in the general case, in angular random walks, the $\bar{\theta}(L^{1/2})$ dependence should have the following form: at small ratios L/l it should not depend on the sample thickness; this should be followed by linear segments according to formula (2) and, finally, when L/l is very large, the scattering should become isotropic. The corrections to the linear part of the curve, given above, correspond to a transition of the dependence $\bar{\theta}(L^{1/2})$ from the linear regime to one of the asymptotic regimes. If ϑ_0 is very small, then the region of linearity $\bar{\theta}(L^{1/2})$ is rather large. We assume below that the L are such that $\vartheta_0 \ll \bar{\theta} \ll 1$ and corrections from the boundaries of the region of low-angle multiple scattering can be neglected.

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However, we note that the picture given here of angular random walk does not always describe correctly multiple low-angle scattering. The fact is that this picture is actually based on an assumption of a sufficiently rapid decay of the cross section of single scattering $\sigma(\vartheta)$ at angles that are large in comparison with the characteristic angles. Actually, in order that the principal contribution to the intensity $I(\theta, L)$ be determined by the diffusion approximation (1), the existence of a mean square angle of single scattering is in the least necessary

$$\overline{\vartheta^2}=\int_0^\infty \vartheta d\vartheta\, \vartheta^2\sigma(\vartheta) \Big/ \int_0^\infty \vartheta d\vartheta \sigma(\vartheta). \tag{4}$$

In other words, a more rapid falloff of $\sigma(\vartheta)$ than ϑ^{-4} is required. In the case in which $\overline{\vartheta^2}$ exists, but $\sigma(\vartheta)$ does not fall off very rapidly, for example, as ϑ^{-5}, corrections to the formulas (2) and (3) arise that can exceed in value those discussed above.

In the present paper, we shall show that the falloff of the cross section as ϑ^{-5} leads to the appearance of a correction to $\overline{\theta}^2(L^{1/2})$ of the order of $L^{1/2}$, and the straight line $\overline{\theta}(L^{1/2})$ does not pass through the origin, i.e., an "intercept" appears. It is important to note that the $\overline{\theta}(L^{1/2})$ dependence is determined here by two quantities—the slope of the straight line and the intercept, and we can in principle derive two parameters from the experimental data which, taken as a whole, characterize the scattering system. At first glance, it appears that this explains the results obtained in Refs. 6 and 7. In fact, the experimental situation in Refs. 6 and 7 was more complicated. The scattering in these experiments takes place from large-scale inhomogeneities having a rather sharp boundary. As will be seen from what appears below, there is actually a term that falls off as ϑ^{-5} in the scattering cross section from such objects; however, the principal term here falls off as[1)] ϑ^{-4}. Multiple scattering in this case remains low-angle ($\overline{\theta} \ll 1$) over a wide range of variation of the quantity L, while the role of large-angle single scattering is that the slope of the line $\overline{\theta}(L^{1/2})$ turns out to be a slowly changing function of L.

In addition, the presence in the cross section of a term proportional to ϑ^{-5}, together with the term ϑ^{-4}, leads to the appearance in $\overline{\theta}(L^{1/2})$ of an increment that depends weakly on L and represents the intercept discussed above.

In the next section of the work, we give a simple derivation of the general Moliere–Bethe formula for the intensity of multiple scattering, which enables us to analyze the corrections to it that arise. Then, in the third section, we discuss multiple scattering by spherical inhomogeneities with sharp boundaries (potential of finite radius). It is shown that here the halfwidth of the angular distribution depends on two parameters—the radius of the potential and the interaction energy, and criteria are advanced for their experimental determination.

In the fourth section, multiple scattering by critical fluctuations is discussed. The corrections to the Moliere formula are analyzed in Appendix I, and Appendix II is devoted to generalization of the results of the third section to the case of multiple scattering by inhomogeneities of arbitrary shape.

2. DERIVATION OF THE MULTIPLE SCATTERING FORMULA

There are in the literature several variants of the derivation of the formula for the particle distribution in multiple scattering at small angles (see the bibliography in Ref. 1, and also the review by Ryazanov[8]).[2)] Here we shall give a rather simple quantum-mechanical derivation of the formula for the angular distribution of the multiple scattering, which allows us to trace all the approximations which lead to the Moliere formula.

We write down the amplitude of the scattering of a particle by a sample in the form of a perturbation-theory series:

$$f_{\mathbf{p}\mathbf{p}_0}=-\frac{1}{4\pi}\int d\mathbf{r}d\mathbf{r}' e^{-i\mathbf{p}\mathbf{r}}\exp(i\mathbf{p}_0\mathbf{r})\{U(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')$$
$$+U(\mathbf{r})G^{(0)}(\mathbf{r}-\mathbf{r}')U(\mathbf{r}')+\ldots\},\qquad G^{(0)}(\mathbf{r})=-\frac{1}{4\pi r}e^{ipr}, \tag{5}$$

where $\mathbf{p}_0$ and $\mathbf{p}$ are the momenta before and after scattering, $U(\mathbf{r})$ is the interaction potential, which is the sum of the potentials from the scattering centers (we use a unit system in which $2m=\hbar=1$). Then, using the averaging procedure described in the book by Abrikosov *et al.*,[9] it is not difficult to obtain the following expression for the cross section, averaged over the distribution of scattering centers:

$$I_{\mathbf{p}\mathbf{p}_0}=\frac{1}{(4\pi)^2}\int d\mathbf{r}d\mathbf{r}'d\mathbf{r}_1d\mathbf{r}_1'\psi_{\mathbf{p}}(\mathbf{r})\psi_{\mathbf{p}}^*(\mathbf{r}')K(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\mathbf{r}_1')\varphi_{\mathbf{p}_0}(\mathbf{r}_1)\varphi_{\mathbf{p}_0}^*(\mathbf{r}_1'); \tag{6a}$$

$$\psi_{\mathbf{p}}(\mathbf{r})=e^{-i\mathbf{p}\mathbf{r}}+\int d\mathbf{r}_1d\mathbf{r}_1'\exp(-i\mathbf{p}\mathbf{r}_1)\Sigma(\mathbf{r}_1,\mathbf{r}_1')G(\mathbf{r}_1',\mathbf{r}), \tag{6b}$$

$$\varphi_{\mathbf{p}_0}(\mathbf{r})=\exp(i\mathbf{p}_0\mathbf{r})+\int d\mathbf{r}_1d\mathbf{r}_1'G(\mathbf{r},\mathbf{r}_1)\Sigma(\mathbf{r}_1,\mathbf{r}_1')\exp(i\mathbf{p}_0\mathbf{r}_1'), \tag{6c}$$

$$K(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\mathbf{r}_1')=W(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\mathbf{r}_1')+\int d\mathbf{r}_2d\mathbf{r}_2'd\mathbf{r}_3d\mathbf{r}_3'$$
$$\times W(\mathbf{r},\mathbf{r}';\mathbf{r}_2,\mathbf{r}_2')G(\mathbf{r}_2,\mathbf{r}_3)G^*(\mathbf{r}_2',\mathbf{r}_3')K(\mathbf{r}_3,\mathbf{r}_3';\mathbf{r}_1,\mathbf{r}_1'). \tag{6d}$$

Here $G(\mathbf{r},\mathbf{r}')$ is the exact Green's function of the particle in the medium (see Ref. 9), Σ is its self-energy part, W is the total irreducible four-particle vertex.

