Quantum effects and regular cosmological models

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Allowance for the quantum nature of matter fields and weak gravitational waves on the background of the classical metric of a cosmological model leads to two main effects—vacuum polarization and particle production. The first of these effects can be taken into account qualitatively by the introduction into the Lagrangian density of the gravitational field of corrections of the type $A + BR^2 + CR^2 \ln|R/R_0|$; the second, by the specification of a local rate of production of particles (gravitons) proportional to the square of the scalar curvature R^2. It is shown that simultaneous allowance for the influence of these effects on the evolution of a homogeneous anisotropic metric of the first Bianchi type eliminates the Einstein singularities. Asymptotic approach to the classical model, however, is attained only if additional assumptions are made. In the contraction stage the solution is close to the anisotropic vacuum Kasner solution; in the expansion stage it tends to the isotropic Friedmann solution, in which matter is produced by the gravitational field.

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§1. INTRODUCTION

The presence of a singularity in the general cosmological solution[1] indicates that in the framework of classical general relativity collapse is not halted. However, at $t \sim t_p$ [$t_p = (G\hbar/c^5)^{1/2}$ is the Planck time], quantum effects must become important. A number of authors[2-4] have shown that these can be manifested at the threshold of the essentially quantum region of gravitation through the quantum nature of the matter fields and weak gravitational waves on the background of a classical metric. The most important effects are the production of particles[2] (see also Ref. 5), including gravitons,[6] and vacuum polarization. The influence of particle production on the evolution of the metric was investigated by various approaches in Refs. 7 and 8. The vacuum polarization can be taken into account qualitatively by the introduction into the Lagrangian density of the gravitational field of corrections quadratic in the curvature. Consideration of different quantum fields leads to the following pairs of corrections: R^2 and $R_{ik}R^{ik}$ (or R^2 and $C_{iklm}C^{iklm}$) and certain coefficients of these terms, which, in general, depend logarithmically on the curvature.[3,9] The corresponding field equations are of fourth order in the components g_{ik} of the metric. It was conjectured in Refs. 9 and 10 that the corresponding theory could admit cosmological solutions without singularity. In such a model, the maximal value of the curvature would never be significantly greater than $l_p^{-2} = c^3/G\hbar$, i.e., the bounce (regular replacement of contraction by expansion) would not lead to the essentially quantum region of gravitation.

The investigation of this problem for an isotropic universe with correction $\Delta L_g = BR^2$ to the Lagrangian density showed[11] that a bounce exists for a definite sign of B. However, as $|t| \to \infty$ we have $R^2 \to \infty$, i.e., a singularity of non-Einstein type arises because of the rapid expansion. In this case, we already have $R^2 \gg l_P^{-4}$ at finite t, which renders the semiclassical theory invalid. Choice of the opposite sign of B ensures a power-law asymptotic behavior of the scale factor $r(t)$ as $t \to \infty$, but it does not eliminate the singularity at $t = 0$. Consideration of the other types of correction indicated above for an anisotropic metric leads to an appreciable complication of the field equations[12,13] and results that agree qualitatively with those of Ref. 11. It was shown subsequently[14] that for the isotropic model one can find a class of solutions without Einstein singularities for all t. The possibility of constructing such models is based on two factors. As was shown in Refs. 3, 5, and 9, the coefficients of the quadratic corrections contain factors of the form $\ln|R/R_c|$. In the region $R^2 > R_c^2$ this ensures a bounce, and for $R^2 < R_c^2$ it gives a power-law asymptotic behavior of the scale factor. Such an asymptotic behavior can be obtained in the limit $t \to \infty$ only with allowance for the production of matter, which was simulated in Ref. 14, as in Ref. 8, by the inclusion of bulk viscosity, though in accordance with Ref. 15 the coefficient of viscosity is assumed to depend on R^2.

This problem requires further investigation in two directions. First, in the contraction stage the isotropic solution is unstable, which results in a transition to an anisotropic "vacuum" stage before the onset of quantum effects. Second, it is not known whether rigorous allowance for matter production will halt the growth of R^2 after a bounce and ensure transition to a power-law asymptotic behavior of the background metric.

In the present paper, we therefore consider a homogeneous cosmological model of the first Bianchi type in a modified theory of gravitation with action prescribed by Eq. (1) below and with self-consistent allowance for the production of particles, the local rate for the increment in their number density being $\propto R^2$.

The additional terms in the gravitational equations considered in the present paper are simplified expressions for some of the single-loop quantum corrections in an external classical gravitational field [except for the phenomenological term prescribed below by Eq. (27), which can evidently arise only in the following approximations]. For simplicity, in the present paper we do not consider the other single-loop corrections which have a similar structure but contain the Weyl tensor C_{iklm}. However, these corrections are small for small anisotropy. The region of applicability of the single-loop approximation is restricted by a condition of the form

$$|R_{iklm}R^{iklm}| \ll l_g^{-4};$$

it is not necessary for the metric of space-time to be close to the Minkowski metric.

It is to be expected that the obtained equations can still be used qualitatively at the limit of applicability of the approximation, i.e., when the two sides of the inequality are of the same order. Therefore, the solutions of these equations for which at all t $(-\infty < t < \infty)$

$$|R_{iklm}R^{iklm}| \lesssim l_g^{-4}$$

can be regarded as having physical meaning. We shall show that such solutions exist and are fairly general. On the other hand, if for some solution this last inequality ceases to be satisfied after some $t = t_0$, then for $t > t_0$ our semiclassical theory ceases to be valid and nothing can be said concerning the further evolution of the model.

§2. BASIC EQUATIONS OF THE GRAVITATIONAL FIELD

In accordance with what we have said above, to obtain the field equations we shall consider an action of the form (here and below $c = \hbar = 1$)

$$S_g = -\frac{1}{16\pi G}\int [R + A + BR^2 + CR^2 \ln|Rl_P^2|](-g)^{1/2} d\Omega. \tag{1}$$

For scalar uncharged nonconformal particles in an isotropic universe $C = l_P^2/144\pi$ (see Appendix I). The choice of A and B, which admit a certain freedom, will be specified below.

The metric of space-time is specified in the form

$$ds^2 = dt^2 - a^2(t)\,dx^2 - b^2(t)\,dy^2 - c^2(t)\,dz^2, \tag{2}$$

where the components of the metric tensor can be conveniently written in the form

$$a = rg_1,\quad b = rg_2,\quad c = rg_3,\quad abc = r^3,$$
$$\prod_{\alpha=1}^{3} g_\alpha = 1,\quad \sum_\alpha \frac{\dot g_\alpha}{g_\alpha} = 0,\quad \frac{dg_\alpha}{dt} = \dot g_\alpha. \tag{3}$$

With this notation, for the components R_i^k we have

$$-R_0^0 = \frac{3\ddot r}{r} + \sum_\alpha \left(\frac{\dot g_\alpha}{g_\alpha}\right)^2,\quad -R = 6\left[\frac{\ddot r}{r} + \left(\frac{\dot r}{r}\right)^2\right] + \sum_\alpha \left(\frac{\dot g_\alpha}{g_\alpha}\right)^2,$$
$$-R_\alpha^\alpha = \frac{\ddot r}{r} + 2\left(\frac{\dot r}{r}\right)^2 + 3\frac{\dot r \dot g_\alpha}{r g_\alpha} + \left(\frac{\dot g_\alpha}{g_\alpha}\right)^{\cdot}. \tag{4}$$

