$$a(t) \approx t + a_1 Gm \ln (mt), \tag{35}$$

and the constant satisfies $a_1 \sim 1$.

As can be seen from (35), the departure of the metric from the Milne metric due to the nonvanishing mass is in this case too negligibly small. Thus, all the models we have found remain self-consistent for massive fields as well.

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Properties of a pion condensate in a magnetic field

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A study is made of the behavior in a magnetic field of a pion condensate that is either homogeneous (with characteristic wave vector $k_0 \approx 0$) or inhomogeneous (the physically interesting case of a pion condensate in a nucleon medium has $k_0 \sim p_F$, where p_F is the nucleon Fermi momentum). An expression is obtained for the spatial distribution of the pion and magnetic fields in a medium with a pion condensate in an external homogeneous magnetic field H. It is shown that the pion condensate is a superconductor of the second type with Ginzburg-Landau parameter $\varkappa \gg 1$. The structure of the mixed state of the system is studied. For a homogeneous condensate, it is the same as for a metallic superconductor of the second type. For an inhomogeneous condensate in the range of variation of the external magnetic field $H_{c1} < H < H'_{c2}$ (where $H_{c1} \sim H_c/\sqrt{\varkappa}$ is the lower critical field, and $H'_{c2} \sim H_c$, where H_c is the thermodynamic critical field) plane layers of the normal phase arise. These layers are parallel to the plane $(\mathbf{k}_0, \mathbf{H})$ $(\mathbf{k}_0 \perp \mathbf{H})$. At values of the magnetic field in the region $H'_{c2} < H < H_{c2}$, where H_{c2} is the upper critical field, the structure of the mixed state for an inhomogeneous condensate is the same as for a homogeneous condensate. It is shown that the value of H_{c2} for an inhomogeneous condensate is finite, irrespective of the amplitude of the condensate field at $H = 0$. The magnetic susceptibility χ of the system is found. It is shown that the qualitative picture of the phenomena that occur does not depend on the actual choice of the model of the pion-nucleon interaction but only on whether the condensate is homogeneous or inhomogeneous.

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INTRODUCTION

The phenomenon of rearrangement of a boson vacuum in strong fields of various types—scalar, electric, nuclear—was first investigated by Migdal in 1971.[1] He showed that in a sufficiently strong field forming for a particle a potential well an instability arises that leads to rearrangement of the ground state of the system, i.e., to a phase transition with the formation of a Bose condensate. The formation of the condensate stabilizes the system and leads to a reduction of its energy. The lightest bosons, for which the instability occurs earlier than for the other particles, are pions. Nuclear matter is a potential well for pions, whose depth increases with increasing density of the nuclear matter. Therefore, at a sufficiently high density n a pion condensate must be formed in a nucleon medium.[1]

In Refs. 2 and 3, and then in Ref. 4, a method was developed for finding the spectrum of pion excitations in nuclear matter with number of neutrons N approximately equal to the number of protons Z, and also in a neutron medium with $Z \ll N$. It was found that in both cases the instability leading to the formation of the pion

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condensate occurs at a density n_c close to the density of normal nuclei: $n_0 \approx 0.17$ F^{-3}. The value of the critical density depends strongly on the phenomenological parameters in the theory of finite Fermi systems, which are inadequately known. Therefore, at the present time it is impossible to say with certainty that a pion condensate is or is not present in nuclei. On the basis of the available experimental data, it can only be concluded that nuclei are close to the phase-transition point. This is indicated by an analysis of all phenomena in which a significant part is played by processes of one-pion exchange.[5,6] On the other hand, the absence of a fairly strong pion condensate field in nuclei is, for example, indicated by analysis of single-nucleon absorption of slow pions by nuclei.[7] Even if there is no pion condensate at the density of nuclear matter $n \approx n_0$, it will arise at a higher density of the medium. In this case, the gain in the energy from the formation of the pion condensate may compensate the loss in energy due to the greater density of the nuclear matter. In such a case, one could have the existence of nuclear systems of a qualitatively new kind—superdense nuclei.[1,8-10] One could attempt to synthesize superdense nuclei (with atomic weight $A \sim 10^2$) in a collision of heavy ions of high energies (of the order of 1 GeV/nucleon). The calculations of Refs. 11 and 12 show that in such collisions nuclear matter may become several times denser than the normal nuclear density. The estimates made indicate that a pion condensate can form during the collision time ($\tau_\pi \sim 10^{-23}$ sec and $\tau_{\rm coll} \gtrsim 10^{-22}$ sec) (Ref. 13), and the resulting heating of the nuclear matter does not apparently lead to its complete disappearance.[14,15] The presence of a strong electric field facilitates the occurrence of a pion condensate. There could therefore also exist supercharged nuclei with "bare" charge $Z \gtrsim 1/e^3$ (e is the electron charge) of the protons that are stable by virtue of pion and electron condensation in the nuclear matter with density $n \sim n_0$ in an electric field.[2,16,17] The existence of superdense (with atomic weight $A \gtrsim 10^3$) and supercharged nuclei could in principle make possible the existence of star nuclei of arbitrary sizes right up to those of neutron stars, these being stable because of the nuclear and electromagnetic forces.[9,10,17]

In nature, dense nuclear matter with density $n \gtrsim n_0$ is probably present in the interior of neutron stars. The presence of a pion condensate significantly softens the equation of state of a neutron star and thus influences important characteristics of neutron stars such as their masses, radii, moments of inertia, etc. For example, the maximal possible mass of a star, moment of inertia, and radius for fixed mass are smaller than in a theory that discounts the possibility of pion condensation. In the presence of a pion condensate, the rate of cooling of a neutron star formed by a supernova explosion is considerably increased.[18] A pion condensate can also have a significant influence on the dynamics of neutron stars. When certain conditions are achieved, some of the star will go over to a superdense state in a hydrodynamic time $\tau \sim 10^{-4}$–10^{-3} sec. In principle, such a phenomenon could provide another mechanism for explaining supernova explosions.[19] A detailed study of these and other consequences of pion condensation in neutron stars and references to the sources can be found in the reviews Refs. 20–22.

Calculations show that in neutron stars there are probably strong magnetic fields of intensity greater than 10^{12} G (on the surface) and 10^{16} G (in the interior). Such strong fields can arise as a result of contraction of the star on the transition of a white dwarf to the state of a neutron star. An estimate of the characteristic magnitude of the magnetic field can be obtained from the condition of conservation of the magnetic flux during contraction ($HR^2 = $ const) due to the freezing of the magnetic field in the matter of the star (because of its high electrical conductivity). In addition, as a result of the motions of the internal layers of the star directly after its formation the magnetic field is twisted and may be increased by several orders of magnitude.[23] We note also that strong magnetic fields could also arise in collisions of heavy ions of high energies (~GeV/nucleon). Indeed, a rough estimate of the characteristic value of the magnetic field h gives

$$hr \sim J/c, \quad r \sim R = r_0 A^{1/3}, \quad J \sim Zec/R,$$

whence

$$h \sim H_\pi (Ze^6)^{1/3}, \quad H_\pi = m_\pi^2 c^3/e\hbar \approx 3.5 \cdot 10^{18} \text{ ΓG}.$$

The following question arises: How do such strong magnetic fields influence the structure of the pion condensate, the presence of which in the system leads to such significant consequences?