For the solution of these equations, we made a number of approximations. We first assume that the interaction energy U is small in comparison with the kinetic energy E; second, that the characteristic radius of interaction r_0 is much greater than the wavelength of the particle λ. If the dimensions of the sample L are large in comparison with r_0, we can then use for Σ and W the expressions obtained for the unbounded medium.[9,10] For a thick sample, in which $L \gg l = p^{-1}$, we can neglect in the calculation of the Green's function G the effect of the boundaries. Since G in this case depends on the difference $\mathbf{r}-\mathbf{r}'$, making a Fourier transformation, we obtain the standard expression $G_p=(E-p^2+i\gamma)^{-1}$.

Let the particles be incident perpendicular on the surface of a sample having the shape of a plate of thickness L. Then, choosing the Z axis along the incident beam we have the following obvious formulas for the functions φ and ψ entering into (6):

$$\varphi_{\mathbf{p}_0}(z)=\exp(ip_0z)e^{-z/2l},\quad \psi_{\mathbf{p}}=e^{-i\mathbf{pr}}e^{-(L-z)/2l'},\quad l'=p_z/\gamma. \tag{7}$$

In an unbounded medium, the function $K(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\mathbf{r}_1')$ depends on three coordinate differences, with respect to which we carry out the Fourier transformation. It then follows from (6d) that

$$K(\mathbf{p},\mathbf{p}';\mathbf{k})=(4\pi)^2n_0\sigma_{\mathbf{pp}'}+\frac{1}{(2\pi)^3}\int d\mathbf{p}_1(4\pi)^2n_0\sigma_{\mathbf{pp}_1}G_{\mathbf{p}_1+\mathbf{k}}G_{\mathbf{p}_1}^*K(\mathbf{p}_1,\mathbf{p}';\mathbf{k}),$$
$$K(\mathbf{r},\mathbf{r}';\mathbf{r}_1,\mathbf{r}_1')=\frac{1}{(2\pi)^9}\int d\mathbf{p}_1\,d\mathbf{p}_1'\exp[i\mathbf{k}(\mathbf{r}-\mathbf{r}_1)]$$
$$\times\exp[i\mathbf{p}_1(\mathbf{r}-\mathbf{r}')+i\mathbf{p}_1'(\mathbf{r}_1-\mathbf{r}_1')]K(\mathbf{p}_1,\mathbf{p}_1';\mathbf{k}). \tag{8}$$

We have taken it into account here that $pr_0\gg 1$, and therefore we have neglected the interference between scatterings from different centers. Moreover, in what follows, we shall be interested in distances $|\mathbf{r}-\mathbf{r}_1|$ that are large in comparison with l and we can therefore assume that $kl\ll 1$. In this approximation $W_{\mathbf{pp}'}=(4\pi)^2n_0\sigma_{\mathbf{pp}'}$, where $\sigma_{\mathbf{pp}'}$ is the mean scattering cross section from a single center and n_0 is the density of scatterers.

It is seen from formulas (6) and (7) that the vector $\mathbf{k}$ in (8) can be regarded as directed along the Z axis. As a result, integrating over the modulus of the vector $\mathbf{p}_1$ and taking into account the smallness of γ in comparison with E, it is not difficult to obtain the expression

$$K(\mu,k)=(4\pi)^2n_0\sigma(\mu)+\frac{1}{2}\int d\mu_1\frac{\sigma(\mu_1)}{\sigma_0}\frac{1}{1+ikl\mu_2}K(\mu_2,k),$$
$$\mu=\frac{\mathbf{pp}'}{p^2},\quad \mu_1=\frac{\mathbf{pp}_1}{p^2},\quad \mu_2=\frac{\mathbf{p}_1\mathbf{p}'}{p^2},\quad \sigma_0=\frac{1}{2}\int d\mu\,\sigma(\mu)=\frac{\sigma_{tot}}{4\pi}. \tag{9}$$

In place of $K(\mu,k)$ we introduce the new function

$$\tilde{K}(\mu,k)=K(\mu,k)(1+ikl\mu)^{-1}$$

and expand the expression (9) in Legendre polynomials. Then

$$\tilde{K}_\lambda(k)\left(1-\frac{\sigma_\lambda}{\sigma_0}\right)=(4\pi)^2n_0\sigma_\lambda-\frac{ikl}{2\lambda+1}\{\lambda\tilde{K}_{\lambda-1}(k)+(\lambda+1)\tilde{K}_{\lambda+1}(k)\}, \tag{10a}$$

$$\tilde{K}_\lambda(k)=\frac{1}{2}\int d\mu\,\tilde{K}(\mu,k)P_\lambda(\mu),\quad \sigma_\lambda=\frac{1}{2}\int d\mu\,\sigma(\mu)P_\lambda(\mu). \tag{10b}$$

If the scattering angles are small, i.e., μ is close to unity, then the basic contribution to $\tilde{K}(\mu)$ will be made by terms with $\lambda\gg 1$. Therefore, in the principal order in λ^{-1} we can set $\tilde{K}_{\lambda-1}\approx\tilde{K}_{\lambda+1}\approx K_\lambda$. In this case, the set of equations (11) is easily solved and

$$\tilde{K}_\lambda(k)=\frac{(4\pi)^2\sigma_\lambda}{1-\sigma_\lambda/\sigma_0+ikl}. \tag{11}$$

Making use of this formula, and also the expressions (7) and (6a), we obtain the Moliere formula[1]:

$$I(\theta,L)=\frac{S}{2\pi}\int_0^\infty\lambda d\lambda\,J_0(\lambda\theta)\exp\left\{-\left(1-\frac{\sigma_\lambda}{\sigma_0}\right)\frac{L}{l}\right\}, \tag{12a}$$

$$\sigma_\lambda=\frac{1}{2}\int_0^\infty\vartheta d\vartheta\,J_0(\lambda\vartheta)\sigma(\vartheta),\quad \sigma_0=\sigma_{\lambda=0}, \tag{12b}$$

where S is the cross-section area of the beam.

In the derivation of this formula, we have replaced the sum over λ by an integral, and the "partial cross section" σ_λ in the numerator of (11) by σ_0. Moreover, we have used the low-angle asymptotic form of the Legendre polynomial $P_\lambda(\vartheta)\approx J_0(\lambda\vartheta)$. The accuracy of all these approximations is discussed in Appendix I.