In accordance with the value of C given above, we introduce a new characteristic length and dimensionless Ricci tensor:

$$l_g = l_P/12(2\pi)^{1/2} \approx l_P/30,\quad \rho_i^k = l_g^2 R_i^k, \tag{5}$$

by means of which the correction to the Lagrangian density in (1) can be written in the form $l_g^{-2} f(\rho)$, where $f(\rho)$ is a dimensionless function. Variation of (1) gives[14]

$$\rho_i^k - \frac{1}{2}\delta_i^k \rho + \left[\left(\frac{\partial f}{\partial \rho}\right)\rho_i^k - \frac{1}{2}\delta_i^k f + (\delta_i^k g^{lm} - \delta_i^l g^{km})\left(\frac{\partial f}{\partial \rho}\right)_{;l;m}\right] = 8\pi G l_g^2 T_i^k, \tag{6}$$

from which it follows that $T^k_{i;k} = 0$. The system (6) together with (3) and (4) makes it possible to obtain an equation containing only the scale factor $r(t)$ and its derivatives. Forming for this the differences of Eqs. (6) with $\alpha, \beta = 1, 2, 3$, we have

$$(1 + f_\rho)(\rho_\alpha^\alpha - \rho_\beta^\beta) = (f_\rho)_{;\alpha}^{;\alpha} - (f_\rho)_{;\beta}^{;\beta},\quad df/d\rho = f_\rho. \tag{7}$$

It is assumed that T_i^k has isotropic spatial components. With allowance for the relation

$$(f_\rho)_{;\alpha}^{;\alpha} = \Gamma_{0\alpha}^{\alpha}(f_\rho)^{\cdot} = \left(\frac{\dot r}{r} + \frac{\dot g_\alpha}{g_\alpha}\right)(f_\rho)^{\cdot} \tag{8}$$

and the expressions (4) and (7), we obtain

$$-\frac{(f_\rho)^{\cdot}}{1 + f_\rho} = 3\frac{\dot r}{r} + \frac{\dot Q_{\alpha\beta}}{Q_{\alpha\beta}},\quad Q_{\alpha\beta} = \frac{\dot g_\alpha}{g_\alpha} - \frac{\dot g_\beta}{g_\beta}. \tag{9}$$

It follows that $Q_{\alpha\beta} = C_{\alpha\beta}/r^3(1 + f_\rho)$, which gives

$$\frac{\dot g_\alpha}{g_\alpha} = \frac{C_\alpha}{r^3(1 + f_\rho)},\quad \sum_\alpha C_\alpha = 0. \tag{10}$$

The last condition follows from (3). Equations (10) determine the g_α responsible for the anisotropy of the metric (2):

$$g_\alpha = A_\alpha \exp\left\{C_\alpha \int \frac{d\theta}{r^3(1 + f_\rho)}\right\},\quad \prod_\alpha A_\alpha = 1,\quad \theta = \frac{t}{l_g}. \tag{11}$$

Using (11), we can express the components R_i^k (4) solely in terms of $r(t)$:

$$-\rho_0^0 = 3\left(\frac{\ddot r}{r}\right) + \frac{K}{r^6(1 + f_\rho)^2},\quad K = \sum_\alpha C_\alpha^2,$$
$$-\rho = 6[(\ddot r/r) + (\dot r/r)^2] + K/r^6(1 + f_\rho)^2. \tag{12}$$

Here, and in what follows, the dot denotes the derivatives with respect to the dimensionless time θ. The last relation specifies ρ implicitly as a function of r, $\dot r$, and $\ddot r$.

As basis for the investigation we choose (6) with T_0^0, which in the new variables

$$y(r) = (\dot r r)^2,\quad E = 8\pi G l_g^2 \varepsilon r^4/3,$$
$$-\rho = \frac{3}{r^3}\frac{dy}{dr} + \frac{K}{r^6(1 + f_\rho)^2} \tag{13}$$

takes the form

$$y + \left[f_\rho\left(y - \frac{r}{2}\frac{dy}{dr}\right) - \frac{r^4}{6}f + ry\frac{df_\rho}{dr}\right] = E + \frac{K(1 + 2f_\rho)}{6r^2(1 + f_\rho)^2}. \tag{14}$$

Note that the solution of the Einstein equations for the isotropic Friedmann model $(\varepsilon = 3p,\ \rho = K = f = f_\rho = 0)$

$$y = (\dot r r)^2 = E = \text{const},\quad r = (2E^{1/2}\theta)^{1/2} \tag{15}$$

and the vacuum anisotropic Kasner solution $(\varepsilon = p = f = f_\rho = 0)$

$$y = K/6r^2,\quad K = 2/3,\quad r = \theta^{1/3} \tag{16}$$

are singular solutions of (14).

The coefficients C_α introduced in (10) are related to the usually employed parameters of the Kasner solution[16] by

$$C_\alpha + 1/3 = p_\alpha,$$

from which, using

$$\sum_a C_a=0 ,$$

we obtain

$$\sum_a p_a=\sum_a p_a^2=1.$$

As was shown in Sec. 1, for $C=0$ there is an asymptotic tending to the Friedmannian solution only for $B < 0$. In this connection, we represent the coefficient B and the correction to the Lagrangian density (1) in the form

$$B=-{}^1/_2 C\ln[\rho_c(l_g/l_P)^2],\quad f(\rho)=\rho^2\ln(\rho/\rho_c)^2.$$

In the limit $\rho\to 0$, this expression leads to an infrared divergence (see Appendix I for a discussion). To avoid this divergence, we modify $f(\rho)$ to make it analytic at the point $\rho=0$:

$$f_\rho=(\rho^2+\lambda)\ln[(\rho^2+\lambda)/\rho_c^2]. \tag{17}$$

For $|\rho|>\rho_c$, the correction is positive and close to BR^2 with $B>0$, which ensures a bounce, i.e., a regular minimum of r (Ref. 11). In the region $|\rho|<\rho_c$, the sign of $f(\rho)$ changes and gives an asymptotic behavior with decreasing value of ρ^2. In the case $\rho^2\ll\lambda$, the correction is close to a constant value (cosmological term) and can be compensated with necessary accuracy by the choice of A in (1). As was noted in Refs. 2 and 3, the validity of the obtained polarization corrections and the expressions for the production rate of particles in the external gravitational field is restricted to the region $|\rho|\ll 1$. It follows that for the investigation of the neighborhood of the singularity, where the curvature invariants diverge, Eq. (14) is inapplicable. Nevertheless, it is of interest to consider the singularities permitted by this equation.