The influence of a magnetic field on the properties of a homogeneous (with characteristic wave vector $k_0 \approx 0$) condensate was studied for the first time in Ref. 24. The treatment was in the framework of the σ model without allowance for the pion–nucleon interaction (the nucleons were prescribed as an external background to satisfy the condition of electrical neutrality of the system). It was found that in this case the homogeneous pion condensate has the properties of an ordinary metallic superconductor of the second type.[25,26] However, in a nucleon medium the pion condensate is inhomogeneous (the wave vector of the condensate is $k_0 \sim p_F$, where p_F is the nucleon Fermi momentum). Therefore, the aim of the present paper is to study the behavior of an inhomogeneous pion condensate in nuclear matter in a strong magnetic field.[1)]

In the first section of the present paper, for the example of the simplest model of pion condensation in a scalar field (the S-wave attractive πN interaction could, for example, play the part of the scalar field) and a magnetic field we consider the properties of a homogeneous condensate. The model is close to the model of Harrington and Shepard[24] and the Ginzburg–Landau theory of superconductivity of metals.[25,26] In Sec. 2, we study the properties of an inhomogeneous condensate in a realistic model of pion condensation in nuclear matter and a homogeneous magnetic field. We find the distribution of the pion and magnetic fields in a nuclear medium filling a half-space (Sec. 2.2); we show that the pion condensate has the properties of a type II superconductor (Secs. 2.2 and 2.4); we investigate the structure of the mixed state, and we find the values of the critical fields and the magnetic susceptibility of the system (Secs. 2.3 and 2.5). In the conclusions, we formulate the results and discuss their possible applications.

1. HOMOGENEOUS CONDENSATE. MODEL OF THE SCALAR FIELD

1.1. Lagrangian of the model. Higgs effect

The Lagrangian density of the complex pion field φ interacting with the external scalar field U and the electromagnetic field A_μ has the form (here and in what follows, we use pion units $\hbar = m_\pi = c = 1$ and $e^2/4\pi = 1/137$)

$$\mathscr{L} = |(\partial_\mu - ieA_\mu)\varphi|^2 - (1+U)|\varphi|^2 - \lambda|\varphi|^4/2 - F_{\mu\nu}^2/4, \tag{1.1}$$

where λ is the constant of the pion–pion interaction, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Suppose $-U = U_0 > 1$ in the whole of space. Then as a result of spontaneous breaking of the symmetry of the ground state of the system, the vacuum expectation value of the pion field is nonzero:

$$\langle|\varphi|^2\rangle = a^2 = -\omega_0^2/\lambda = (U_0 - 1)/\lambda > 0. \tag{1.2}$$

We represent the pion field φ in the form $\varphi = \rho\exp(i\chi)$, where ρ and χ are arbitrary real functions of the coordinates and the time, and we go over in the Lagrangian density (1.1) to the fields $\rho' = a - \rho$ and $\chi' = \chi$, which are calculated from the new vacuum $\rho = a, \chi = 0$. After this, the equations for ρ' and $A'_\mu = A_\mu - \partial_\mu\chi'$ in the approximation linear in ρ' and A' take the form

$$\Box\rho' + 2\omega_0^2\rho' = 0, \tag{1.3}$$

$$\Box A_\mu' = j_\mu, \quad j_\mu = \frac{\partial\mathscr{L}}{\partial A_\mu} = 2e^2a^2A_\mu', \quad \partial_\mu A_\mu = 0. \tag{1.3'}$$

As can be seen from (1.3) and (1.3′), charged particles do not remain in the system, the Goldstone bonson (χ') has been absorbed by the gauge transformation, and the photon has acquired the mass $m_{\text{ph}} = \sqrt{2}ea$. Thus, we have the Higgs effect. Because the photon acquires mass in a weak external magnetic field, the system has the property of superconductivity. The Meissner effort arises—the magnetic field is expelled from the volume of the superconductor. The analogy between different models of quantum field theory with spontaneous symmetry breaking of the ground state of the system and the theory of superconductivity is well known.[29]

We now turn to the study of superconductivity of the pion condensate in finite volume. Suppose there is only an external static magnetic field $\mathbf{H} = \operatorname{curl}\mathbf{A}_0$, $\mathbf{A}_0 = (0, Hx, 0)$, and the scalar field

$$U = \begin{cases} -U_0 & \text{for} \quad x<0 \\ 0 & \text{for} \quad x>0 \end{cases}$$

is such that $\omega_0^2 < 0$ in the region $x < 0$ and $\omega_0^2 = 0$ for $x > 0$.

We seek the pion field $\varphi(t)$ in the form $\varphi(t) = \varphi\exp(i\omega t)$. The frequency of the field in a system of sufficiently large volume can be found from the condition of total electrical neutrality of the system. In our case, it follows from the condition $\rho_\pi = \partial\mathscr{L}/\partial\omega = 0$ that $\omega = 0$. In accordance with (1.1), the energy density is

$$E = \omega\frac{\partial\mathscr{L}}{\partial\omega} - \mathscr{L} = |(\nabla - ie\mathbf{A})\varphi|^2 + (1+U)|\varphi|^2 + \lambda|\varphi|^4/2 + \mathbf{h}^2/2, \tag{1.4}$$

where $\mathbf{h} = \operatorname{curl}\mathbf{A}$ is the microscopic magnetic field. From (1.1) and (1.4), we obtain equations for the pion and magnetic fields:

$$(\nabla - ie\mathbf{A})^2\varphi - \omega_0^2\varphi - \lambda|\varphi|^2\varphi = 0; \tag{1.5}$$

$$\Delta\mathbf{A} = -\mathbf{j}, \ \mathbf{j} = \mathbf{j}_p + \mathbf{j}_d, \quad \operatorname{div}\mathbf{A} = 0,$$

$$\mathbf{j}_p = -ie(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*), \quad \mathbf{j}_d = -2e^2\mathbf{A}|\varphi|^2. \tag{1.5'}$$

Equations (1.5) and (1.5′) are completely analogous to the equations of Landau and Ginzburg's phenomenological theory of superconductivity.[25] When there is no spontaneous symmetry breaking of the ground state of the system, the "paramagnetic," $\mathbf{j}_p$, and "diamagnetic," $\mathbf{j}_d$, contributions to the current compensate each other, so that the superconducting part proportional to the vector potential $\mathbf{A}$ does not remain in the current. Therefore, the phenomenon of superconductivity is absent in "normal" systems. This is the case, for example, for an electron gas in the absence of pairing.[30] In our model, the part of the current $\mathbf{j}_p$ does not make a contribution proportional to $\mathbf{A}$ when the "corrected" wave function of the ground state of the system is substituted in it, i.e., the wave function of the ground state has a "hardness" property.

1.2. Penetration of magnetic field into a region with pion condensate. Two characteristic lengths

Suppose the field $\mathbf{H}$ is parallel to the z axis, i.e., along the medium–vacuum interface. As follows from (1.1), φ and φ' are continuous on the boundary $x = 0$. Instead of these boundary conditions, we shall use the simpler condition $\varphi|_{x=0} \approx 0$, which, as is readily seen, is approximately satisfied for sufficiently smooth variation of φ. For the magnetic field, $A_y'(x)|_{x=0} = H$.