Before proceeding to what follows, we shall demonstrate how the Gaussian distribution (1) arises from (12) in the case of a single-scattering cross section which falls off sufficiently rapidly with increase in ϑ. We first assume that $\overline{\vartheta^2}$ defined by Eq. (11) exists. Then, for not very large λ we can expand the Bessel function in the definition (12b) and obtain the following expression for σ_λ:

$$\sigma_\lambda=\sigma_0-\frac{\lambda^2}{8}\int\vartheta d\vartheta\,\vartheta^2\sigma(\vartheta)=\sigma_0\left[1-\frac{(\lambda\vartheta_0)^2}{4}\right],$$
$$\vartheta_0^2=\frac{\sigma_0^{-1}}{2}\int\vartheta d\vartheta\,\vartheta^2\sigma(\vartheta) \tag{13}$$

substituting σ_λ in such form in (12a), we obtain the diffusion formula (1). We note that we can use the expansion (13) in the calculation of the integral in (12a) for the reason that at $L/l\gg 1$ the principal contribution to this integral arises from the region of small $\lambda\vartheta_0$. Actually, $J_0(\lambda\vartheta)$ falls off as $\lambda^{-1/2}$ at $\lambda\vartheta\gg 1$; therefore, σ_λ becomes small in comparison with σ_0 and consequently, the integral over the region $\lambda>\vartheta_0^{-1}$ turns out to be of the order of $e^{-L/l}$. If there exist higher moments of the angular distribution in single scattering for example, $\overline{\vartheta^4}$ and so on, then we can find corrections to the formula (13) of order $(\lambda\vartheta_0)^4$ and so on; [$J_0(\lambda\vartheta)$ is expanded only in even powers of $\lambda\vartheta$]. Now, substituting σ_λ in (12a) and expanding the exponential under the integral sign in $(\lambda\vartheta_0)^4L/l$ and higher powers of $(\lambda\vartheta_0)^2$, we obtain an asymptotic series for $I(\theta,L)$ in $l/L\ll 1$. Here, however, it must be noted that even in the calculation of σ_λ with accuracy to $(\lambda\vartheta_0)^4$, we must also take into account the corrections to the Moliere formula (12) itself, which turn out to be of the same order (see Appendix I).

We now discuss the situation in which there exists only $\overline{\vartheta^2}$ and the higher moments cannot be calculated, i.e., when $\sigma(\vartheta)$ falls off more slowly than ϑ^{-6} (for example, as ϑ^{-5}). Then the definition (12) can be rewritten in the following form:

$$\sigma_\lambda=\sigma_0-\frac{(\lambda\vartheta_0)^2}{4}\sigma_0+\frac{1}{2}\int_0^\infty\vartheta d\vartheta\,\sigma(\vartheta)\left[J_0(\lambda\vartheta)-1+\frac{(\lambda\vartheta)^2}{4}\right]. \tag{14}$$

The basic role in the latter integral is played by the large angles $\vartheta>\vartheta_0$; therefore, we can substitute the asymptotic form of $\sigma(\vartheta)$ in its calculation. It is obvious here that if $\sigma(\vartheta)\propto\vartheta^{-5}$, then a term appears in the expansion of σ_λ that is proportional to $(\lambda\vartheta_0)^3$. It is not difficult to verify, by calculating the integral in formula (12a), that the halfwidth of the distribution $I(\theta,L)$ has in this case a constant term along with the term that depends linearly on $L^{1/2}$.

3. MULTIPLE SCATTERING FROM A POTENTIAL OF FINITE RADIUS

As was noted in the Introduction, a term proportional to ϑ^{-5} appears in the cross section $\sigma(\vartheta)$ at large angles if the scattering takes place from inhomogeneities having sharp boundaries. We now discuss multiple scattering from spherical inhomogeneities. It is intuitively clear that the distribution in multiple scattering from inhomogeneities of arbitrary shape with sharp boundaries, randomly oriented in space, does not differ significantly from the distribution in scattering from spheres. This is shown rigorously in Appendix II.

Thus, let the energy of interaction of a particle with the inhomogeneity be equal to U_0 if the distance from its center is less than r_0, and equal to zero at $r > r_0$, and let $U_0 \ll E$, where $E = p^2$ is the kinetic energy of the particles (we use a system of units with $\hbar = 2m = 1$). Then the eikonal approximation is valid for the scattering amplitude (see, for example, Refs. 1 and 11):

$$f(\vartheta) = -ip \left\{ \rho d\rho J_0(p\rho\vartheta) \left\{ \exp\left[-\frac{i}{2p} \int_{-\infty}^{\infty} dz\, U(\rho^2+z^2) \right] - 1 \right\} \right.$$

$$= -\frac{ir_0}{\vartheta_0} \int_0^1 y dy\, J_0\left(y \frac{\vartheta}{\vartheta_0} \right) \{\exp[-i\alpha(1-y^2)^{1/2}] - 1\}, \qquad (15)$$

where $\alpha = U_0 r_0/p = U_0 p r_0/E$, $\vartheta_0 = (pr_0)^{-1}$ is the characteristic diffraction angle. If $\alpha \ll 1$, then, expanding this expression to second order in α, we obtain for the cross section $\sigma(\vartheta) = |f(\vartheta)|^2$

$$\sigma(\vartheta) = r_0^2 \alpha^2 \frac{\vartheta_0^2}{\vartheta^4} \left\{ \frac{\pi}{2} \left(\frac{\vartheta}{\vartheta_0} \right) J_{3/2}^2 \left(\frac{\vartheta}{\vartheta_0} \right) + \alpha^2 J_2^2 \left(\frac{\vartheta}{\vartheta_0} \right) \right\}. \qquad (16)$$

Here the first term, which corresponds to the Born approximation, falls off as ϑ^{-4} at $\vartheta \gg \vartheta_0$ if we disregard insignificant oscillations, and the second term falls off as ϑ^{-5}. Substituting Eq. (16) in the definition (12b), we have

$$\sigma_\lambda = \sigma_0 + \frac{r_0^2 \alpha^2}{2} \left\{ \frac{(\lambda\vartheta_0)^2}{8} \ln \frac{\lambda\vartheta_0}{4} + \alpha^2 \frac{(\lambda\vartheta_0)^3}{9\pi} \right\}. \qquad (17)$$

The appearance of $\ln \lambda\vartheta_0$ as a factor of $(\lambda\vartheta_0^2)^2$ [see (14)] is connected with the first term in (16). A similar logarithm appears also in the scattering by a Coulomb potential (see Ref. 1), since in this case the cross section also falls off as ϑ^{-4}.

Formula (17), which is obtained at $\alpha \ll 1$, turns out to be valid over a wide range of variation of α. We can convince ourselves of this if we write out the scattering cross section in the form of a series in α, substitute the expression obtained in the definition (12b) and, integrating termwise, collect all the terms in front of each of the powers of α. This procedure was carried out by us in Appendix III, where it is shown that the expression (17) holds for arbitrary α, even at $\alpha \gg 1$, if only $\alpha^2 \ll \vartheta_0^{-1}$, and

$$\sigma_0 = \frac{r_0^2}{2} \left[1 - 2\frac{\sin\alpha}{\alpha} - 2\frac{\cos\alpha - 1}{\alpha^2} \right];$$

$$\sigma_0 = {}^1/_8 r_0^2 \alpha^2 = {}^1/_8 r_0^2 (U_0/E)^2 (pr_0)^2, \qquad \alpha \ll 1, \qquad (18)$$
$$\sigma_0 = {}^1/_2 r_0^2, \qquad \alpha \gg 1.$$

We note that the conditions $\alpha \ll 1$ and $\alpha \gg 1$ correspond to two different physical situations. In the first case, the particles are diffracted by inhomogeneities, and in the second, passing through them, they experience refraction. Therefore, the region $\alpha \ll 1$, i.e., $U_0/E \ll (pr_0)^{-1}$ is known as the diffraction region, while the region $\alpha \gg 1$ is known as the refraction region. It is evident that at $\alpha \gg 1$ the single scattering takes place principally at the characteristic refraction angle $\vartheta_a = U_0/E = \alpha\vartheta_0 \gg \vartheta_0$. In the diffraction region, the scattering cross section is small and therefore the free path length $l = (4\pi n_0 \sigma_0)^{-1}$ is much greater than the distance between the scatterers than their dimensions. In the refraction region, if the density of inhomogeneities n_0 is of the order of r_0^{-3}, the free path length is $l \sim r_0$.