First, (14) admits singularities of Einstein type, since the Friedmann (15) and Kasner (16) solutions satisfy this equation. Second, (14) admits singularities of non-Einstein type. One of them was indicated earlier for the isotropic case ($K=0$). For $\rho^2>\rho_c^2$, the correction (17) is close to BR^2, which gives $\rho^2\to\infty$ as $r\to\infty$ (Ref. 11). In the anisotropic case, there is a singularity for $\rho=\rho_a=\text{const}$, where ρ_a is determined by the equation $f_\rho(\rho_a)=-1$ [see (12)]. At the same time $\theta\to 0$, $r=\theta^b$, where $0<b<\frac{1}{3}$, and for the coefficients determining the anisotropy of the metric we have

$$g_a=A_a\theta^{\eta_a},\quad \eta_a=[6b(1-2b)/K]^{1/2}C_a. \tag{18}$$

We introduce the notation

$$\varphi=\ln(\rho^2+\lambda)/\rho_*^2,\quad \rho_*=\rho_c e^{-1/2}, \tag{19}$$

by means of which, under the conditions (see Appendix II)

$$|\rho_{max}|\ll 1/\varphi_{max},\quad |\rho|\gg K/r^6, \tag{20}$$

it follows from (14) and (17) that

$$y''y\varphi=\frac{r^2}{6}\left(y-E-\frac{K}{6r^2}\right)+\frac{(y')^2}{4}(\varphi-1)+\frac{2yy'}{r}(\varphi+1)+O(r^{-3}),\quad \frac{dy}{dr}=y'. \tag{21}$$

In this approximation, the equation is close to the isotropic case,[14] and the contribution of the anisotropy can be regarded as admixture of a gas with maximally hard equation of state. Indeed, in accordance with (13),

$$E\sim\varepsilon r^4,\quad \varepsilon=3p\sim n^{4/3}\sim r^{-4},$$

which gives $E=\text{const}$. If $\varepsilon=p\sim n^2\sim r^{-6}$, then $E\sim r^{-2}$. Before we investigate the complete class of solutions, we consider solutions with power-law asymptotic behavior of the background metric.

§3. INVESTIGATION OF OSCILLATION REGIME

As will be shown below, Eq. (21) admits solutions with $\rho\to 0$. To investigate them, we set $\lambda\gg\rho^2$, with

$$\varphi=\ln(\lambda/\rho_*^2)=-\varphi_0,\quad \varphi_0\gg 1. \tag{22}$$

With this notation, Eq. (21) can be written as

$$y\frac{d^2y}{d\xi^2}+\left(y-E-\frac{\tilde K}{\xi}\right)=\frac{1}{4}\left(\frac{dy}{d\xi}\right)^2+\frac{y}{2\xi}\frac{dy}{d\xi},\qquad \xi=r^2/(24\varphi_0)^{1/2},\ \tilde K=K/12(6\varphi_0)^{1/2}. \tag{23}$$

We investigate first small oscillations on the background of the Friedmann solution $y=E=\text{const}$. Setting $y=E+\psi$, where $|\psi|\ll E$, we obtain from (23) for $\xi\gg 1$

$$E\frac{d^2\psi}{d\xi^2}=\frac{E}{2\xi}\frac{d\psi}{d\xi}-\psi,\quad \psi\approx\psi_0\xi^{1/4}\sin\left[\frac{\xi+\xi_0}{E^{1/2}}\right]. \tag{24}$$

As $r\to\infty$, we have $\overline{\rho^2}\sim r^{-3}$ in accordance with (13), which justifies the assumption (22).

Note, however, that $\overline{\rho^2}$ occurs in (14) together with $\varepsilon\sim r^{-4}$. It can be seen from this that with increasing r the contribution of $\overline{\rho^2}$ will be decisive in the dependence $r(\theta)$. Allowance for nonlinearity does not halt the growth in the amplitude of the oscillations (see Sec. 4).

To establish the cause of the growth of the oscillations, we return to the basic equation (6), whose contraction in dimensional form gives

$$-R+l_g^{-2}\left\{\frac{\partial f}{\partial R}R-2f+3\left(\frac{\partial f}{\partial R}\right)^{;i}_{;i}\right\}=8\pi GT_i^i. \tag{25}$$

For the case (22), we have $f=-\varphi_0 l_g^4R^2$, and it then follows from (25) that

$$\frac{d^2R}{d\theta^2}+\frac{3\dot r}{r}\frac{dR}{d\theta}+\frac{R}{6\varphi_0}=0,\quad R\approx R_0r^{-3/2}\sin\left[\frac{\theta+\theta_0}{(6\varphi_0)^{1/2}}\right], \tag{26}$$

which agrees with (24). As was shown above, the oscillations of R are damped too slowly as $r\to\infty$. This is evidently due to the fact that we have ignored terms of the type $(d\ln r/dt)(dR/dt)$ in the expectation value $\langle T_i^k\rangle$ of the energy-momentum tensor of the quantum field in the ground state. To eliminate this shortcoming, we introduce phenomenologically in (6) the term

$$T_i^{k(v)}=D(u_iu^k-\delta_i^k)(u^lR_{;l})(u_{;l}^l). \tag{27}$$

This quantity is written down by analogy with bulk viscosity of matter. It follows from (27) that

$$T_i^{k(v)}u^i=0,\quad T_i^{i(v)}=-9D\frac{r_t}{r}\frac{dR}{dt}. \tag{28}$$

Using the relations we have obtained, we write Eq. (26) in the form

$$\frac{d^2R}{d\theta^2}+\frac{(3+\gamma)\dot r}{r}\frac{dR}{d\theta}+\frac{R}{6\varphi_0}=0,\quad D=-\frac{\gamma l_g^2\varphi_0}{12\pi G}, \tag{29}$$

whose solution is

$$R=R_0r^{-(3+\gamma)/2}\sin[(\theta+\theta_0)/(6\varphi_0)^{1/2}]. \tag{30}$$

The value of γ must be calculated from the exact theory; we shall regard it as an arbitrary parameter. For asymptotic disappearance of the influence of oscillations, it is necessary to take $\gamma \geqslant 1, \overline{R^2} \leqslant r^{-4}$.

The presence of (27) does not change the form of Eq. (21), since $T_0^{0(v)} = 0$. However, we now have $E \neq \text{const}$, since (27) changes the equation of state for the relativistic gas with $T_i^{k(m)}$. We now have

$$T_{i;k}^{k(m)} u^i = -T_{i;k}^{k(v)} u^i, \tag{31}$$

from which, using the relations

$$u^i T_{i;k}^{k(m)} = (\varepsilon + p) u_{;k}^{k} + \varepsilon_{,k} u^k, \quad T_{i;k}^{k(v)} u^i = -T_i^{k(v)} u_{;k}^{i} \tag{32}$$

we find that for $u^\alpha = 0, u^0 u_0 = 1$

$$(r^4 \varepsilon)_t = -9D(r_t r)^2 R_t. \tag{33}$$

Hence, using the adiabaticity of the variation of the background metric and (23), we obtain

$$E = E - 6\varphi_0 \gamma \frac{y}{r^3} \frac{dy}{dr}. \tag{34}$$

Substitution of this expression in (21) gives

$$y \frac{d^2 y}{d\xi^2} = (E - y) + \frac{1}{4}\left(\frac{dy}{d\xi}\right)^2 + \frac{y(1-\gamma)}{2\xi} \frac{dy}{d\xi}, \tag{35}$$

where ξ is determined in (23). Saturation of the amplitude of the solutions of (35) is attained for $\gamma = 1$, which agrees with the investigation of the linear equation. The results derived below are obtained for arbitrary γ, and some of the consequences do not depend on its value.