As can be seen from (1.5), under the condition that the magnetic field is sufficiently weak, namely, is such that the length over which the trajectory of the motion of a particle in the homogeneous magnetic field is curved, $R = 1/\sqrt{eH}$, satisfies $R^2 \gg l_H l_\varphi$, where l_H and l_φ are, respectively, the characteristic length of variation of the magnetic and pion fields in the medium, the dependence on $\mathbf{A}$ in this equation can be ignored, after which its solution can be readily found:

$$\varphi = a\,\operatorname{th}(-x/\sqrt{2}l_\varphi), \quad x<0, \quad l_\varphi = 1/|\omega_0^2|. \tag{1.6}$$

Under the condition $\varkappa \equiv l_H/l_\varphi \gg 1$, Eq. (1.5′) has the solution

$$A_y(x) = Hl_He^{x/l_H}, \quad l_H = (\lambda/2e^2)^{1/2}|\omega_0^2|^{-1}, \quad x<0. \tag{1.6'}$$

As can be seen from (1.6) and (1.6′), at the critical point ($\omega_0^2 = 0$) of pion condensation $l_H, l_\varphi \to \infty$, but the ratio of these lengths remains constant:

$$\varkappa = (\lambda/2e^2)^{1/2}. \tag{1.7}$$

In the theory of superconductivity of metals, the parameter $\varkappa$ is called the Ginzburg–Landau parameter. Since the constant λ characterizes the hadron interaction, $\lambda \gg e^2$ and $\varkappa \gg 1$. In this case, we have a type II superconductor. It is shown in Sec. 2.4 that for $\varkappa \gg 1$ the surface energy of the interface between the normal and superconducting phases is negative. It is this in fact that is the cause of the distinction between type-I and type-II superconductors. The presence of a negative surface energy in type-II superconductors is responsible for the existence of a mixed state in a certain range of values of the magnetic field.

1.3. Thermodynamic critical field H_c

The critical field H_c is defined as the intensity of the external magnetic field at which the superconducting phase ($|\varphi|=a$, $\overline{\mathbf{B}}=\overline{\mathrm{curl}\,\mathbf{A}}=0$, where $\mathbf{B}$ is the magnetic induction vector, and the bar denotes averaging over the volume of the system) and the normal phase ($|\varphi|=0, \mathbf{B}=\mathbf{H}$) are in equilibrium. We shall study the equilibrium for given volume, magnetic field, and temperature $T=0$. At the same time, the Gibbs free energies of the phases must be equal.

The Gibbs free energy is

$$G=E-\mathbf{MH}, \tag{1.8}$$

where $\mathbf{M}=\mathbf{B}-\mathbf{H}$ is the magnetization and E is the energy. Here and in what follows, we take the volume V equal to unity. In the normal region ($\varphi=0$, $\mathbf{B}=\mathbf{H}$)

$$G=H^2/2, \tag{1.9}$$

and in the superconducting region, in accordance with (1.4) and (1.6)

$$G=-\omega_0^4/2\lambda+H^2. \tag{1.10}$$

Equating (1.9) and (1.10), we obtain

$$H_c=-\omega_0^2/\sqrt{\lambda}. \tag{1.11}$$

For $H<H_c$ and $H>H_c$ superconducting and normal states, respectively, are advantageous. In deriving (1.6), we have used the condition $R^2 \gg l_H l_\varphi$, which now takes the form $H \ll H_c$. Generally speaking, it is only in this case that the ground state of the system is not disturbed by the magnetic field and the complete Meissner effect occurs.

1.4. Structure of the mixed state

The analysis of the structure of the mixed state of the system for a homogeneous pion condensate differs in no way from that in the case of ordinary metallic type-II superconductors, given, for example, in Ref. 26 or the monograph Ref. 31. Therefore, we shall restrict ourselves to a brief description of the physical picture and give some results, which are required for comparison with the corresponding result for an inhomogeneous condensate.

Since the surface energy of the interface between the normal and the superconducting phases is negative, the regions of penetration of the normal phase must have the maximal possible surface. Under conditions of cylindrical symmetry of the system ($\mathbf{H}\parallel z$), such a structure corresponds to parallel, periodically arranged vortex filaments, within which the magnetic field is concentrated and decreases smoothly in space over distances $\sim l_H$, and the pion field is absent and reaches an equilibrium value at distances $\sim l_\varphi$ from each filament. An estimate of the lower critical field H_{c1}, at which the first filament appears, can be readily obtained by comparing the Gibbs energy of one filament of radius $\sim l_\varphi$ (of order $-H_{c1}{}^2 l_H^2$) and the energy of the pion condensate displaced from the volume of the filament (of order $-(\omega_0^4/\lambda)l_\varphi^2$). The calculation gives

$$H_{c1}=\frac{H_c}{\varkappa\sqrt{2}}\ln\varkappa.$$

Taking into account the interaction between neighboring filaments, one can find the magnetic susceptibility of the system and the structure of the lattice formed by the filaments in the plane perpendicular to the direction of the magnetic field. As $H\to H_{c1}$, the magnetic susceptibility behaves as

$$\chi\sim-(H-H_{c1})^{-1}\ln^{-3}(H-H_{c1})\to\infty. \tag{1.12}$$

A triangular lattice of filaments is energetically the most advantageous. With a further increase of the external magnetic field, the number of vortex filaments increases, and the distance d between the neighboring filaments decreases and when $d\sim l_\varphi$ the superconducting state disappears. This occurs when the magnetic field increases to the upper critical field H_{c2}. The calculation gives

$$H_{c2}=H_c\varkappa\sqrt{2}.$$

It can be seen from the above values of the critical fields that $H_{c1}\ll H_c \ll H_{c2}$. At the critical point of pion condensation ($\omega_0^2\to 0$), H_{c1} and H_{c2} vanish. Analysis shows that for values of the magnetic field in the interval $H_{c2}-H\ll H_{c2}$ the regions with pion field $\varphi\neq 0$ are arranged at the points of a triangular lattice formed in the plane perpendicular to the direction of the magnetic field. The value of H_{c2} and the structure of the lattice for a homogeneous condensate are calculated in the same way as is done in Sec. 2.5b for an inhomogeneous condensate. Finally, for the free energy of the system when $H_{c2}-H\ll H_{c2}$ we have

$$F=\frac{B^2}{2}-\frac{(H_{c2}-B)^2}{1+\eta(2\varkappa^2-1)}, \tag{1.13}$$

where $\eta=\overline{\varphi^4}/(\overline{\varphi^2})^2$. The magnetic susceptibility of the system tends to a constant as $H\to H_{c2}$:

$$\chi=[\eta(2\varkappa^2-1)]^{-1}. \tag{1.14}$$

2. INHOMOGENEOUS CONDENSATE. PION CONDENSATION IN A NUCLEON MEDIUM

2.1. The Lagrangian. The equations for the pion and magnetic fields

We proceed from the Lagrangian density

$$\mathscr{L}=D^{-1}(\omega,\mathbf{k}^2)|\varphi|^2-\lambda|\varphi|^4/2+\mathscr{L}_B+\mathscr{L}_e. \tag{2.1}$$

Here

$$D^{-1}(\omega,\mathbf{k}^2)=\omega^2-1-\mathbf{k}^2-\Pi(\omega,\mathbf{k}^2);$$

ω and $\mathbf{k}$ are the frequency and wave vector, $\Pi(\omega,\mathbf{k}^2)$ is the pion polarization operator,[4] λ is the pion–pion coupling constant; and $\mathscr{L}_B$ and $\mathscr{L}_e$ are the Lagrangian densities of the baryon and electron subsystems, respectively. Varying (2.1) with respect to φ, we obtain the equation for the pion field:

$$D^{-1}(\omega,\mathbf{k}^2)\varphi-\lambda|\varphi|^2\varphi=0,\quad \mathbf{k}=-i\nabla. \tag{2.2}$$