On the other hand, for multiple scattering, satisfaction of the condition $L/l \gg 1$ is required, i.e., at $\alpha \ll 1$, it takes place at substantially greater thicknesses than at $\alpha \gg 1$. In the case of the same ratios L/l the halfwidth of the distribution for diffraction will be significantly smaller than in the case of refraction. Nevertheless, in an arbitrary case, the transmitted beam in multiple scattering should be entirely diffuse. Here we can monitor the multiplicity through the dependence $[I(0, L^{1/2})]^{-1/2}$, which should be linear in the region $(L/l)^{1/2} \gg 1$.

We now turn to the calculation of $I(\theta, L)$ by formulas (12a) and (17). We introduce a new variable of integration

$$y = \lambda\vartheta_0 (BL/l_1)^{1/2}, \qquad l_1 = \{{}^1/_2 \pi r_0^2 \alpha^2 n_0\}^{-1}, \qquad B - \ln B = \ln(4L/l_1);$$

we then obtain from (12a)

$$I(\theta, L) = \frac{S}{2\pi} \frac{1}{\vartheta_0^2} \frac{l_1}{BL} \int_0^{y_{max}} y dy\, J_0\left(\frac{\theta}{\vartheta} \left(\frac{l_1}{BL} \right)^{1/2} y \right)$$

$$\times \exp\left(-\frac{y^2}{4} + \frac{1}{4B} y^2 \ln\frac{y^2}{4} + C \left(\frac{l_1}{l} \right)^{1/2} y^3 \right), \quad C = \frac{4\alpha^2}{9\pi B^{3/2}}. \qquad (19)$$

The value of the upper limit y_{max} in this formula is determined by the region of applicability of the expression (17). Actually, formula (17) is valid at small $\lambda\vartheta_0$. Therefore, the region of integration over λ in the expression (12a) must be divided into two parts, such that in one of them the expansion (17) is applicable, while in the other, we can use for σ_λ its values at large $\lambda\vartheta_0$.

As has already been noted, the region of integration in (12a) in which $\lambda\vartheta_0 > 1$ gives a very small contribution, of the order of $e^{-L/l}$, and this contribution can be neglected. On the other hand, in the region in which $\lambda\vartheta_0 < 1$, the values of λ that are significant turn out to be those at which $y \lesssim 1$, $\lambda\vartheta_0 \ll 1$. In the case of further increase of λ the integrand begins to fall off rapidly and reaches a minimum close to the boundary of the region of applicability of formula (17); it then becomes of the order of $e^{-L/l}$, Here

$$y = {}^1/_2 e^{(B-1)/2} \sim (L/l_1)^{1/2} \gg 1.$$

In the vicinity of the boundary, i.e., in the region in which $\lambda\vartheta_0 \sim 1$, the integrand begins to increase again, but in fact this growth is fictitious and is connected with the approximate character of the formula (17). At $y \sim 1$, the integrand should remain small and its values in this region should match up with the values obtained at $y > 1$, (i.e., $\lambda\vartheta_0 > 1$).

Thus, with accuracy to a quantity of order $\exp(-L/l_1)$ the value of the integral in (19) is practically independent of the upper limit if $y_{max} \lesssim (L/l_1)^{1/2}$ and the point of the minimum of the exponential in (19) can be taken as the value of y_{max}. This question is discussed by us in such detail because at not very large B it is difficult to calculate the integral analytically with great accuracy. For practical purposes, it is best to obtain the distribution $I(\theta, L)$ and its halfwidth from (19) with the help of an electronic computer.

However, if B is sufficiently large, then the integral in formula (19) can be calculated by expanding the integrand in the series in $y^2 B^{-1} \ln(y^2/4)$ and $C(l_1/L)^{1/2} y^3$. Here, limiting ourselves to the first orders in both quantitites, we obtain

$$I(\theta,L)=\frac{S}{2\pi}\frac{1}{\vartheta_0^2}\frac{l_1}{BL}\int_0 y\,dy\,J_0\left(\frac{\theta}{\vartheta_0}\left(\frac{l_1}{BL}\right)^{1/2}y\right)e^{-y^2/4}$$

$$\times\left\{1+\frac{1}{4B}y^2\ln\frac{y^2}{4}+C\left(\frac{l_1}{L}\right)^{1/2}y^3\right\},\qquad(20)$$

$$I(\theta,L)=I_0(\theta,L)+I_1(\theta,L)+I_2(\theta,L).$$

Thus, in the zeroth approximation the distribution of particles by angle is described by the Gaussian curve

$$I_0(\theta,L)=\frac{S}{\pi}\frac{1}{\vartheta_0^2}\frac{l_1}{BL}\exp\left\{-\left[\frac{\theta^2}{\vartheta_0^2}\frac{l_1}{BL}\right]\right\}\qquad(21)$$

with a width varying as $LB(L)$. The equation for b, generally speaking, is solved numerically (the corresponding tables are given in the work of Marion and Zimmerman[12]); however, if B is large, then, with accuracy to several percent, the second iteration of this equation is already satisfactory:

$$B=\ln\frac{4L}{l_1}+\ln\left\{\ln\frac{4L}{l_1}+\ln\ln\frac{4L}{l_1}\right\}.\qquad(22)$$

Thus, it follows from the expression (21) that the slope of the halfwidth of the distribution $\bar{\theta}(L^{1/2})$ depends logarithmically on L in this approximation.