§4. NONLINEAR OSCILLATIONS

A picture of the oscillations in the nonlinear regime can be obtained by analogy with the investigation of a nonlinear oscillator with slowly varying parameters (see, for example, Ref. 17). Introduction of the new variables

$$y = q^{4/3}, \quad \xi = (\beta \zeta)^{2/(3-\gamma)} \tag{36}$$

reduces (35) to the form

$$\frac{d^2 q}{d\zeta^2} = \frac{1}{3\zeta^\delta}(E q^{-5/3} - q^{-1/3}), \quad \delta = \frac{2(1-\gamma)}{3-\gamma}, \quad \beta = \left(\frac{3-\gamma}{3}\right)^{(3-\gamma)/2}. \tag{37}$$

The "Hamilton function" of this equation is

$$H(q_\zeta, q; \zeta) = \frac{1}{2} q_\zeta^2 + \frac{1}{2\zeta^\delta}(E q^{-2/3} + q^{2/3}) = \mathscr{E}(\zeta), \tag{38}$$

and the momentum is determined by the expression

$$q_\zeta = \pm [2\mathscr{E} - \zeta^{-\delta}(E q^{-2/3} + q^{2/3})]^{1/2}. \tag{39}$$

The extremal values of the coordinate q and the values averaged over the oscillations have the form

$$(q^{2/3})_{\min}^{\max} = \mathscr{E}\zeta^\delta \pm [(\mathscr{E}\zeta^\delta)^2 - E(\zeta)]^{1/2}, \quad \overline{q^{2/3}} = \mathscr{E}(\zeta)\zeta^\delta. \tag{40}$$

For the oscillation period T and the dependence $\mathscr{E}(\zeta)$ we have

$$T(\zeta) = \oint \frac{dq}{q_\zeta} = 3\pi \zeta^{3\delta/2} \mathscr{E}(\zeta), \quad \frac{d\mathscr{E}}{d\zeta} = -\frac{1}{T} \oint \frac{\partial q_\zeta}{\partial \zeta} dq. \tag{41}$$

The solution of the last equation is

$$\mathscr{E}^2(\zeta) = C\zeta^{-3\delta/2} + \zeta^{-3\delta/2} \int_{\zeta_0}^{\zeta} \left(\zeta \frac{dE}{d\zeta} - \frac{\delta}{2} E\right) \zeta^{-(1+\delta/2)} d\zeta. \tag{42}$$

Here, we assume adiabatic variation of $E(\zeta)$ due to the inclusion in E of an anisotropic correction or allowance for particle production (see Sec. 5).

We investigate in more detail oscillations with $\gamma = 0 (\delta = \frac{2}{3})$. From (42),

$$\mathscr{E}(\zeta) = E^{1/2} \zeta^{-2/3} [1 + (\zeta/\zeta_c)^{1/3}]^{1/2}, \tag{43}$$

where we have introduced a new constant ζ_c formed from C. The new constant determines the value of ζ at which the oscillations go over into the essentially nonlinear regime. In the case $\zeta \ll \zeta_c$, we obtain from (40)–(43)

$$\mathscr{E}(\zeta) = E^{1/2} \zeta^{-2/3}, \quad T_\zeta = 3\pi \zeta^{1/3} E^{1/2}, \quad \overline{q^{2/3}} = E^{1/2}. \tag{44}$$

In accordance with (36), this last equation means that the oscillations take place around the Friedmann solution with $p = \varepsilon/3$, whence, using (23) for the background metric, we have $\overline{\xi} = \overline{\zeta^{2/3}} = [E\theta^2/6\varphi_0]^{1/2}$. Substitution of this expression in T_ζ on the transition to the variable θ gives the period $T_\theta = 2\pi(6\varphi_0)^{1/2}$, which agrees with (30). The law of growth of the oscillation amplitude is given in (24).

If $\zeta \gg \zeta_c$, it follows from (40) and (43) that

$$\mathscr{E} = \overline{E}^{1/2} \zeta^{-1/2}, \quad T_\zeta = 3\pi \overline{E}^{1/2} \zeta^{1/2}, \quad \overline{q^{2/3}} = \overline{E}^{1/2} \zeta^{1/6}, \tag{45}$$

where $\overline{E} = E\zeta_c^{-1/3}$. The mean value $\overline{q}$ now increases with increasing ζ. Transition to the variables $\overline{y}$ and $\overline{r}$ in accordance with (23) and (36) gives

$$\overline{y} = \frac{\overline{E}}{(24\varphi_0)^{1/4}} \overline{r}, \quad \overline{r} = \left(\frac{9}{4} \frac{E}{(24\varphi_0)^{1/4}}\right)^{1/3} \theta^{2/3}, \tag{46}$$

i.e., the background metric expands like a Friedmann model with $p \ll \varepsilon$, although the expansion is determined by fluctuations of the curvature. The oscillation period T_θ remains the same as in the linear oscillations. For the extremal values of the amplitude, we obtain $y_{\max} = 4\overline{y}$ and $y_{\min} = E^2/4\overline{y}$ from (40). Thus, at the minimum of each oscillation the expansion almost stops, and at the points of maximum the rate of expansion is twice the rate averaged over a period.

We now consider oscillations for $\gamma = 1 (\delta = 0)$. From (40)–(42),

$$\mathscr{E}^2 = C + E, \quad (y^{1/2})_{\min}^{\max} = (C+E)^{1/2} \pm C^{1/2}, \quad T_\zeta = 3\pi (C+E)^{1/2}. \tag{47}$$

In the case $\gamma > 1, \delta < 0$, and $E = \text{const}$

$$\mathscr{E}^2 = C\zeta^{3|\delta|/2} + E\zeta^{2|\delta|}, \quad (y^{1/2})_{\min}^{\max} = (C\zeta^{-|\delta|/2} + E)^{1/2} \pm (C\zeta^{-|\delta|/2})^{1/2}, \tag{48}$$

i.e., the oscillations tend to the Friedmann solution $y = E$.

Note also that the term (27) leads to the absence of other power singularities in r and g_α at finite t apart from the Kasner singularity and (only for $K = 0$) Friedmann singularity with $r \propto (|t|)^{1/2}$. As $r \to 0$ in the Kasner asymptotic behavior, the value of $|R|$ is bounded.

§5. INVESTIGATION OF SOLUTIONS WITH REGULAR MINIMUM

We investigate qualitatively the solutions in the region of the minimum, where $y_0 = (\dot{r}_0 r_0)^2 = 0$. The basic equation (21) gives, with allowance for the conditions $K/r_0^8 \ll |\rho| \ll 1$ in the lowest approximation in $(r - r_0)/r_0$,

$$y''y\varphi=\frac{r_0^2}{6}\left(y-E-\frac{K}{6r_0^2}\right)+\frac{(y')^2}{4}(\varphi-1). \tag{49}$$

The indicated inequalities lead to an equation in which the highest derivative has a small parameter; to investigate it, we introduce the deformed coordinate

$$x=(r-r_0)r_0/6^{1/2}\ll r_0^2,\quad p=dy/dx,\quad p_*=\rho_*r_0^2 6^{1/2}/3. \tag{50}$$

Series expansion of the solution (49) in the neighborhood of $x=0$ gives

$$y(x)=a_0x+d_0x^{3/2}+O(x^2),\ a_0=\left(\frac{4\tilde{E}}{\varphi_0-1}\right)^{1/2},\quad \tilde{E}=E+K/6r_0^2. \tag{51}$$

This last is possible if $\varphi(r_0)>1$, which in accordance with (19) holds for $\rho_0^2>e\rho_*^2$ [the sign of the correction (17) is positive]. The general solution of Eqs. (6) with T_i^k from (27) and $T_i^{k(m)}$ in the class of metrics (2) depends on four physical constants. The other four constants determine the scales with respect to the three spatial coordinates and the t origin. The regular solution (51) depends on the four physical constants d_0, C_1, C_2, E (or r_0) and, thus, is the general solution.

