Weak supergravity

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A weak-field approximation is constructed in the theory of pure supergravity. The gravitational supermultiplet combines small perturbations of an arbitrary background gravitational field (spin $s = 2$) and a weak fermion field ψ_μ (spin $s = 3/2$). In the lowest approximation, the gravitational perturbations and the ψ_μ field do not have a dynamical influence on the background space-time. It is shown that the approximate theory is locally or globally supersymmetric depending on the assumptions made concerning the magnitude of the ψ_μ field compared with the gravitational perturbations. The properties of the linear and nonlinear (supergravitational) equations for the particles of spin $s = 3/2$ under conformal scale transformations are investigated. It is shown that by using the freedom in the definition of the spinorial connection these equations can be made conformally invariant. The theory of weak supergravity is considered on a Friedmannian (conformally flat) background. Because of the conformal invariance of the equations, spin 3/2 particles are not produced, and the conformal noninvariance of the equations for the gravitons leads to the possibility of amplification of classical gravitational waves and the production of gravitons, as also occurs in Einstein's theory of gravitation.

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Supergravity theories, which combine the gravitational field with one or several fermion and boson fields, attract interest through their mathematical elegance and some practical achievements, among which the most important is the reduction in the number of divergences.[1)] Although the physical reality of supersymmetric theories and, in particular, supergravity must still be demonstrated, the achievements of these theories are sufficient ground for attempting to extract from them physical consequences with a view to gaining a deeper understanding of the theories themselves and possibly verifying them. Among the supergravity theories, the one most fully developed is the simplest variant, namely, so called pure supergravity, which combines the gravitational field (spin 2) with one massless fermion field (spin 3/2) (Refs. 2–4). The exact theory of pure supergravity couples the nonlinear gravitational field to a nonlinear fermion field (we shall call it the ψ_μ field) and presupposes that the source of the gravitational field is the energy-momentum tensor of the ψ_μ field.

From the point of view of the use of supergravity in applications (for example, in some astrophysical situations), the deduction of comprehensible theoretical conclusions, and the comparison of them with the conclusions of standard general relativity, it is necessary to give a formulation of the theory in which supergravity acts on a curved gravitational background produced by an external source, rather than the ψ_μ field. Thus, we need to preserve the supersymmetry but represent the basic equations in the form of series in some small parameter that describes the deviation of the geometry from the background geometry. For the background, one could use the Schwarzschild metric, the metrics of cosmological models, etc. Truncating the obtained series (not necessarily at the quadratic approximation in the Lagrangian, as in the present paper), one could obtain expressions amenable to practical calculations and describing the investigated system with a certain degree of accuracy. In this connection, it must be recalled that many years elapsed after the creation of Einstein's theory of gravitation before fundamental physical consequences were drawn from it; these conclusions, which also relate to the experimental verification, were obtained in the main by means of approximate methods. We shall call pure supergravity acting on the background of an external classical gravitational field weak supergravity (by analogy with the weak-field approximation in standard general relativity). The present paper is devoted to the formulation of such an approximation and the derivation of certain conclusions.

In Sec. I, by decomposing the gravitational variables into background and dynamical parts, we obtain an approximate supersymmetric Lagrangian and the equations of motion. Conceptually, this procedure is simple and reduces to the derivation of approximate equations from the exact equations, but technically the finding and combining of terms of a given approximation with preservation of the supersymmetry is not so easy. It turns out that the equations of motion of a given approximation can be derived by taking an appropriately truncated Lagrangian, but verification of its supersymmetry requires allowance for terms ignored by virture of their smallness in the verification of the supersymmetry of the Lagrangian of the preceding approximation.

In Sec. 2 we investigate a variant in which we require the approximate Lagrangian to satisfy more exacting requirements, by virtue of which supersymmetry must be manifested when a smaller number of terms is retained in the Lagrangian $L_{3/2}$ of the ψ_μ field. In this variant, we arrive only at a globally supersymmetric theory which couples on a flat background free gravitational waves (two degrees of freedom) and free ψ_μ waves (two degrees of freedom). The gauge conditions that distinguish the physical degrees of freedom of these fields also remain superinvariant. In Sec. 3, the attempt is made to include the Lagrangian L_m of matter as well in the total Lagrangian $L_2 + L_{3/2}$. The purpose of this term is to produce a non-vacuum external