The integrals in the expressions for $I_1(\theta,L)$ and $I_2(\theta,L)$ are easily reduced to tabulated ones. It is shown[3)] that

$$I_1(\theta,L)=\frac{S}{\pi}\frac{l_1}{\vartheta_0^2L}\frac{1}{B^2}\{e^{-x^2}(x^2-1)[\overline{\mathrm{Ei}}(x^2)-\ln x^2]-(1-2e^{-x^2})\}.\qquad(23a)$$

$$I_2(\theta,L)=\frac{S}{\pi^{1/2}}\frac{6C}{\vartheta_0^2B}\left(\frac{l_1}{L}\right)^{3/2}{}_1F_1\left(\frac{5}{2},1,-x^2\right),\quad x^2=\frac{\theta^2}{\vartheta_0^2}\frac{l_1}{BL}.\qquad(23b)$$

Tables of the exponential integral $\overline{\mathrm{Ei}}(x)$ and graphs of the confluent hypergeometric function ${}_1F_1(5/2,1,-x^2)$ can be found in the book of Jahnke, Emde and Lösch.[13]

We calculate the halfwidth of the scattering curve $\bar{\theta}(L^{1/2})$, determined by formulas (21) and (23), iterating the equation

$$I(\bar{\theta},L)={}^1/_2I(0,L)\qquad(24)$$

over B^{-1}. Limiting ourselves to the first iteration approximation, we get

$$\bar{\theta}(L^{1/2})=DL^{1/2}+T,\qquad(25)$$

where $D=D(L)$ and $T=T(L)$ are slowly changing functions of the thickness:

$$D\approx\frac{U_0}{E}r_0(n_0B)^{1/2},\qquad(26a)$$

$$T\approx-\left(\frac{U_0}{E}\right)^2\frac{pr_0}{B}.\qquad(26b)$$

In order to have the right to make further iterations and to consider the higher powers in B^{-1} of the obtained solution, we must take into account in the expression (16) the higher powers of the expansion in the quantity $B^{-1}\ln(y^2/4)$ under the integral sign. We must keep in mind here that the accuracy of the expression (19) itself is limited by the approximate character of the form of (17) and the accuracy of the Moliere formula (12) (see Appendix I). Therefore, inclusion of higher orders of the expansion in $B^{-1}\ln(y^2/4)$ in Eq. (19) does not always make sense.

Finally, we shall show that if the multiple scattering takes place from a random structure of inhomogeneities of arbitrary shape, then we must understand r_0 in the expressions (19)-(26) to be the mean linear dimension of these inhomogeneities, i.e., the quantity r_0 is of the order of the cube root of their volume (see Appendix II).

In conclusion to this section, we return to the question of the intercept, which was discussed above in connection with the results of the researches of Dexter and Beeman[6] and Shil'shteĭn *et al.*[7] It follows from the formula (25) that if we draw the tangent to the curve $\bar{\theta}(L^{1/2})$ at the point with the abscissa $L_1^{1/2}$, then, by virtue of the fact that D and T depend on $L^{1/2}$ (even though weakly), the angle of inclination of this tangent will depend on $L_1^{1/2}$. The tangent, continued to small values of $L^{1/2}$, intercepts on the abscissa an interval $L_0^{1/2}$, the value of which will also depend on $L_1^{1/2}$, and the fact that this intercept is not equal to zero is connected both with the dependence of D on $L^{1/2}$ and with the presence of the quantity T in formula (25). On the other hand, in the case in which we can separate from the intercept the contribution connected with T, we can determine from the experimental data two of the three parameters U_0, r_0, and n_0 of the problem.

It is not difficult to formulate criteria for which the principal contribution to the intercept is connected with T. For this, in addition to the condition that $B\gg1$, it is necessary that the inequality

$$L_0\ll L_1\ll(BL_0)^{1/2}\qquad(27)$$

be satisfied. Unfortunately, it is not possible to draw any final conclusion from the experimental data[6,7] on the applicability to them of the results of the present research, since it is not clear how well the conditions of multiple scattering are satisfied. If these conditions were poorly satisfied, the result can be a non-informative imitation of the intercept, due to trivial corrections (see Appendix I).

4. SCATTERING FROM CRITICAL FLUCTUATIONS

In this section, we discuss multiple, low-angle scattering of particles from systems which experience a second-order phase transition (ferromagnetics near T_c, liquids near the gas-liquid transition, and so on). It is well known that as the Curie temperature is approached, the fluctuations of the order parameter increase strongly in such systems. In this connection, the scattering from the fluctuations increases, while the free path length decreases and, in principle, can become less than the dimensions of the scattering system. As an example, we consider the multiple scattering of slow neutrons from fluctuations of the magnetization in ferromagnets above T_c. (The results obtained will be valid also for other systems if the wavelength of the scattered particle is small in comparison with the characteristic dimensions of the critical fluctuations.)

As has already been shown,[14,15] in the case of critical scattering in iron at $\tau=(T-T_c)T_c^{-1}\sim10^{-4}$ the contribution of double scattering by a sample of thickness of several millimeters reaches tens of percent. Thus, at large sample thicknesses, the scattering in this temperature range can become multiple. Here, as is well known (see, for example, Ref. 15), the transfer of energy in the scattering is small and therefore we can use the formulas introduced above for the description of multiple scattering.

The cross section of single scattering has the well known form[16]

$$\sigma(\mu)=\frac{g}{1+\vartheta_0^2/2-\mu},\quad g=\frac{(R_0\gamma_n)^2Z}{(pa)^2}\frac{T}{T_c},\quad \vartheta_0=\frac{\varkappa}{p},\tag{28}$$

where R_0 is the classical radius of the electron, γ_n is the gyromagnetic ratio of the neutron, $Z\sim 1$, p is the momentum of the neutron, a is a quantity of the order of the lattice constant, $\varkappa=a^{-1}\tau^2$ is the inverse correlation radius, $\nu\approx 2/3$, $\mu=\cos\vartheta$ is the cosine of the scattering angle.

In order to calculate the distribution $I(\theta, L)$ by means of formula (12a), we must first find σ_λ. However, the expression (12b) now is unsuitable for these purposes, since at $\vartheta\gg\vartheta_0$ (but $\vartheta\ll 1$) the value of $\sigma(\vartheta)$, defined by Eq. (28), falls off as ϑ^{-2}. Therefore, σ_λ must be determined from Eq. (10b). Substituting $\sigma(\mu)$ from (28) in (19b) and using the Heumann integral representation for the Legendre functions (see, for example, Ref. 13), we obtain

$$\sigma_\lambda=\frac{g}{2}\int_{-1}^{1}d\mu\frac{P_\lambda(\mu)}{1+\vartheta_0^2/2-\mu}=gQ_\lambda\left(1+\frac{\vartheta_0^2}{2}\right),\tag{29}$$

where Q_λ is a Legendre function of the second kind. Since $\lambda\vartheta_0\ll 1$, we can then write the following approximate equality for $Q_\lambda(1+\vartheta_0^2/2)$:

$$Q_\lambda\left(1+\frac{\vartheta_0^2}{2}\right)\approx-\frac{1}{2}\ln\frac{\vartheta_0^2}{4}+\Psi(1)-\Psi(\lambda+1)\approx-\ln\frac{\lambda\vartheta_0}{2}.\tag{30}$$

Since we are interested in σ_λ at $\lambda\gg 1$, then we replace the logarithmic derivative of the gamma function $\Psi(\lambda)$ by its asymptotic form $\psi(\lambda)\approx\ln\lambda$.

We first calculate the free path length $l=(4\pi n\sigma_0)^{-1}$ entering into formula (12a), where n is the density of magnetic atoms. According to (30), we have

$$l=\left\{4\pi ng\ln\frac{2p}{\varkappa}\right\}^{-1}=\pi\left\{\lambda^2\left(\frac{R_0\gamma_n}{a}\right)^2nZ\ln\frac{2p}{\varkappa}\right\}^{-1}.\tag{31}$$

In this formula, $R_0\gamma_n/a$ is a quantity of the order of 10^{-10} cm; therefore, the free path length for neutrons with a wavelength of the order of 20 Å can become less than one millimeter, i.e., the region of multiple scattering, generally speaking, is completely accessible to experiment.