Equation (49) with (50) takes the form

$$py\varphi\frac{dp}{dy}=(y-\tilde{E})+\frac{p^2}{4}(\varphi-1),\quad \varphi=\ln\left(\frac{p}{p_*}\right)^2, \tag{52}$$

whose general solution is

$$\Psi(p)={}^1/_4p^2[\ln(p/p_*)^2-1]=y+\tilde{E}-2Cy^{1/2}, \tag{53}$$

where C is a constant of integration. The dependence $y(x)$ is specified in the parametric form

$$y(p)=[C\pm(C^2+\Psi(p)-\tilde{E})^{1/2}]^2,$$
$$x=\frac{1}{2}\int_{p_0}^{p}\left[1\pm\frac{C}{(C^2+\Psi(p)-\tilde{E})^{1/2}}\right]\ln\left(\frac{p}{p_*}\right)^2dp. \tag{54}$$

It follows from (54) that our condition $x\ll r_0^2$ is satisfied if the inequalities (20) are true. The phase diagram for (52) is shown in Fig. 1. The point of regular minimum $\Psi(\rho_0)=\tilde{E}$ is a node. Comparison of the expansions (51) and (53) near $p=p_0$ gives

$$a_0=p_0,\quad C=\pm{}^3/_8\varphi_0p_0^{1/2}d_0,$$

i.e., for given C two branches of the solution leave the point of minimum. For the solutions of type 2 and 3, increase in p leads to growth of x until the condition $x\ll r_0^2$ is violated. Their further evolution will be indicated below.

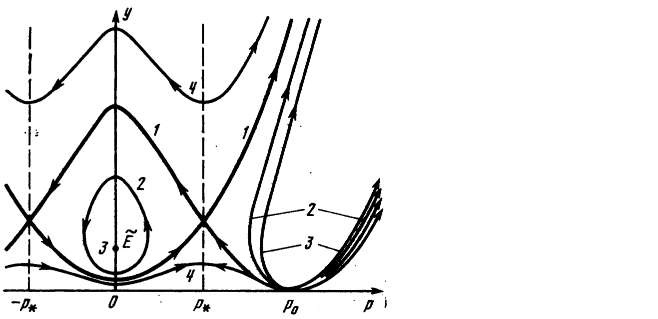

FIG. 1. Phase trajectories of Eq. (52): 1) separatrix with $C_s^2=\tilde{E}+p_*^2/4$; 2) solutions for which $\tilde{E}<C_2^2<C_s^2$; 3) solution with $C_3^2=\tilde{E}$; 4) solutions with $C_4^2>C_s^2$. The direction of the arrows corresponds to $\dot{r}>0$ and is reversed when the sign of the derivative changes.

For $p<p_*$, the solutions have an oscillatory regime of type 2. They correspond to the nonlinear oscillations investigated in detail in Secs. 3 and 4. The transition of the model to the oscillation regime is the necessary condition of asymptotic transition to the classical cosmological models. For transition from the region of the regular minimum to the oscillation region it is necessary that there be an intersection of the dashed line $p=p_*$ $(\varphi(p_*)=0)$, i.e., vanishing of the factor multiplying the highest derivative in (21) and (49). In accordance with Fig. 1, this is possible for curves of the type 1 and 4. The separatrices 1, which form a saddle in the neighborhood of p_*, are the only solutions that can be analytically continued into the oscillation region. Allowance for the small corrections to (49) may lead to a reduction of C and pulling of the given solution into the oscillation region. As can be seen from the direction of the arrows in Fig. 1, the solutions of type 4 lead to a singularity. To establish the nature of this singularity, we expand the solution in a series near the point $x=x_*, y=y_*, p=p_*$:

$$y(x)=y_*+p_*(\Delta x)+d_*(\Delta x)^{3/2}+O[(\Delta x)^2], \tag{55}$$

where the value of the parameter d_* is determined by the choice of C in the solution (53):

$$d_*=\pm\frac{2}{3}\left(2p_*\left|\frac{|C|}{y_*^{1/2}}-1\right|\right)^{1/2},$$
$$|C|=\frac{1}{2y_*^{1/2}}\left(y_*+\tilde{E}+\frac{p_*^2}{4}\right). \tag{56}$$

As is indicated in Fig. 1, the separatrix corresponds to $y_*=\tilde{E}+p_*^2/4$, $|C|\,y_*^{1/2}$, $d_*=0$. For trajectories of the type 4, $d_*\neq 0$, which leads to the dependence

$$r(t)=r_{reg}(t)+\text{const}\,|t-t_*|^{5/2}, \tag{57}$$

where const ~ d_*, and invariants of the type $R_{ik;l}R^{ik;l}\sim(t-t_*)^{-1}$ diverge, although all the invariants without derivatives are finite. This is the singularity mentioned above.

§6. SOLUTIONS OF TYPES 2 AND 3 AND THE INFLUENCE OF PARTICLE PRODUCTION

Numerical analysis of these solutions reveals an increase in $|\rho|$ (Ref. 14). The explicit form of the asymptotic behavior can be obtained from the original equation (21) (see below):

$$y\sim r^4\exp\ln^{1/2}r,\ r\gg r_0. \tag{58}$$

The corresponding law of increase of $|\rho|$ follows from (13):

$$-\rho(r)\sim[\exp\ln^{1/2}r](1+O(1/\ln^{1/2}r)). \tag{59}$$

However, this asymptotic behavior cannot be continued to $r=\infty$. In accordance with (59), we arrive sooner or later at $\rho=\rho_a, f_\rho(\rho_a)=-1$ [see (18)], at which the anisotropy of the metric again becomes important.

We now investigate the influence of particle production on the expansion stage. We note first that the fulfillment of the inequalities given in Sec. 5 has the consequence that the contribution of anisotropy is small in the complete region in which the effects of vacuum polarization and particle production are important. This justifies the calculation of the corrections in the Lagrangian of the field and the particle production in accordance with the isotropic model. In accordance with Ref. 15, the rate of increase in the number density of ultrarelativistic particles can be written in the form

$$\frac{d(r^3 n)}{r^3 dt}=\Pi R^2. \tag{60}$$

We restrict ourselves to spontaneous production and do not take into account induced production, since the population numbers n_k in the modes in which significant production [with frequency $\omega \sim (\ln r)_t$] occurs are of order unity. The coefficient is determined by the particle species; for nonconformal scalar particles $\Pi = 1/576\pi$, and for gravitons Π is twice as large.

From this there follows approximately an expression for the rate of increase of the energy density additional to (33) due to the production of real particles:

$$\frac{d(\varepsilon r^4)}{r^4 dt}=\Pi_1\left|\frac{\dot r}{r}\right| R^2, \tag{61}$$

where $\Pi_1 \sim \Pi$. Using (13), we write the last expression in the form ($\dot r > 0$)

$$\frac{dE}{dr}=\bar\Pi \rho^2 r^3, \qquad \bar\Pi=\frac{8\pi G}{3l_g^2}\Pi_1=768\pi^2\Pi_1. \tag{62}$$

Allowance for (61) changes the expression for $T^\alpha_\alpha = -p$, where p is now a functional of $\varepsilon(t)$ and t. For its determination, in accordance with the rigorous equation $\langle T^k_i\rangle_{;k} = 0$ for the expectation value of the energy-momentum tensor of the quantized field, we require $T^{k(m)}_{i;k} = 0$ [see (32)]. Hence, for the determination of p we have

$$\frac{d(r^4\varepsilon)}{r^4 dt}=(\varepsilon-3p)\frac{d\ln r}{dt},$$

where the left-hand side is determined by Eq. (61).