This equation in an infinite nucleon medium with baryon density $n>n_c$ has the solution[2)]

$$\varphi=a\exp(i\mathbf{k}_0\mathbf{r}-i\omega_c t),\qquad a=|\omega_0|^2/\sqrt{\lambda},\quad \omega_0^2=-D^{-1}(\omega_c,k_0^2). \tag{2.3}$$

Here, ω_c and $\mathbf{k}_0$ are the frequency and wave vector of the condensate, which are determined from the condition of total electrical neutrality $\partial\mathscr{L}/\partial\omega=0$ and the

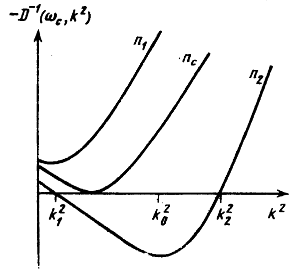

FIG. 1. Dependence of $-D^{-1}(\omega_c, k^2)$ on k^2 for three values of the nucleon density n: n_1, n_c, n_2 ($n_1 < n_c < n_2$); n_c is the density at which a pion condensate arises.

minimum of the energy with respect to k_0, which is identical with the condition that there be no current in the ground state of the system. In Fig. 1, $-D^{-1}(\omega_c, \mathbf{k}^2)$ is plotted against k^2. The critical point of pion condensation ($n = n_c$) is determined by the condition

$$D^{-1}(\omega_c, k_0^2)=0. \quad (2.4)$$

For the energy density of the system, we obtain from (2.1)

$$E=\omega\partial\mathscr{L}/\partial\omega-\mathscr{L}=-\omega_0^4/2\lambda+E_B+E_e, \quad (2.5)$$

where E_B and E_e are the energy densities of the baryon and electron subsystems.

As a rule, we shall be interested in wave numbers k near k_0. It is sufficient to expand $D^{-1}(\omega_c, \mathbf{k}^2)$ near the minimum in $\mathbf{k}^2$ and retain the terms of the expansion that are $\sim(\mathbf{k}^2 - \mathbf{k}_0^2)$. We have

$$D^{-1}(\omega_c, \mathbf{k}^2)=-\omega_0^2-\frac{\gamma}{2}(\mathbf{k}^2-\mathbf{k}_0^2)^2, \quad \gamma=\frac{d^2\Pi}{d(\mathbf{k}^2)^2}\Big|_{\omega=\omega_c, \mathbf{k}=\mathbf{k}_0}. \quad (2.6)$$

Then the Lagrangian density (2.1) in the coordinate representation can be rewritten in a form convenient for calculations ($\omega = \omega_c$):

$$\mathscr{L}=(-\omega_0^2-{}^1/_2\gamma k_0^4)|\varphi|^2+{}^1/_2\gamma[(\nabla-ie\mathbf{A})^2\varphi(\nabla+ie\mathbf{A})^2\varphi^* +(\nabla-ie\mathbf{A})\varphi(\nabla+ie\mathbf{A})^3\varphi^*+(\nabla-ie\mathbf{A})^3\varphi(\nabla+ie\mathbf{A})\varphi^*] +\gamma k_0^2(\nabla-ie\mathbf{A})\varphi(\nabla+ie\mathbf{A})\varphi^*-{}^1/_2\lambda(\varphi^*\varphi)^2-{}^1/_2\mathbf{h}^2+\mathscr{L}_B+\mathscr{L}_e. \quad (2.7)$$

In the Lagrangian density (2.7), we have introduced the interaction of the pion field with the static magnetic field. As can be seen from (2.7), the magnetic field is introduced into the Lagrangian density by the standard gauge substitution and the addition of the free-field Lagrangian density $L_{fr} = -\mathbf{h}^2/2$, where $\mathbf{h} = \operatorname{curl}\mathbf{A}$, as before, is the microscopic magnetic field. The term in the Lagrangian containing the differentiation in fourth order is written down such that $\operatorname{div}\mathbf{j} = 0$. The equations for the pion and magnetic fields have the form

$$D^{-1}(\omega, (\mathbf{k}-e\mathbf{A})^2)\varphi-\lambda|\varphi|^2\varphi=0, \quad (2.8)$$

$$\Delta\mathbf{A}=-\mathbf{j}, \quad \mathbf{j}=\partial\mathscr{L}/\partial\mathbf{A}, \quad \operatorname{div}\mathbf{A}=0. \quad (2.8')$$

Making the substitution $\varphi = \varphi\exp(i\mathbf{k}\cdot\mathbf{r})$, we obtain

$$\mathbf{j}=2e(\mathbf{k}-e\mathbf{A})\left\{1+\frac{\partial\Pi(\omega,(\mathbf{k}-e\mathbf{A})^2)}{\partial(\mathbf{k}-e\mathbf{A})^2}\right\}|\varphi|^2.$$

For simplicity, we have omitted the nucleon and electron parts of the current:

$$\mathbf{j}_B=\partial\mathscr{L}_B/\partial\mathbf{A}, \quad \mathbf{j}_e=\partial\mathscr{L}_e/\partial\mathbf{A}.$$

Evidently, allowance for them will basically lead to an additional additive contribution to the magnetic susceptibility of the system that we obtain below.

We introduce the fields $\rho'(\mathbf{r})$ and $\chi'(\mathbf{r})$, which are measured from the new vacuum, so that

$$\varphi=(a-\rho')\exp(i\mathbf{k}_0\mathbf{r}+i\chi').$$

Then Eqs. (2.8) and (2.8′) up to terms linear in ρ' and χ' are transformed to

$${}^1/_2\gamma\Delta^2\rho'-2\gamma(\mathbf{k}_0\nabla)^2\rho'-2\omega_0^2\rho'=0, \quad (2.9)$$

$$\Delta\mathbf{A}'=-\mathbf{j}, \quad \mathbf{j}=-4\gamma a^2\mathbf{k}_0(\mathbf{A}'\mathbf{k}_0)e^2+2\gamma\mathbf{k}_0 a\Delta\rho' e, \quad \mathbf{k}_0\perp\nabla, \quad (2.9')$$

where $\mathbf{A}' = \mathbf{A} - \nabla\chi'/e$. As can be seen from (2.9) and (2.9′), the Goldstone boson has been absorbed by the gauge transformation, and the photon has become massive. Thus, we have a Higgs effect. By virtue of the Higgs effect, the ground state of the system is superconducting.

2.2. Distribution of pion and magnetic fields in a nuclear medium filling a half-space

Suppose the nucleon medium fills the half-space $x < 0$, so that the density of nucleons is $n_N = n\theta(-x)$, $n = \text{const} > n_c$, where θ is the unit function, and suppose that there is a magnetic field which is homogeneous in the whole of space and is parallel to the medium-vacuum interface, $\mathbf{H}\parallel z$. Note that these simplifying assumptions probably correspond to a realistic situation.