In order to obtain the distribution in the case of multiple scattering, we substitute σ_λ from formulas (29) and (30) in (12a). Here

$$I(\theta,L)=\frac{S}{2\pi}\int_0^\infty\lambda d\lambda J_0(\lambda\theta)\lambda^{-\nu},\quad \nu=\left|L/l\ln\frac{\vartheta_0}{2}\right|.\tag{32}$$

In order that the scattering be low-angle in this integral, the principal contribution should be made by the region in which $\lambda\gg 1$. This requirement leads to the condition $L/l<2|\ln(\vartheta_0/2)|$.

On the other hand, multiple scattering is realized at $L/l\gg 1$, so that the region of low-angle multiple scattering turns out to be rather narrow:

$$1\ll L/l<2|\ln{}^1/_2\vartheta_0|,$$
$$\pi\left\{\lambda^2\left(\frac{R_0\gamma_n}{a}\right)^2nZ\ln\frac{2p}{\varkappa}\right\}^{-1}\ll L<2\pi\left\{\lambda^2\left(\frac{R_0\gamma_n}{a}\right)^2nZ\right\}^{-1}.\tag{33}$$

We estimate the integral (32) in two limiting cases: $\theta\gg\vartheta_0$ and $\theta\ll\vartheta_0$. Since the expression (32) holds only at $\lambda\vartheta_0<1$, we need to assume for these estimates that the upper limit in the integral is a finite quantity, less than ϑ_0^{-1}. At $\lambda\gg\vartheta_0^{-1}$, the quantity σ_λ decays as $\lambda^{-1/2}$ and the contribution to the integral from the region $\lambda>\vartheta_0^{-1}$ will be exponentially small at large L/l. Furthermore, the region $\lambda<\theta^{-1}$, i.e., the region in which $J_0(\lambda\theta)\sim 1$, plays the principal role in the integral. Thus, the following expression is suitable for the estimates:

$$I(\theta,L)\infty\frac{S}{2\pi}\begin{cases}\theta^{\nu-2}, & \theta\gg\vartheta_0\\ \vartheta_0^{\nu-2}, & \theta\ll\vartheta_0\end{cases}.\tag{34}$$

Here we note a curious circumstance. The cross section of each of the elementary scattering acts depends on ϑ logarithmically (a logarithmic situation arises) and the power-law dependence of the cross section on θ arises as a result of the summation of the principal logarithms.

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APPENDIX I

We now discuss the corrections to formula (12). For this purpose, we write down the expression for the multiple scattering cross section that follows from formulas (6a) and (7):

$$I(\theta,L)=\frac{Sl^2}{(4\pi)^2}\left(1-\frac{l'-l}{l'}\right)\Sigma(2\lambda+1)P_\lambda(\theta)\frac{1}{2\pi}\int dk\,e^{ikL}\frac{(1+ikl\mu)}{(1+ikl)^2}\tilde{K}_\lambda(k),\tag{A.I.1}$$

where the quantity $\tilde{K}_\lambda(k)$ is defined by the set of equations (10a). In order to obtain formula (12) from this expression, we must make the following simplifications:

1) neglect the quantity $(l'-l)/l'\approx\theta^2/2$ in comparison with unity;

2) neglect the kl component in the factors $(1+ikl\mu)$, $(1+ikl)^2$ under the integral sign;

3) decouple the infinite set of equations (10a), assuming $K_{\lambda\pm 1}\approx K_\lambda$;

4) replace the quantity σ_λ in the numerator of the solution obtained for this system by σ_0;

5) replace the summation over λ by integration;

6) use the low-angle asymptotic forms of the Legendre polynomials: $P_\lambda(\vartheta)\approx J_0(\lambda\vartheta)$;

7) neglect unity in the factor $2\lambda+1$.

The corrections to the cross section from the first two approximations, $I^{(1)}(\theta, L)$ and $I^{(2)}(\theta, L)$, are calculated in trivial fashion and have the following form:

$$I^{(1)}(\theta, L) = -\frac{S}{2\pi}\frac{\theta^2}{2} I(\theta, L), \quad I^{(2)} = -\frac{S}{2\pi} l \frac{\partial}{\partial L} I(\theta, L), \tag{A.I.2}$$

where $I(\theta, L)$ is defined by formula (12).

The form of the corrections that arise when account is taken of the other approximations mentioned depends on the character of the interaction. Therefore, we carry out the subsequent calculations for the case of a sufficiently smooth potential, in which $\sigma(\vartheta)$ falls off more rapidly than ϑ^{-4}. In order to take into account the inaccuracy due to the decoupling of the set of equations (10a), we iterate these equations with respect to λ^{-1}; then

$$I^{(3)}(\theta, L) + I^{(4)}(\theta, L) = \frac{S}{2\pi} l \frac{\partial}{\partial L}\left\{-\frac{\vartheta_0^2}{2}\left(\frac{L}{l}\right)^2 + 1\right\} I(\theta, L). \tag{A.I.3}$$

In the derivation of this formula, it turns out to be convenient to take into account at the same time the correction connected with the replacement of σ_λ by σ_0 in the numerator of the forumla (11) for K_λ.

We now consider the effect of the replacement of the summation over a discrete set of values λ by integration (12a). We write down the sum in (A.I.1) in the form of a Sommerfeld–Watson integral (see Ref. 17):

$$I(\theta, L) = -i\frac{Sl^2}{(4\pi)^2}\int \frac{\lambda\, d\lambda}{\cos \pi\lambda} P_{\lambda-1/2}(-\mu) K_{\lambda-1/2}(L)$$

$$= -i\frac{S}{4\pi}\int_c \frac{\lambda\, d\lambda}{\cos \pi\lambda} P_{\lambda-1/2}(-\mu) \exp\left[-\frac{(\lambda\vartheta_0)^2 L}{4l}\right]\left[1 + \left(\frac{\vartheta_0}{4}\right)^2 \frac{L}{l}\right], \tag{A.I.4}$$

where the integration contour c bypasses the real positive semiaxis, and excludes the origin of the coordinates. In the integral written in such a form, it is not possible to detour the contour along the imaginary axis, as is usually done. We shall therefore calculate the integrals along the upper and lower parts of the contour separately. We continue each of the integrals along L into the complex plane: the first of them, $I^+(L)$ on the imaginary negative semiaxis, and the second, $I^-(L)$, on the positive one. After this, we can detour the path of integration over λ in each of the integrals. In $I^+(L)$ along the positive imaginary semiaxis, and in $I^-(L)$ along the negative, i.e., we can write for I^+

$$I^+(\theta, L) = -\frac{iS}{4\pi}\int_0^\infty \frac{\lambda\, d\lambda}{\operatorname{ch} \pi\lambda} P_{i\lambda-1/2}(-\mu) \exp\left[-i\frac{(\lambda\vartheta_0)^2 L}{4l}\right]\left[1 + \left(\frac{\vartheta_0}{4}\right)^2 \frac{L}{l}\right]. \tag{A.I.5}$$