We note first that the system of equations (21) and (62) [because the asymptotic behavior (59) is independent of γ, we restrict ourselves to the case $\gamma = 0$] for $\bar\Pi > \frac{2}{3}$ and with neglect of the anisotropic correction admits the solution

$$y=mr^4, \qquad \rho_m=-\frac{3}{r^3}\frac{dy}{dr}=-12m,$$
$$\varphi_m=\ln(144m^2/\rho_*^2), \qquad E=nr^4, \qquad n=36m^2\bar\Pi. \tag{63}$$

The value of the parameter m is determined by substituting (63) in (21); $m^{-1} = 12(3\bar\Pi - 2)$. Note that this relation does not contain φ_m. The condition of applicability of the semiclassical theory (20) requires $m \ll 1$. Setting $m \approx 0.01$, we obtain $\bar\Pi \approx 3.44$ and $\Pi_1 \approx 4.5\times 10^{-4}$. This last result agrees with the order of magnitude of this coefficient found above.

We now investigate the stability of the solutions (63), setting for this

$$y=mr^4(1+h), \qquad \rho=-12m[1+(hr^4)'/4r^3], \tag{64}$$

where $|h| \ll 1$. Using (64), for the perturbations δE we find from (62)

$$\delta E=72m^2\bar\Pi r^4 h=Fmr^4 h. \tag{65}$$

It follows from (21) for h that

$$\varphi_m r^2 h''=\frac{h}{6m}(1-F)+8h-4\varphi_m r h'. \tag{66}$$

We calculate first the sign of the parameter in the round brackets. From (63) and (65),

$$1-F=-(2+3\bar\Pi)/(3\bar\Pi-2)=-1/\alpha<0. \tag{67}$$

Setting further $|\varphi_m| \sim 1$ and using the condition $m \ll 1$, we obtain from (66)

$$(\operatorname{sign}\varphi_m)\frac{d^2h}{dx^2}=-h+O(\mu), \qquad x=\frac{\ln r}{\mu}, \qquad \mu^2=6m|\varphi_m|\alpha. \tag{68}$$

The solution of this equation is

$$h=\begin{cases} h_0\sin[(\ln r)/\mu+\psi_0], & \varphi_m>0,\\ h_0\exp[(\ln r)/\mu]=h_0 r^{1/\mu}, & \varphi_m<0.\end{cases} \tag{69}$$

Thus, the solution (63) is stable for $(12m)^2 > \rho_*^2$ and unstable otherwise.

We now investigate the branches of the solutions of (21) that leave the point of the regular minimum $y(r_0) = 0$ with increasing value of p (see Fig. 1). We choose the only parameter that is as yet undetermined, ρ_* (or p_*), on the basis of the condition $|\rho_*| > 12m(\varphi_m < 0)$. To attain a regular minimum at r_0, it is necessary in accordance with Sec. 5 to have $\rho_0^2 > e\rho_*^2$, i.e., the derivative $y'(r)$ at this point is greater than the corresponding value in the solution (63). This then leads to a rapid increase of $y(r)$ compared with (63) but to an even more rapid increase in $E(r)$. This can be seen by substituting $y = mr^{4+\nu}$ with $\nu > 0$ in (62). In this case $E = \text{const}\ r^{4+2\nu}$, and this last leads to a change in sign of the round bracket in (21) and a decrease of $y'(r)$, and with it ρ^2. One can also show that if $\bar\Pi > \frac{2}{3}$ then it is impossible to have a regime of growth $y = f(r)r^4$, where $f(r)$ is a monotonically increasing function that increases slower than any power of r. We note for completeness that when $\bar\Pi < \frac{2}{3}$ such a growth regime is possible, and

$$f(r)=\text{const}\exp[[(1-3\bar\Pi/2)\ln r]^{1/2}], \qquad \bar\Pi<2/3,$$
$$f(r)=\frac{\ln r}{48\ln(\ln r)}, \qquad \bar\Pi=\frac{2}{3}. \tag{70}$$

Indeed, we seek a solution of (21) in the form $y = r^4 f(r)$, where $f(r)$ is a slowly varying function of r, $|rf'| \ll f$, $f > 0$; in this case, $\rho \approx -12f$. Substituting $y(r)$ in (21), in which we take into account the change in E due to the particle production (62), and ignoring the terms containing f'', we obtain the equation

$$rff'\ln(12f/\rho_*)=\tfrac{1}{48}f-\tfrac{1}{4}f^2(3\bar\Pi-2). \tag{71}$$

For $\bar\Pi > \frac{2}{3}$, the particular solution has the form (63), and if $\bar\Pi < \frac{2}{3}$, the asymptotic behavior of the general solution in the limit $r\to\infty, f\to\infty$ is described by (70). Thus, when $\bar\Pi > \frac{2}{3}$ allowance for the matter production (62) eliminates the asymptotic behavior (59) leading to violation of the conditions (20).

In the case $|\rho_*| < 12m$, $\varphi_m > 0$ this decrease of ρ would lead to a tending of $y(r)$ to the stable solution (63), i.e., to a stationary cosmological model in which the particle production (61) plays the part of the hypothetical C field. For this choice of ρ_* with $\varphi_m < 0$, the curvature will continue to decrease, and the solution will approach ρ_* (in Fig. 1 this is p_*) in accordance with type 1.

Qualitatively, this last result follows from the fact that, in accordance with (51), an increase in E leads to a displacement of the point $a_0 = p_0$ in Fig. 1 to the right, and the solution, which is initially to the right of this point, moves to the left of it with increasing r (or x).

§7. CONCLUSIONS

Thus, allowance for the polarization corrections and particle production for all branches leaving the regular minimum leads (except in the case of the separatrix) to the above weak singularity in the invariant $R_{ik;l}R^{ik;l}$ [see (57)]. In the considered approximation, it is not possible to find a unique continuation of the solution beyond the dashed line in Fig. 1. According to the classification of singularities proposed in Ref. 18, such singularities are of the Newtonian type. Therefore, near them the true quantum-gravitational effects are not large. The occurrence of such singularities is related to the poor model of T_i^k near $\rho \sim \rho_*$, and the finding of the correct continuation of the solution requires study here of a more exact nonlocal energy-momentum tensor (see Appendix I), which leads to integrodifferential equations whose solutions do not contain singularities of the type (57).

If we remain in the framework of differential equations, an acceptable hypothesis would consist of replacing $f(\rho)$ in (6) in the neighborhood of ρ_* in such a way as to make the saddle in Fig. 1 become a complicated state of equilibrium of saddle-node type.[19] Then the solutions of type 4 in Fig. 1 will for $C^2 > C_s^2$ enter a sector of node type and have a unique continuation into the region $p < p_*$. The solutions given at the end of the preceding section will then enter the oscillation regime. The rapid decrease of ρ^2 leads to elimination of the particle production process and to oscillations around the value of E accumulated in the preceding stages. For $\gamma \geq 1$, the oscillations tend to the Friedmann solution $y = E$, $r = (2E^{1/2}\theta)^{1/2}$, and in accordance with (11)

$$g_\alpha = \bar{A}_\alpha \exp[2^{-1/2} C_\alpha (\theta_*^{-1/2} - \theta^{-1/2}) E^{-3/4}]. \tag{72}$$

Here $\bar{A}_\alpha = g_\alpha(\theta_*)$, and θ_* is the time of arrival at the oscillation regime. For $\theta \gg \theta_*$, we have $g_\alpha \to \text{const}$, which indicates a tending to the isotropic model.