There are several lengths that characterize the variation in space of the pion and magnetic fields: l_φ, which is the length over which there is a variation of the amplitude of the pion field in the nucleon medium: l_H, which is the length over which the magnetic field h changes in the medium; $R = 1/\sqrt{eH}$, which is the radius of curvature of the trajectory of a particle in the homogeneous magnetic field; and $l_k = 1/k_0$, which is the length associated with the variation of the phase of the condensate field. Thus, for an inhomogeneous pion condensate we have the additional length l_k. All the characteristic values of the magnetic fields at which the various changes of the physical properties of the system occur are, as we have already partly seen in Sec. 1, due to the existence of these lengths. Because of the appearance of the new length l_k, it is natural to expect new properties of a pion condensate in a magnetic field associated with the existence of this length. In addition, in the system there is, besides the direction $\mathbf{H}\parallel z$ of the magnetic field, a further distinguished direction $\mathbf{k}_0$, which is due to the one-dimensionality of the pion condensate and along which the superconducting current flows in accordance with (2.9′). As we shall shortly see, this also leads to important features in the behavior of an inhomogeneous condensate. Note that the length l_k, in contrast to l_φ and l_H (see Sec. 1.2), does not become infinite at the critical point of pion condensation (for $\omega_0 = 0$). As can be seen from Fig. 1, the inhomogeneous pion condensate acquires a finite wave vector $k_0 \neq 0$ abruptly at $n = n_c$.

We shall seek the distribution of the pion and magnetic fields under the following simplifying assumptions: 1) $R^2 \gg l_H l_k$. In this case, as can be seen from the expressions (2.8′) and (2.9′), the current can be treated in the approximation linear in A; 2) $R^2 \gg l_H l_\varphi^2/l_k$. Under this condition, the dependence on the magnetic field in Eq. (2.8) for the pion field φ can be ignored; 3) $l_H \gg l_\varphi$.

Then in the expression for the current, the pion field can be assumed equal to its equilibrium value $|\varphi|=a$; 4) $l_\varphi \gg l_k$. In this case, to find the spatial dependence of the amplitude of the pion field, we can assume that the amplitude is a weak function of the coordinates.

In the absence of a magnetic field, the direction $\mathbf{k}_0$ would be in no way distinguished. To find the pion field φ in a finite system, it was assumed in Ref. 32 that $\mathbf{k}_0 \| x$ (in our notation). In the equation for the amplitude of the pion field, the retained terms had derivatives of lowest order [they were of the type $(\mathbf{k}_0\nabla)^2\rho'$, and terms of the type $\Delta^2\rho'$ were ignored]. Then the solution to Eq. (2.8) that decreases at the boundary has the form

$$\varphi=a\,\mathrm{th}\,[(|x|-x_0)/\sqrt{2}\tilde{l}_\varphi]\exp[i(k_0x-\omega_c t)],\quad x<0,\quad \tilde{l}_\varphi=\gamma^{1/2}k_0|\omega_0|^{-1},\quad x_0=0. \tag{2.10}$$

In our case, there is a distinguished direction—the direction of the magnetic field, whose presence destroys the initial symmetry of the system. As can be seen from (2.8′) and (2.9′), $\mathbf{k}_0$ is parallel to $\mathbf{A}$. We shall seek the pion field φ in the form

$$\varphi=af(x)\exp(ik_0y-i\omega_c t), \tag{2.11}$$

where f is a function that varies over the length l_φ. The equation for $f(x)$ can be obtained from Eq. (2.9) for ρ' by the substitution $\rho'\to -af$. It can then be seen from (2.9) that $(\mathbf{k}_0\nabla)f(x)=0$ and (2.10) is not a solution to the problem. We seek a solution to (2.11) perturbively, setting

$$f=1-\psi_1-\psi_2-\ldots,\quad 1\gg\psi_1\gg\psi_2\gg\ldots\ .$$

We obtain

$$\varphi\approx a\{1-C_1\exp(x/l_\varphi)\cos(x/l_\varphi+C_2)-{}^1/_{10}C_1^2\exp(2x/l_\varphi)[1+\sin^2(x/l_\varphi+C_2)]-\ldots\}\exp[i(k_0y-\omega_c t)];\quad l_\varphi=\gamma^{1/4}/|\omega_0|^{1/2};\quad C_1, C_2=\mathrm{const}. \tag{2.12}$$

Note that both (2.10) and (2.12) are obtained under the assumption $\tilde{l}_\varphi$, $l_\varphi \gg l_k$, which is justified in the case of weak supercriticality $n-n_c \ll n_c(|\omega_0|^2\ll 1)$. In what follows, we shall also assume that this condition is satisfied.

The pion field (2.12) decreases over a shorter length than (2.10) $(l_\varphi \ll \tilde{l}_\varphi)$. Therefore, the surface part of the energy of the system is less for a pion field of the form (2.12) than for (2.10). [Of course, in the y direction in a finite system for $H=0$ the characteristic length of variation of the pion field is, as before, the length $\tilde{l}_\varphi$, and in a spherically symmetric nucleus a solution of the type (2.10) with $x_0=(R^2-y^2-z^2)^{1/2}$ is, as before, valid.] To determine the constants C_1 and C_2, we must use the boundary conditions to Eq. (2.9). As follows from the Lagrangian (2.7), the boundary conditions to the equation for the pion field φ (for $\mathbf{A}=0$) consist of continuity of the quantities

$$\alpha_2\nabla\varphi+\alpha_3\nabla\Delta\varphi,\quad \Delta\varphi,\quad \alpha_3\nabla\varphi,\quad \varphi,$$

where

$$\alpha_1=\omega_0^2+{}^1/_2\gamma k_0^4,\quad \alpha_2=\gamma k_0^2,\quad \alpha_3=\gamma/2$$

within the system $(n>n_c)$ and, respectively, 1, −1, 0 outside it $(n=0)$.[3)] It is easy to show that for $l_\varphi \gg l_k$ one can use instead of them the simpler approximate conditions $\varphi(0)=\varphi_x'(0)=0$. Hence, we obtain $C_1\approx\sqrt{2}$, $C_2\approx\pi/4$.

The equation for determining the magnetic field (2.8′) [(2.9′)] with allowance for the boundary condition $A_x'(0)=H$ has the solution

$$A_y(x)=Hl_H\exp(x/l_H),\quad x<0,\quad l_H=(\lambda/e^2\gamma)^{1/2}(2k_0|\omega_0|)^{-1}. \tag{2.13}$$

For $x>0$, we have $\mathbf{A}=\mathbf{A}_0=(0, Hx, 0)$.

As can be seen from (2.12) and (2.13), the Ginzburg-Landau parameter is

$$\varkappa=l_H/l_\varphi=(\lambda/4e^2)^{1/2}\gamma^{-3/4}|\omega_0|^{-1/2}k_0^{-1}\gg 1, \tag{2.14}$$

since $|\omega_0|\ll 1$ for $n-n_c\ll n_c$ and, in addition, $\lambda\gg e^2$. Note that the parameter $\varkappa$ (2.14), in contrast to the corresponding quantity (1.7) for a homogeneous condensate, becomes infinite at the critical point $(n=n_c)$ of pion condensation. In finding (2.13), we have, to simplify the calculations, ignored the second term in the current (2.9′) compared with the first. As can be seen from the result, this is valid for not too small H $(H\gg H_c/\varkappa$, where $H_c=|\omega_0|^2/\sqrt{\lambda})$. Using the obtained solutions (2.12) and (2.13), we can readily verify the applicability of the approximations used in their derivation. All the approximations are satisfied if

$$H_c/\varkappa\ll H\ll H_c. \tag{2.15}$$

As will be seen from what follows, H_c plays the part of a thermodynamic critical field. For $H\ll H_c$, a complete Meissner effect occurs.