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background on which weak supergravity can act. This could be achieved with complete preservation of supersymmetry in a variant of extended supergravity that combines not only the 2 and 3/2 spin fields but also fields of other spins.[5] These fields could then provide a model of the energy-momentum tensor of all the matter producing the background curvature. However, we consider a simplified variant, accepting in advance that not all the properties of supersymmetric theories will be satisfied. In our variant, it is assumed that L_m depends on certain fields which do not participate in the supersymmetry, and on a tetrad field $V^a_\mu(x)$, which does participate in the symmetry. As an example of such a L_m, we can take the Lagrangian of a hydrodynamic fluid. Of course, in such a scheme the variation of the action with respect to supertransformations does not vanish identically, and its vanishing requires additional constraints: $T_{\mu\nu}\gamma^\mu\psi^\nu=0$, where $T_{\mu\nu}$ is the energy-momentum tensor of the matter. However, these conditions themselves are not invariant under supertransformations, and, in general, the second variation of the action does not vanish.

In Sec. 4, we consider the conformal invariance properties of the equations for the ψ_μ field. The conclusions obtained in this section are of interest both in their own right and in connection with the subsequent study (in Sec. 5) of weak supergravity in the linear approximation on the background of isotropic metrics. The point is that, as is well known, the equations for weak gravitational waves are not conformally invariant.[6] Because of this, it is possible to have effects that amplify classical gravitational waves and create gravitons even in the simplest geometries, i.e., conformally flat nonstationary geometries, i.e., in situations when analogous effects are absent for all the remaining massless fields.[6,7] On the other hand, the equations for massless particles of spin 3/2 (which are sometimes called gravitinos), which were considered in the literature before the creation of supergravity, are conformally invariant but are not completely integrable in a non-vacuum curved universe.

In supergravity, the inconsistency of the equations for the gravitino is eliminated by a modification of the equations. At the first glance, this modification would seem to change their property of conformal invariance, but we shall show that these equations can be made conformally invariant by exploiting the freedom in the definition of the spinorial connection. Thus, in a single supermultiplet one can combine gravitons described by conformally noninvariant equations and gravitinos described by conformally invariant equations[2)].

In Sec. 5, the theory of weak supergravity is constructed on the background of an isotropic cosmological model. It is shown that the equation for weak gravitational fields contains a source in the form of the operator of the energy-momentum tensor of the ψ_μ field. It is natural to take the vacuum state as initial state. Then gravitinos are not created, while gravitons are created in complete agreement with Einstein's theory considered in the same approximation. Thus, the conclusions concerning the production of gravitons near the cosmological singularity are not affected by the replacement of ordinary gravity by supergravity.

1. CONSTRUCTION OF WEAK SUPERGRAVITY

The exact theory of pure supergravity developed in Refs. 2, 3, and 4 combines in a supermultiplet the gravitational field and the spin 3/2 field. The gravitational part is represented in the tetrad formalism, and the independent variables of the gravitational field are the field of tetrad vectors V^a_μ and the tetrad connection $\Omega_{\mu ab}$. The spin 3/2 field is described by a Majorana spinor ($\psi_\mu=C\bar{\psi}^T_\mu$) with anticommutation properties. The action is the sum of the ordinary gravitational action and the generalized Rarita—Schwinger action:

$$I=\int d^4x\,(L_2+L_{3/2}),$$

$$L_2=\frac{1}{4k^2}VR(V,\Omega),\quad L_{3/2}=-\frac{1}{2}e^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho, \tag{1.1}$$

where $R(V,\Omega)$ is the scalar curvature. Our notation and definitions basically follow Ref. 4:

$$R_{\mu\nu ab}=(\partial_\mu\Omega_{\nu ab}+\Omega_{\mu a}{}^c\Omega_{\nu cb})-(\mu\leftrightarrow\nu),$$
$$R_{\mu a}=V^{b\nu}R_{\mu\nu ab},\quad R=V^{a\mu}R_{\mu a},\quad V=\det V_\mu{}^a,$$
$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu},\quad V_{a\mu}V^a{}_\nu=g_{\mu\nu},\quad \gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3,$$
$$\sigma^{ab}={}^1/_4(\gamma^a\gamma^b-\gamma^b\gamma^a),\quad g_{\mu\nu}(+,-,-,-),\quad e^{0123}=1.$$