We indicate the qualitative solution of the model if the solutions are matched in the neighborhood of $\rho = \rho_*$ in accordance with the type of the separtrix (see Fig. 2). The solution that begins with anisotropic vacuum stage oscillates around the Kasner solution $y = K/6r^2$. Decrease of r during contraction leads to increase of ρ^2 in the oscillations and a change in the form of the solution at $\rho = \rho_*$, after which $y(r)$ decreases monotonically to $y(r_0) = 0$. The influence of the growth of E due to the particle production is not important in this stage, since it does not halt the growth of ρ^2. After the bounce ($y = 0$), i.e., the transition through the regular minimum $r = r_0$, the growth of E slows down the growth of the curvature and leads to $\rho^2 < \rho_*^2$. Again an oscillation regime commences, but $y(r)$ now oscillates around

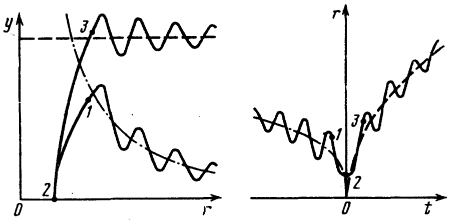

FIG. 2. Regular cosmological solution on the phase plane and the physical plane (continuous curves); the dashed line is the Friedmann solution for $E = \text{const}$; the chain curves represent the "vacuum" Kasner solution: 1 and 3 are the points of vanishing of φ in (21); 2-3 is the interval of significant influence of the produced particles on the evolution of the model.

the isotropic Friedmann solution $y = E$, the value of E having been accumulated in the preceding stages. In the oscillations, the particle production rapidly ceases, since $\rho^2 \to 0$. This picture exists for arbitrary $\gamma \geq 1$, and the value of γ determines only the law of approach of the oscillations to the corresponding solutions of the Einstein equations in the limit $r \to \infty$. In the R, t plane, the solution has the form of a soliton with oscillating "tails." The crown of the soliton is at the point $r_1 \gtrsim r_0$. The maximal value of the curvature at the point r_1 is such that we remain in the region of applicability of the semiclassical theory, which takes into account only the single-loop quantum-gravitational corrections. The magnitude of the corrections in the Lagrangian density remains smaller than that of the main term.

Investigation of the homogeneous anisotropic regular model (which is more general than the isotropic model considered in Ref. 14) makes it possible to elucidate the part played by the different additional terms in the gravitational equations. The bounce (replacement of contraction by expansion) is due to the polarization correction in (1). The particle production (60) leads to a halting of the growth of the curvature and rapid isotropization of the model during the expansion stage after the bounce. Note that particle production cannot isotropize the model during the contraction stage. The term (27) does not influence the bounce and the isotropization, but it ensures damping of the oscillations around the classical power-law asymptotic behaviors as $t \to \pm\infty$.

It is interesting to note that in the expansion stage between the bounce and the transition to the $r \propto t^{1/2}$ regime the model spends a long time ($\Delta t \gg l_g$) near the de Sitter solution (63) with constant curvature. This could be important for analyzing observational consequences, in particular, the spectrum of relic gravitational waves.

Note also that we can put matter with inhomogeneously distributed baryon charge into the constructed regular homogeneous model during the stage of anisotropic contraction, and this will not have an appreciable influence on the metric. Since the matter produced by the gravitational field is charge symmetric and its energy density appreciably exceeds that of the charge-asymmetric matter, during the expansion stage we may have entropy perturbations of any magnitude with very small adiabatic perturbations. The spectrum of the entropy perturbations is determined by the distribution of the baryon charge during the contraction stage. The occurrence of entropy perturbations is a typical feature of models with regular replacement of anisotropic contraction by isotropic expansion.

We thank Ya. B. Zel'dovich for his constant interest in the work and valuable advice.

APPENDIX I

The coefficient $C = l_P^2/144\pi$ for a real nonconformal massless scalar field was obtained in Ref. 4 in the case when space-time has constant negative curvature (the anti-de Sitter metric). Below, we shall indicate how C is obtained in the case of the homogeneous and time-dependent metric (2). We consider the more general case when the scalar field φ has mass m. The method of calculation and the regularization is entirely equivalent to that used in Ref. 2 in the case of scalar field with conformal coupling. The regularized result is

$$\varepsilon = \langle T_0^0 \rangle = \frac{1}{(2\pi r)^3} \int d^3k\, \omega_k [s_k - s_k^{(2)} - s_k^{(4)}] \quad (I.1)$$

with similar results for the other diagonal components of $\langle T_i^k \rangle$; all the nondiagonal components of $\langle T_i^k \rangle$ are zero. In (I.1)

$$\omega_k = \left(\sum_\alpha k_\alpha^2 (r g_\alpha)^{-2} + m^2 \right)^{1/2};$$

further, $s_k(t) = |\beta_k(t)|^2$, where $\beta_k(t)$ and $\alpha_k(t)$ are solutions of the system of linear differential equations

$$\dot\alpha_k = \frac{1}{2} \left(\frac{\dot\omega_k}{\omega_k} + 3\frac{\dot r}{r} \right) \beta_k \exp\left(2i \int \omega_k dt \right),$$
$$\dot\beta_k = \frac{1}{2} \left(\frac{\dot\omega_k}{\omega_k} + 3\frac{\dot r}{r} \right) \alpha_k \exp\left(-2i \int \omega_k dt \right) \quad (I.2)$$

with initial conditions $\alpha_k = 1, \beta_k = 0$ for $t = -\infty$. The quantities α_k and β_k are related by

$$|\alpha_k(t)|^2 - |\beta_k(t)|^2 = 1. \quad (I.3)$$

The renormalization terms $s_k^{(2)}(t)$ and $s_k^{(4)}(t)$ are local, i.e., they depend on the metric coefficients only at the time t, and they have the form (for simplicity, the subscript k is omitted)

$$s^{(2)} = \frac{1}{16\omega^2} \left(\frac{\dot\omega}{\omega} + 3\frac{\dot r}{r} \right)^2,$$
$$s^{(4)} = \frac{3}{256\omega^4} \left(\frac{\dot\omega}{\omega} + 3\frac{\dot r}{r} \right)^4 + \frac{1}{64} \left\{ \frac{1}{\omega} \frac{d}{dt} \left[\frac{1}{\omega} \left(\frac{\dot\omega}{\omega} + 3\frac{\dot r}{r} \right) \right] \right\}^2$$
$$- \frac{1}{32} \frac{1}{\omega^2} \left(\frac{\dot\omega}{\omega} + 3\frac{\dot r}{r} \right) \frac{d}{dt} \left\{ \frac{1}{\omega} \frac{d}{dt} \left[\frac{1}{\omega} \left(\frac{\dot\omega}{\omega} + 3\frac{\dot r}{r} \right) \right] \right\}. \quad (I.4)$$

In the special isotropic case ($a \sim b \sim c$), these terms agree with the ones obtained in Ref. 20. They are analyzed in detail in the isotropic case in Ref. 21.