2.3. Thermodynamic critical field H_c

We now establish whether it is energetically more advantageous for given external field H for the superconducting $(|\varphi|=a,\ \mathbf{B}=0)$ or normal $(\varphi=0,\ \mathbf{B}=\mathbf{H})$ phase to exist. For this, as in Sec. 1.3, we compare their Gibbs free energies. According to (1.8) and (2.5), in the superconducting phase

$$G=-\omega_0^4/2\lambda+H^2, \tag{2.16}$$

and in the normal phase $G=H^2/2$. Equating these expressions, we find

$$H_c=|\omega_0|^2/\sqrt{\lambda}. \tag{2.17}$$

For $H<H_c$, the superconducting state in the interior of the system is more advantageous than the normal state. Note that if a pion condensate were a superconductor of the first and not the second type, H_c would have the physical meaning of the field that destroys the superconductivity.

2.4. Surface energy of the interface between the superconducting and normal phases

Since $\varkappa\gg 1$, it is natural to expect that the pion condensate is a type II superconductor. Indeed, we shall show that the surface energy of the interface between the normal and superconducting phases is negative.

Suppose the interface is the plane (y, z), and $H\approx H_c$. Because of the presence of the surface, the free energy contains additional surface terms: 1) the negative correction

$$-\int_0^\infty hH_c\,dx\,S\sim H_c^2 l_H;$$

2) the positive correction

$$\int_0^\infty (\lambda\varphi^4/2-|\omega_0|^4/2\lambda)\, dx S\sim H_c^2 l_\varphi$$

(S is the area of the surface). Therefore,

$$\delta G\sim -H_c^2 l_\varphi S\varkappa<0.$$

This result is valid for both a homogeneous and an inhomogeneous pion condensate. The presence of the negative surface energy is responsible for the partial penetration of the magnetic field into the nucleon medium in the range of fields $H_{c1}<H<H_{c2}$.

2.5. Structure of the mixed state

A. *Beginning of the penetration of the magnetic field into the condensate.* The inhomogeneity of the pion condensate leads to qualitative changes in the structure of the mixed state. It can no longer have a filamentary structure, as in the case of a homogeneous condensate, since, besides the direction of the magnetic field **H**, there is the further distinguished direction $\mathbf{k}_0\perp\mathbf{H}$, along which the superconducting current **j** flows. Therefore, the system is a structure of periodically arranged normal plane ($\mathbf{k}_0$, **H** plane) layers of width $\sim l_\varphi$, between which there are superconducting regions. In the theory of the superconductivity of metals, such a structure was considered in Ref. 33, though for natural physical reasons it was found there that this structure is less advantageous than the filamentary structure.[26]

We obtain an estimate of the lower critical field first from simple qualitative considerations: at $H\sim H_{c1}$, the energy per unit area of a normal layer, which is of order $-H^2 l_H$, must be comparable with the energy of the pion condensate that occupied this layer (of order $-\omega_0^4 l_\varphi/\lambda$). This yields

$$H_{c1}\sim H_c/\sqrt{\varkappa}.$$

We now consider the determination of H_c in more detail. For the vector potential **A**, we choose the gauge $\mathbf{A}=A(x)\mathbf{e}_y$, where $\mathbf{e}_y$ is a unit vector in the direction of y, and we take the origin $x=0$ in the middle of the distance d between the centers of the layers. We transform Eq. (2.8′) [or (2.9′)] into an equation for the magnetic field h. Outside a layer, at distances from it much greater than l_φ, this equation has the form

$$l_H^2\frac{d^2h}{dx^2}=h,\quad h\left(\pm\frac{d}{2}\right)=H_n,\quad H_n=\text{const}. \tag{2.18}$$

We shall show below that $H_n=H$. From this we obtain

$$h(x)=H_n\frac{\operatorname{ch}(x/l_H)}{\operatorname{ch}(d/2l_H)}. \tag{2.19}$$

In accordance with (2.7) and (2.9′), the energy density can be written in the form

$$E=-\lambda|\varphi|^4/2+h^2/2+\mathbf{jA}. \tag{2.20}$$

Using (2.12), (2.18), and (2.9′), we obtain from (2.20) an expression for the total energy:

$$E\approx-\frac{|\omega_0|^4}{2\lambda}\left(1-\frac{cl_\varphi}{d}\right)+\frac{2}{d}\int_0^{d/2}\frac{1}{2}\left[h^2+l_H^2\left(\frac{dh}{dx}\right)^2\right]dx; \tag{2.21}$$

$c\approx 3$, the area is $S=1$. The magnetic induction is

$$B=\bar{h}=H_n \operatorname{th}\tau/\tau,\quad \tau=d/2l_H. \tag{2.22}$$

Substituting (2.19) in (2.21) and using (2.22) for the Gibbs free energy of the system (1.8), we obtain the expression

$$G=-\frac{H_c^2}{2}\left(1-\frac{cl_\varphi}{d}\right)+H_n\left(\frac{H_n}{2}-H\right)\frac{\operatorname{th}\tau}{\tau}+H^2. \tag{2.23}$$

From the condition $\partial G/\partial H_n=0$ it follows that $H_n=H$. Finally,

$$G=-\frac{H_c^2}{2}\left(1-\frac{cl_H}{\varkappa d}\right)+H^2\left(1-\frac{\operatorname{th}\tau}{2\tau}\right). \tag{2.24}$$

Minimizing (2.24) with respect to d, we find

$$\operatorname{th}\tau-\tau/\operatorname{ch}^2\tau=cH_c^2/2\varkappa H^2. \tag{2.25}$$

This equation has a solution only for

$$H>H_{c1}\approx(c/2\varkappa)^{1/2}H_c. \tag{2.26}$$

Thus, when $H=H_{c1}$ the formation of the first layer becomes advantageous. We find the magnetic susceptibility of the system under the condition that $H-H_{c1}\ll H_{c1}$, i.e., there are still only a few layers and the distance d between them is much greater than l_H. From (2.25) and (2.26), we obtain

$$\tau\approx\frac{1}{2}\ln\frac{4H^2}{H^2-H_{c1}^2},\quad \tau\gg 1. \tag{2.27}$$

Substituting (2.27) in (2.22), we obtain

$$B\approx-2H/\ln[{}^1/_2(1-H_{c1}/H)], \tag{2.28}$$

$$\chi\approx-1+\frac{4H_{c1}}{(H-H_{c1})\ln^2[{}^1/_2(1-H_{c1}/H)]}. \tag{2.29}$$

As can be seen from the last expression, at the point H_{c1} the magnetic susceptibility is singular. Comparing (1.12) and (2.29), we see that for an inhomogeneous condensate the singularity is barely stronger than in the case of a homogeneous condensate:

$$\chi_{\text{inhom}}/\chi_{\text{hom}}\sim\ln[(H-H_{c1})^{-1}].$$

When there is a further increase in the strength of the magnetic field (above the value H_{c1}), more layers appear, the distance between them decreases, and at $d\sim l_\varphi$ the layer structure is destroyed. An estimate of the corresponding magnetic field strength H_{c2}' at which the layered structure completely disappears can be readily obtained from the expression (2.25). We have

$$H_{c2}'\sim\varkappa H_c. \tag{2.30}$$

Despite the disappearance of the layered structure for $H>H_{c2}'$, there still remain regions with a weak pion field in the system.