We must now use the identity (see Ref. 17)

$$\frac{P_{\lambda-1/2}(-\mu)}{\cos \pi\lambda} = \frac{1}{\pi}\int_0^\infty d\xi \frac{P_{\lambda-1/2}(1+\xi)}{1+\xi-\mu}. \tag{A.I.6}$$

We then obtain the following formula for $I^+(\theta, L)$:

$$I^+(\theta, L) = -i\frac{S}{2\pi}\int_0^\infty \lambda d\lambda \frac{1}{2\pi}\int_0^\infty d\xi \frac{P_{i\lambda-1/2}(1+\xi)}{1+\xi-\mu}$$

$$\times \exp\left[\frac{i(\lambda\vartheta_0)^2 L}{4l}\right]\left[1 + \left(\frac{\vartheta_0}{4}\right)^2 \frac{L}{l}\right] = -\frac{iS}{2\pi}\int_0^\infty \lambda d\lambda \frac{1}{2\pi}\int_0^\infty d\xi$$

$$\times \frac{\exp\{-i(\lambda\vartheta_0)^2 L/4l\}}{\xi + 2\sin^2(\theta/2)}\left[1 + \left(\frac{\vartheta_0}{4}\right)^2 \frac{L}{l}\right] J_0(\lambda(2\xi)^{1/2}). \tag{A.I.7}$$

Here we have replaced the Legendre function in the principal term by its low-angle asymptotic form.

We immediately note that corrections are of the same order as the corrections which follow from the replacement of formula (A.I.7) by formula (12) for the succeeding terms of the low-angle asymptotic form of the Legendre functions. But these corrections can be calculated by substituting the following expression for $J_0(\lambda\theta)$ in (12):

$$\tfrac{1}{2}\theta^2[J_1(\lambda\theta)/2\lambda - J_2(\lambda\theta) + \tfrac{1}{6}\lambda\theta J_3(\lambda\theta)]. \tag{A.I.8}$$

We continue the transformations in (A.I.7). After integration over λ we have

$$I^+(\theta, L) = \frac{S}{\pi}\frac{l}{\vartheta_0^2 L}\frac{1}{2\pi}\int_0^\infty d\xi \frac{\exp(2i\xi l/\vartheta_0^2 L)}{\xi + 2\sin^2(\theta/2)}\left[1 + \left(\frac{\vartheta_0}{4}\right)^2 \frac{L}{l}\right]$$

$$= \frac{S}{\pi}\frac{l}{\vartheta_0^2 L}\frac{1}{2\pi}\int_0^\infty d\xi_1 \exp(i\xi_1)\left[1 + \left(\frac{\vartheta_0}{4}\right)^2 \frac{L}{l}\right] \Big/ \left(\xi_1 + 4\frac{l}{\vartheta_0^2 L}\sin^2\frac{\theta}{2}\right). \tag{A.I.9}$$

Further, returning to real L, adding $I^-(\theta, L)$ to the obtained expression, and integrating over ξ we obtain

$$I(\theta, L) = \frac{S}{\pi}\frac{l}{\vartheta_0^2 L}\exp\left\{-4\sin^2\frac{\theta}{2}\frac{l}{\vartheta_0^2 L}\right\}\left[1 + \left(\frac{\vartheta_0}{4}\right)^2 \frac{L}{l}\right]. \tag{A.I.10}$$

APPENDIX II

We now show that for inhomogeneities of arbitrary shape, provided only that they are randomly distributed, formula (17) applies, in which r_0 is defined in order of magnitude as the cube root of the volume of the inhomogeneities. We first establish the fact that the principal term $\sigma(\vartheta)$ in this case falls off as ϑ^{-4}. The scattering cross section from the inhomogeneities can be written in the Born approximation in the following form:

$$\sigma_q = \frac{U^2}{(4\pi)^2}\frac{1}{(2p)^2}\int d\mathbf{r}\, d\mathbf{r}'\, e^{-i\mathbf{q}(\mathbf{r}-\mathbf{r}')}, \tag{A.II.1}$$

where the integration over $\mathbf{r}$ and $\mathbf{r}'$ is carried out over the volume of the inhomogeneity. We introduce the variable $\boldsymbol{\zeta} = \mathbf{r} - \mathbf{r}'$; then

$$\sigma_q = \frac{U^2}{(4\pi)^2}\frac{1}{(2p)^2}\int d\zeta\, e^{-iq\zeta}\Delta V(\zeta). \tag{A.II.2}$$

Here $\Delta V(\boldsymbol{\zeta})$ is the volume bounded by the intersection of the surface of the inhomogeneity with the surface of the same inhomogeneity but displaced by the vector $\boldsymbol{\zeta}$.

If the orientations of the inhomogeneities are equally probable, then all the formulas obtained above contain the cross section averaged over these orientations. But the mean $\langle\Delta V(\boldsymbol{\zeta})\rangle$ depends only on $|\boldsymbol{\zeta}|$; therefore, the integral over the angles in the expression for the mean cross section is easily carried out, and by integrating the remaining expression by parts, we obtain

$$\sigma_q = \frac{U^2}{(4\pi)}\left\{-\frac{1}{q^2}\cos q\zeta[\zeta\langle\Delta V(\zeta)\rangle]\Big|_0^{2\zeta_{max}} + \frac{1}{q^3}\sin q\zeta[\zeta\langle\Delta V(\zeta)\rangle]'\Big|_0^{2\zeta_{max}}\right.$$

$$\left.-\frac{1}{q^4}\cos q\zeta[\zeta\langle\Delta V(\zeta)\rangle]''\Big|_0^{2\zeta_{max}}\right\}\ldots \approx -2\frac{U^2}{4\pi}\frac{\langle\Delta V'(0)\rangle}{q^4}, \tag{A.II.3}$$

where $\zeta_{\max}$ is the value of ζ corresponding to the tangent to the surface. [In the case of spheres $\Delta V(\zeta) = 2\pi\{2/3r_0^3 - 1/2r_0^2\zeta + 1/3(\zeta/2)^3\}$.]

The first term in (A.II.3) is equal to zero. The second term is also equal to zero at the lower limit. Further, since we are interested in large $q = p\vartheta$, the upper limit in these expressions contains a rapidly oscillating function, and therefore its contribution to the integral

(12b) will be small. On the other hand, it is obvious that

$$\frac{d}{d\zeta}\langle\Delta V(\zeta)\rangle|_{\zeta=0}\sim\frac{V}{r_0}\sim r_0^2. \qquad (A.II.4)$$

Similarly, starting out from formula (15), we can show that there is a term in second order perturbation theory for the mean scattering cross section from the inhomogeneities that falls off as ϑ^{-5} which also reduces in this case to an expression for σ_λ of the same structure as (17).