Analysis of the Einstein equations with $\langle T_i^k \rangle$ from (I.4) shows that in the framework of these equations the unphysical singularities of the form (57) described in Sec. 5 do not arise.

We consider the asymptotic behavior of $\langle T_i^k \rangle$ near the singularity, when r and g_α vary in accordance with power laws (this is the case for $|t| \gg t_P$). Suppose, in addition, $m \ll |\dot r / r|$. To obtain the polarization correction to the Lagrangian density, which, of course, is only part of the total value of $\langle T_i^k \rangle$, we must retain in (I.1) only the local terms $s_k^{(2)}$ and $s_k^{(4)}$. The integration must be performed over the region $\omega_k^2 < (\dot r / r)^2$, since at higher frequencies the entire integrand in (I.1) tends rapidly to zero and its contribution to the integral is small. Making the necessary calculations, we obtain for the considered part of $\langle T_i^k \rangle$ (which we denote by $\langle \tilde T_i^k \rangle$)

$$\langle \tilde T_i^k \rangle = \left[\frac{1}{576\pi^2} M_i^{k(1)} + \frac{1}{960\pi^2} M_i^{k(2)} \right] \ln m|t|, \quad (I.5)$$

here

$$M_{ik}^{(1)} = \frac{1}{(-g)^{1/2}} \frac{\delta}{\delta g^{ik}} \int (-g)^{1/2} R^2 d\Omega$$
$$M_{ik}^{(2)} = \frac{1}{(-g)^{1/2}} \frac{\delta}{\delta g^{ik}} \int (-g)^{1/2} C_{iklm} C^{iklm} d\Omega, \quad (I.6)$$

and C_{iklm} is the conformal Weyl tensor. In the power-law regime

$$\langle \tilde T_i^k \rangle \propto t^{-4} \ln |t| m.$$

The first term in (I.6) was also obtained in the recent paper Ref. 22 in the isotropic case. Using $|R| \propto t^{-2}$ we see that the first term in (I.5) can be regarded approximately as deriving from the variation of the following correction to the Lagrangian density:

$$\Delta L_g = -\frac{1}{4608\pi^2} R^2 \ln (R l_P^2)^2 = -\frac{C}{32\pi G} R^2 \ln (R l_P^2)^2, \quad (I.7)$$

where $C = l_P^2/144\pi$.

Note that in the case of a weak but rapidly varying gravitational field ($g_{ik} = \eta_{ik} + h_{ik}$, where η_{ik} is the Minkowski metric, and $|h_{ik}| \ll 1$) T_i^k in the Fourier representation has the form

$$T_{ik}(q) = \Pi_{iklm}(q) h^{lm}(q), \quad \Pi_{iklm} \propto q^4 \ln q^2,$$

for $|q^2| \gg m^2 (q^2 = q_i q^i)$. The relationship between this expression and the effective Lagrangian density (I.7) in the case of strong but slowly varying gravitational fields is analogous to the situation in quantum electrodynamics, for which in weak rapidly varying external electromagnetic fields the photon polarization operator satisfies

$$\Pi(q^2) \propto q^2 \ln q^2, \quad |q^2| \gg m,$$

and in a quasihomogeneous magnetic (or electric) field we have the Euler–Heisenberg correction to the Lagrangian density

$$\Delta L \propto H^2 \ln H.$$

On the other hand, analysis of the total value of $\langle T_i^k \rangle$ for $|R| \ll m^2$ shows that $\langle T_i^k \rangle$ is analytic with respect to R at $R = 0$. We simulate this property by including the constant λ in (17). In the case $m \neq 0$, it is natural to assume $\lambda = m^4$, and for massless fields, when an infrared divergence arises in the limit $\omega_k \to 0$ (which leads to the necessity to retain in $\langle T_i^k \rangle$ a fictitious mass $\mu \neq 0$), λ may be chosen arbitrarily.

APPENDIX II

For the polarization correction (17), we have

$$f(\rho)=(\rho^2+\lambda)\varphi_1, \qquad \varphi_1=\ln[(\rho^2+\lambda)/\rho_c^2],$$
$$f_\rho=2\rho(\varphi_1+1), \qquad f_{\rho\rho}=2(\varphi_1+3)-4\lambda/(\rho^2+\lambda). \tag{II.1}$$

By virtue of the inequality (17), the second term in $f_{\rho\rho}$ can be ignored. Indeed, in the case $\rho^2 \gg \lambda$ it is small. For $\rho^2 \lesssim \lambda$, this term is of order unity, but then by virtue of $\lambda \ll \rho_c^2$, $|\varphi_1| \gg 1$ determines the value of $f_{\rho\rho}$. Substitution of (II.1) in (14) gives

$$y+\Big\{2(\varphi_1+1)\Big[\frac{3y'}{r^3}+\frac{K}{r^6(1+f_\rho)^2}\Big]\Big(\frac{ry'}{2}-y\Big)$$
$$-\frac{r^4}{6}\Big[\frac{3y'}{r^3}+\frac{K}{r^6(1+f_\rho)^2}\Big]^2\varphi_1+2ry(\varphi_1+3)$$
$$\times\Big[-\frac{3y''}{r^3}+\frac{9y'}{r^4}+\frac{6K}{r^7(1+f_\rho)^2}\Big]\Big[1-\frac{2Kf_{\rho\rho}}{r^6(1+f_\rho)^3}\Big]^{-1}\Big\}=E+\frac{K(1+2f_\rho)}{6r^2(1+f_\rho)^2}. \tag{II.2}$$

If

$$|\rho_{max}|\ll\varphi_1^{-1}(\rho_{max}), \quad K/r_{min}^6\ll|\rho_{max}| \tag{II.3}$$

then from (II.2) and (II.1) we have

$$y+\Big\{6(\varphi_1+1)\frac{y'}{r^3}\Big(\frac{ry'}{2}-y\Big)-\frac{3}{2}\frac{(y')^2}{r^2}\varphi_1$$
$$-ry(\varphi_1+3)\Big(\frac{6y''}{r^3}-\frac{18y'}{r^4}\Big)\Big\}=E+\frac{K}{6r^2}+O(r^{-5}). \tag{II.4}$$

Hence, after regrouping and introduction of ρ_* instead of ρ_c we obtain (21).

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The Cavendish experiment at large distances

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The ratio of the gravitational constants G is measured for masses interacting at distances r_1 and r_0 or r_2 and r_0 ($r_0 \approx 0.4$ m, $r_1 \approx 0.3$ m, $r_2 \approx 10$ m) for the purpose of verifying the hypothesis that G depends on distance. The measurements were made with a highly sensitive torsion balance and an electronic indicating system. The values obtained are $G(r_1)/G(r_0) = 1.003 \pm 0.006$ and $G(r_2)/G(r_0) = 0.998 \pm 0.013$ (the limits correspond to the level of one standard deviation). These results do not confirm the experimental data of D. I. Long [Nature 260, 417 (1976)], according to which spatial variations of G do exist. The possible imitating effects are analyzed and prospects for other procedures of experimentally verifying the indicated hypothesis are discussed.

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1. INTRODUCTION

The possible existence of not only temporal (in accordance with the well-known hypothesis of Dirac et al.[1-3]) but also spatial variations of the gravitational constant G[4,5] is being discussed of late. The dependence of G on the distance follows, in particular, from the scalar-tensor variants of gravitation theory[6-8] if one adds to