B. *Complete destruction of superconductivity.* The magnetic field completely destroys the condensate if the radius of curvature of the trajectory along which a particle moves in the magnetic field becomes of the order of the length $l_k=1/k_0$ associated with the phase of the condensate field. The estimate $R\sim l_k$ gives $H_{c2}\sim k_0^2/e$. We consider the finding of H_{c2} and the behavior of the system in the region of magnetic fields $H_{c2}-H\ll H_{c2}$ in more detail.

Under the condition $H_{c2}-H\ll H_{c2}$, the pion field φ is weak irrespective of the value of its amplitude for $H=0$. Therefore, for its determination we can ignore the nonlinear term $\lambda|\varphi|^2\varphi$ in Eq. (2.8), after which it takes the form

$$D^{-1}(\omega_c,-(\nabla-ie\mathbf{A})^2)\varphi=0. \tag{2.31}$$

In Eq. (2.31), to determine H_{c2} we can set $\mathbf{A}=\mathbf{A}_0$ because the difference $\mathbf{A}-\mathbf{A}_0$ is quadratic in φ [see

(2.8′)]. We shall seek φ as the solution to the equation

$$(\nabla - ie\mathbf{A}_0)^2\varphi = -k^2\varphi, \tag{2.32}$$

where the constant k^2 is determined by the condition

$$D^{-1}(\omega_c, k^2) = 0. \tag{2.33}$$

The solution to Eq. (2.32) is exponentially damped as $x \to \pm\infty$, and therefore it also satisfies all the requirements imposed on the solution of Eq. (2.31). In (2.32), $k^2/2$ plays the role of the energy of a nonrelativistic particle in a homogeneous magnetic field. Therefore[31]

$$k^2 = (2m+1)eH + p_z^2, \quad m = 0, 1, \ldots . \tag{2.34}$$

To find H_{c2}, we must set $m = 0$, $p_z = 0$. Equation (2.33) has two real roots: k_1^2 and k_2^2 (see Fig. 1). We choose the larger. Note that, for example, with increasing density n of the system k_2^2 increases, whereas k_1^2 decreases. Thus, we obtain

$$eH_{c2} = k_2^2 . \tag{2.35}$$

A nontrivial circumstance is the nonvanishing of H_{c2} at the critical point of pion condensation (at $n = n_c$, $\omega_0 = 0$). In other words, the mixed state disappears completely only at a finite value of H_{c2}, irrespective of the amplitude of the pion field for $H = 0$. Such behavior, like the layered structure, is peculiar to the inhomogeneous condensate.

We now turn to the calculation of the free energy and the magnetic susceptibility of the system for $H_{c2} - H \ll H_{c2}$. As before, we shall for simplicity assume $n - n_c \ll n_c$ and use the Lagrangian (2.7) instead of (2.1). The solution to Eq. (2.32) corresponding to $H = H_{c2}(k^2 = k_2^2)$ has the form

$$\varphi = \sum_m e^{iqz} C_m \exp[imky - (x - x_m)^2/2l^2], \qquad x_m = mk/eH_{c2}, \quad l = (eH_{c2})^{-1/2}; \tag{2.36}$$

$q = 0$, since (2.36) corresponds to the smallest eigenvalue of Eq. (2.32). The choice of k will be discussed below.

We require periodicity of (2.36) with respect to x and y. This imposes a restriction on the coefficients C_m:

$$C_m = C_{m+\nu}. \tag{2.37}$$

We find the coefficients C_m and the structure of the lattice formed by the regions with $\varphi \neq 0$ in the plane (x, y) from the condition of vanishing of the variation of the energy. Replacing φ in (2.7) by $\varphi(1 + \varepsilon)$ and equating to zero the terms linear in ε, and using Eq. (2.32) and also an expansion of k_2^2 as $n \to n_c$,

$$k_2^2 \approx k_0^2 + |\omega_0| (\gamma/2)_,^{-1/2}$$

we obtain

$$\lambda\overline{|\varphi|^4} = \overline{\mathbf{A}_1\mathbf{j}^0}, \quad \mathbf{j}^0 \equiv \operatorname{rot}\mathbf{h}_s = \sqrt{2\gamma}|\omega_0|\mathbf{j}_0, \qquad \mathbf{j}_0 = -ie[\varphi^*(\nabla - ie\mathbf{A}_0)\varphi + \text{c.c.}]. \tag{2.38}$$

Here, $\mathbf{A}_1 = \mathbf{A} - \mathbf{A}_0$ is the correction in the first order of perturbation theory to the vector potential of the homogeneous magnetic field $\mathbf{A}_0$ for $H_{c2} - H \ll H_{c2}$, and the bar, as before, denotes averaging over the volume of the system. Integrating (2.38) by parts, and introducing $\mathbf{h}_1 \equiv \operatorname{curl}\mathbf{A}_1$, we obtain

(2.39)

From (2.38) and (2.36), we can obtain the relations

$$j_x^0 = -e\sqrt{2\gamma}|\omega_0|\frac{\partial}{\partial y}|\varphi|^2, \quad j_y^0 = e\sqrt{2\gamma}|\omega_0|\frac{\partial}{\partial x}|\varphi|^2. \tag{2.40}$$

Using the fact that in the absence of a condensate $\overline{|\varphi|^2} = 0$ and $h_s = 0$, we obtain from (2.40) and (2.38)

$$h_s = -e\sqrt{2\gamma}|\omega_0||\varphi|^2. \tag{2.41}$$

Substituting h_s from (2.41) in (2.39) and using the obvious relation

$$h_1 = H - H_{c2} + h_s,$$

and also (2.3), we have

$$\overline{f^2} = \frac{e(2\gamma)^{1/2}}{|\omega_0|\eta}\frac{H_{c2} - H}{1 - 2e^2\gamma|\omega_0|^2/\lambda}, \qquad \eta = \overline{f^4}/(\overline{f^2})^2, \quad |\varphi| = af. \tag{2.42}$$

For the magnetic induction B and the magnetic susceptibility χ of the system it follows from (2.41) and (2.42) that

$$B = H + \bar{h}_s = H - \frac{2e^2\gamma|\omega_0|^2(H_{c2} - H)}{\lambda\eta(1 - 2e^2\gamma|\omega_0|^2/\lambda)} \tag{2.43}$$

$$\chi \approx 2e^2\gamma|\omega_0|^2/\lambda\eta. \tag{2.44}$$

The free energy of the system (2.7), with allowance for (2.8) and (2.43), takes the form

$$F(B) = \frac{B^2}{2} - \frac{2e^2\gamma|\omega_0|^2}{\lambda\eta}(H_{c2} - B)^2, \tag{2.45}$$

i.e., has a negative correction for $B < H_{c2}$.

Note that our derivation is also suitable for the case of a homogeneous condensate. In this case, it is necessary to replace $\sqrt{2\gamma}|\omega_0|$ by unity in the expressions (2.38)–(2.45).

In the theory of the superconductivity of metals the values of η were compared by a numerical calculation for two types of lattice: square (all C_m equal) and triangular ($C_{m+2} = C_m$, $C_1 = iC_0$). It was found that the parameter η is smaller for the triangular lattice.[31] The quantity k in (2.36) can also be found from the minimum of η. Since the solution (2.36) in the case of an inhomogeneous condensate differs from the analogous solution for a homogeneous condensate only by the different value of l, this result remains valid. Thus, the triangular lattice with $\varphi \neq 0$ at its points formed in the plane perpendicular to the direction of the magnetic field is energetically the most advantageous. In contrast to the expression (1.14) obtained for a homogeneous condensate, for an inhomogeneous condensate the magnetic susceptibility of the system vanishes at the critical point of pion condensation (at $n = n_c$). Note that the correction to the free energy of the system is proportional to $|\omega_0|^2$ and vanishes as $n \to n_c$. Therefore, the finiteness of H_{c2} as $n \to n_c$ does not lead to any contradictions.