APPENDIX III

We expand the expression (15) in powers of α and integrate with respect to y. Then the "partial cross section" can be written down in the form of a double sum:

$$\sigma_\lambda^{(a)}=\frac{r_0^2}{2}\sum_{n,m=1}\frac{(-i\alpha)^m}{m!}\frac{(i\alpha)^n}{n!}\cdot 2^{(n+m)/2}\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m}{2}+1\right)$$
$$\times\int z\,dz\,J_0(\lambda\vartheta_0 z)J_{n/2+1}(z)J_{m/2+1}(z)z^{-(m+n)/2-2}. \qquad (A.III.1)$$

In the obtained expression, with the help of subtractions of the first terms of the expansion of $J_0(\lambda\vartheta_0 z)$ in $\lambda\vartheta_0 z$ from the corresponding terms of the series, we separate a term that is independent of λ (total cross section), and also separate the terms proportional in the principal orders to $\lambda^2\ln\lambda$, λ^2 and λ^3:

$$\sigma_\lambda^{(a)}=\sigma_0^{(a)}+\frac{r_0^2\alpha^2}{2}\int\frac{dz}{z^3}\cdot 2\Gamma^2\left(\frac{3}{2}\right)J_{3/2}^2(z)\,[J_0(\lambda\vartheta_0 z)-1]$$
$$-\frac{r_0^2}{2}\sum_{n,m=2}^{\infty}\frac{(i\alpha)^n}{n!}\frac{(-i\alpha)^m}{m!}\cdot 2^{(n+m)/2}\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{m}{2}+1\right)$$
$$\times\int_0^\infty dz\,z^{-(n+m)/2-1}J_{n/2+1}(z)J_{m/2+1}(z)\frac{(\lambda\vartheta_0)^2}{4}z^2$$
$$-2\,\mathrm{Re}\,\frac{r_0^2}{2}\sum_{n=2}^{\infty}\frac{i\alpha(-i\alpha)^n}{n!}\cdot 2^{(n+1)/2}\Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{3}{2}\right)$$
$$\times\int_0^\infty dz\,z^{-n/2-3/2}J_{n/2+1}(z)J_{3/2}(z)\frac{(\lambda\vartheta_0)^2z^2}{4}$$
$$+\frac{r_0^2}{2}\alpha^4\Gamma(2)\int\frac{dz}{z^3}J_2^2(z)\left[J_0(\lambda\vartheta_0 z)-1+\frac{(\lambda\vartheta_0)^2}{4}z^2\right]+\dots \qquad (A.III.2)$$

It is easy to obtain the value of σ_0 by summing the series (A.III.1) at $\lambda=0$ or starting out directly from the formula (15). It is also not difficult to calculate the second and last terms [see (14) and (17)]. It remains to sum the series in the case of $(\lambda\vartheta_0)^2$ [we neglect the higher powers of $\lambda\vartheta_0$ in the expression (A.III.2)]. After carrying out integration over z we obtain for the contribution to the cross section that is proportional to λ^2

$$\sigma_\lambda^{(2)}=-\frac{r_0^2(\lambda\vartheta_0)^2}{8}\left\{\sum_{n,m=2}^{\infty}2\frac{(i\alpha)^n}{(n-1)!}\frac{(-i\alpha)^m}{(m-1)!}\frac{1}{n+m}\frac{1}{n+m-2}\right.$$
$$\left.+2\,\mathrm{Re}\sum_{n=2}^{\infty}\frac{(i\alpha)(-i\alpha)^n}{(n-1)!}\frac{2}{(n+1)(n-1)}\right\}. \qquad (A.III.3)$$

We first note that terms of order α^4 in this expression cancel each other exactly. This cancellation turns out to take place in all orders of perturbation theory, i.e., $\sigma_\lambda^{(2)}=0$. This can be verified by direct summation of the series in (A.III.3). We shall now show how this is done. We write

$$2(n+m)^{-1}(m+n-2)^{-1}=(n+m-2)^{-1}-(n+m)^{-1},$$
$$2(n+1)^{-1}(n-1)^{-1}=(n-1)^{-1}-(n+1)^{-1}$$

and calculate one of the four sums obtained, for example,

$$F(\alpha)=-\sum_{m=2}^{\infty}\frac{(-i\alpha)^{m+1}}{(m+1)(m-1)!}. \qquad (A.III.4)$$

Differentiating $F(\alpha)$ with respect to α, we get

$$\frac{d}{d\alpha}F(\alpha)=\alpha(e^{i\alpha}-1),\quad 2\,\mathrm{Re}\,F(\alpha)=-2\int d\alpha\,\alpha(1-\cos\alpha). \qquad (A.III.5)$$

Calculating the remaining sums from (A.III.3) in similar fashion and combining them, we obtain $\sigma_\lambda^{(2)}=0$.

[1)] The falloff of the scattering cross section as ϑ^{-4} is characteristic of interactions that are described by Coulomb's law at small distances and was discussed in detail in Ref. 1.

[2)] See also the papers of Kalashnikov and Ryazanov cited in Ref. 8.

[3)] The expression for $I_1(\theta, L)$ was found by Moliere.

[1]N. Mott and H. Massey, Theory of Atomic Collisions, 3d ed. Oxford, 1965.
[2]W. T. Scott, Rev. Mod. Phys. 35, 231 (1963).
[3]Review of Particle Properties, Rev. Mod. Phys. 48, S45 (1976).
[4]C. T. Case and E. L. Battle, Phys. Rev. 169, 201 (1968).
[5]V. L. Highland, Nucl. Instr. Meth. 129, 497 (1975).
[6]D. L. Dexter and W. W. Beeman, Phys. Rev. 76, 1782 (1949).
[7]S. Sh. Shil'shteĭn, V. A. Somenkov, M. Kalanov and I. O. Elyutin, Fiz. Tverd. Tela (Leningrad) 18, 3231 (1976) [Sov. Phys. Solid State 18, 1886 (1976)].
[8]M. I. Ryazanov. Usp. Fiz. Nauk 114, 393 (1974) [Sov. Phys. Uspekhi 17, 815 (1975)].
[9]A. A. Abrikosov, L. P. Gor'kov and I. E. Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Quantum Field Theoretical Methods in Statistical Physics) Fizmatgiz. 1963 [Pergamon, 1965]
[10]S. V. Maleev and B. P. Toperverg, Zh. Eksp. Teor. Fiz. 69, 1440 (1975) [Sov. Phys. JETP 42, 734 (1975)].
[11]L. D. Landau and E. M. Lifshitz, Kvantovaya Mekhanika (Quantum Mechanics) Fizmatgiz, 1963 [Pergamon 1968].
[12]J. B. Marion and B. Z. Zimmerman, Nucl. Instr. Meth. 51, 93 (1967).
[13]E. Jahnke, F. Emde and F. Lösch, Tables of Higher Functions, McGraw, 1960.
[14]W. A. Holm and H. A. Gersh, Phys. Rev. 13B, 4984 (1976).
[15]B. P. Toperverg, V. V. Runov, A. G. Gukasov and A. I. Okorokov, Phys. Lett. 71A, 289 (1979).
[16]M. F. Collins, V. J. Minkiewicz, P. Nathans, L. Passel and G. Shirane Phys. Rev. 179, 417 (1969).
[17]V. de Alvaro and T. Regge, Potential Scattering (Russian translation, Mir, 1966).

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