CONCLUSIONS

Thus, studying the properties of a homogeneous (in a simple model) and inhomogeneous (realistic case of pion condensation in a nucleon medium) pion condensate in an external magnetic field, we have obtained the following results.

1) The pion condensate in the magnetic field has the

properties of a superconductor of the second type with Ginzburg–Landau parameter $\varkappa \gg 1$. We have found that for an inhomogeneous condensate $\varkappa \to \infty$ as $n \to n_c$, whereas for a homogeneous condensate $\varkappa \to$ const as $n \to n_c$.

2) The structure of the mixed state of the system for a homogeneous condensate when $H > H_{c1}$ is the structure of vortex filaments arranged (in the section) at the points of a triangular lattice. For the inhomogeneous condensate there is formed a structure of plane ($\mathbf{k}_0$, $\mathbf{H}$ plane) "normal" layers, this disappearing when the magnetic field H is raised above the value $H'_{c2} \sim H_c$. For $H_{c2} - H \ll H_{c2}$ in both cases the system takes the form of a triangular lattice formed by regions with nonzero pion field in the plane perpendicular to the direction of the magnetic field.

3) For a homogeneous condensate, the critical magnetic fields H_{c1}, H_c, H_{c2} vanish at the critical point of pion condensation. For the inhomogeneous condensate, H_{c1}, H_c, H'_{c2} also vanish as $n \to n_c$, and H_{c2} remains finite irrespective of the initial amplitude of the pion field for $H=0$. This is due to the presence of the additional length $l_k \sim 1/k_0$, which characterizes the change in the phase of the condensate field and arises abruptly at $n = n_c$.

4) We have found the magnetic susceptibility of the system. In the case $H \to H_{c1}$, it is singular. For homogeneous condensate, the singularity is barely smoothed:

$$\chi_{\text{inhom}}/\chi_{\text{hom}} \sim \ln[(H-H_{c1})^{-1}].$$

As $H \to H_{c2}$, the magnetic susceptibility tends to a constant. For an inhomogeneous condensate, it tends to zero as $n \to n_c$, whereas for a homogeneous condensate it remains finite.

Thus, there are significant differences between the behavior of homogeneous and inhomogeneous condensates in the magnetic field. Note that we have nowhere used the explicit form of the dependence of the polarization operator on k^2 and ω. Therefore, the results are fairly general.

The values of the critical fields obtained here have the order of magnitude 10^{16}–10^{19} G. If in the interior regions of neutron stars there are magnetic fields with $H > H_{c2}$, the pion condensate in this case will disappear completely, which renders the equation of state of the star harder. For $H \sim H_c$, calculations of the equation of state must be made with allowance for the structure of the mixed state of the system studied in the present paper. For $H < H_{c1}$, the magnetic field is completely repelled from the region with the pion condensate. As we have already said, knowledge of the equation of state is needed to determine very important characteristics such as the radii of neutron stars, their moments of inertia, masses, and so forth. It is known that the nucleons in a star can have the property of superconductivity,[22] though $H_c^{\text{nucl}} \ll H_c^{\text{pion}}$, i.e., the superconductivity of the pion condensate "survives" at much higher values of the magnetic field; in addition, it exists in a different range of nucleon densities.

It is possible that the superconductivity of the pion condensate influences the dynamics of the transition of a supernova to the state of a neutron star. The point is that the Meissner effect must hinder the penetration of the magnetic field, which is frozen into the matter and increases as a result of the motion of the internal layers of the star, into the interior of the system, and this may lead to a change in the distribution of the magnetic field in the star and to additional dissipative processes. In heavy-ion collisions, strong magnetic fields can probably also arise (see the rough estimate of the magnetic field in the Introduction). In our opinion, a calculation of the magnetic field that arises in this process would be helpful in elucidating the possibility of a condensate's existing in such a system. In nuclei, if there is a pion condensate in them, its superconductivity could lead to a number of effects, for example, it could change the magnetic moments of the nuclei and their moments of inertia.

Finally, we note that in neutron stars the pion condensate could have a domain structure with different directions of the wave vector in different domains. Then besides the considered interaction of the magnetic field with the pion condensate, one could also have partial "flowing" of magnetic lines of force between the domains, and also alignment of domains. However, the possibility of formation of such domains has not hitherto been investigated.

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Note added in proof (November 25, 1979). While the present paper was in press, a paper was published by V. L. Ginzburg (Pis'ma Zh. Eksp. Teor. **30**, 345 (1979) [JETP Lett. **30**, (1979)]), in which he proposes a generalization of the phenomenological Ginzburg–Landau model to the case when the order parameter may have a nonzero wave vector. He expresses the hope that his model could be helpful in describing so-called superdiamagnets. We note that the expression for the free energy of this model is very similar to the expression (2.7) of the present work. Because of this, the qualitative results of our paper are, under certain assumptions, also obtained in the framework of the model proposed by Ginzburg.

[1)]Some of the results of the present paper were published in a shortened form in Ref. 27. Recently, Harrington and Shepard,[28] in the framework of the σ model, also attempted to take into account nucleons in the problem of the superconductivity of a pion condensate. They found the value of H_c, the critical field for destruction of the pion condensate, if it is a type I superconductor. The lower critical field H_{c1}, if the pion condensate is a type II superconductor, was estimated under the assumption of a filamentary vortex structure of the mixed state of the system and, as can be seen from what follows, does not agree with the result of the present paper. In Ref. 28, fields satisfying $H > H_{c1}$ were not considered.

[2)]Here and in what follows, we assume φ_1, $\varphi_2 \neq 0$, $\varphi_3 = 0$. Such a simplified description is accepted (see, for example, Refs. 18 and 24).

[3)]To obtain the boundary conditions to the equation for the pion field, we regard the abrupt boundary as the limit of a smoothed boundary when the transition region tends to zero. We write down an equation for φ with variable coefficients that hold in the whole of space, and the boundary conditions are obtained by integrating it and then going to the limit of a sharp boundary.

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Pendellosung radiation of an electron diffracted in a single crystal

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A quantum-mechanical analysis is made of Pendellosung radiation produced by diffraction of an electron in a single crystal, with account taken of the deviation from the Bragg condition in the final state of the electron. The formulas obtained for the angular distribution duplicate the result of I. M. Frank for the radiation of an oscillator moving in a refracting medium and oriented perpendicular to the velocity.

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1. Frank[1] has developed the theory of the emission of a classical oscillator moving in a refracting medium. He has shown that at $n\beta > 1$ ($\beta = v/c$, v is the oscillator velocity, and n is the refractive index) a number of new phenomena appear, namely the anomalous and complex Doppler effects.

It was previously noted[2] that the Pendellosung effect causes an electron diffracted in a crystal to behave, with respect to emission, like a moving oscillator, i.e., when the electron is diffracted Pendellosung radiation is produced at a frequency and polarization that are determined by the frequency and direction of the oscillations of the diffracted electron. It was shown in Ref. 3 that $n\beta_{\parallel} > 1$ ($\beta_{\parallel} = v\cos\theta_B/c$, θ_B is the Bragg angle), just as in the case considered by Frank,[1] the dependence