We retain the notation D_ν for the derivative taking into account only spinor indices,

$$D_\nu\psi_\rho=\partial_\nu\psi_\rho-\Gamma_\nu\psi_\rho,$$

in which Γ_ν is the spinorial connection:

$$\Gamma_\nu=-{}^1/_2\Omega_{\nu ab}\sigma^{ab}.$$

The action (1.1) is invariant under the following supersymmetry transformations with coordinate-dependent parameter[3)] $\varepsilon(x)$:

$$\delta V^a{}_\mu=ik(\bar{\varepsilon}\gamma^a\psi_\mu),\quad \delta\psi_\mu=\varepsilon_{;\mu},$$
$$\delta\Omega_{\mu ab}=B_{\mu ab}-{}^1/_2V_{b\mu}B_{ca}{}^c+{}^1/_2V_{a\mu}B_{cb}{}^c, \tag{1.2}$$
$$B_c{}^{\sigma\tau}=-kV^{-1}(\bar{\varepsilon}\gamma_5\gamma_c D_\nu\psi_\rho)e^{\sigma\tau\nu\rho}.$$

The tetrad connection can be represented in the form (this follows from the equations $g_{\mu\nu:\lambda}=0$, $V_{a\mu:\nu}=0$)

$$\Omega_{\mu ab}=\Omega_{\mu ab}(S=0)+K_{\mu ab}, \tag{1.3a}$$

where

$$\Omega_{\mu ab}(S=0)={}^1/_2[V_a{}^\nu(\partial_\mu V_{b\nu}-\partial_\nu V_{b\mu})+V_a{}^\rho V_b{}^\sigma(\partial_\sigma V_{c\rho})V^c{}_\mu]-[a\leftrightarrow b]$$

is the connection in the absence of torsion and, by definition,

$$K_{\mu\nu\rho}=-S_{\mu\nu\rho}+S_{\nu\rho\mu}-S_{\rho\mu\nu},\quad S_{\mu\nu}{}^\rho={}^1/_2(\Gamma_{\mu\nu}{}^\rho-\Gamma_{\nu\mu}{}^\rho).$$

The equation of motion obtained by varying the action with respect to $\Omega_{\mu ab}$ gives for the torsion the expression

$$S_{\mu\nu\rho}={}^1/_2ik^2(\bar{\psi}_\mu\gamma_\rho\psi_\nu). \tag{1.3b}$$

Variation of the action with respect to the variables V^a_μ and ψ_μ leads, respectively, to the equations of motion

$$R_a^{\mu}-{}^1/_2 V_a^{\mu} R=-k^2 V^{-1} e^{\lambda\mu\nu\rho}\bar{\psi}_\lambda \gamma_5 \gamma_a D_\nu \psi_\rho, \tag{1.3c}$$

$$e^{\lambda\mu\nu\rho}(\gamma_\mu D_\nu \psi_\rho-{}^1/_2 S_{\mu\nu}{}^{\sigma}\gamma_\sigma\psi_\rho)=0. \tag{1.3d}$$

Since the torsion satisfies algebraic equations, and not propagation equations, it can be eliminated from the remaining equations (1.3c) and (1.3d), the last term in (1.3d) then vanishing by virtue of the identities

$$e^{\lambda\mu\nu\rho}(\bar{\psi}_\mu Q\psi_\nu) Q\psi_\rho=0, \tag{1.4}$$

where Q is any one of the 16 matrices I, γ^5, γ^μ, $i\gamma^5\gamma^\mu$, and $i\sigma^{\mu\nu}$, and (1.3d) can be reduced to the form

$$\gamma^\mu(D_\mu\psi_\nu-D_\nu\psi_\mu)=0. \tag{1.5}$$

This equation contains the torsion only through the connection.

Note that in the exact theory of supergravity the geometry (nonvacuum) is produced by the ψ_μ field. For $\psi_\mu=0$, the system (1.3) degenerates into Einstein's vacuum equations.

We now turn directly to the construction of weak supergravity.

Expansion of the total Lagrangian

We represent the gravitational variables in the form of the sum of a principal (background) term and a small (dynamical) correction[4]:

$$V_{a\mu}=V_{a\mu}^{(0)}+V_{a\mu}^{(1)}, \quad \Omega_{\mu ab}=\Omega_{\mu ab}^{(0)}+\Omega_{\mu ab}^{(1)}. \tag{1.6}$$

The superscripts 0 and 1 indicate the order of magnitude. The expansions (1.6) can be regarded, not as approximate, but as exact in the sense that in the expansion of $V_{a\mu}$ and $\Omega_{\mu ab}$ in powers of a small parameter all the corrections to $V_{a\mu}^{(0)}$ and $\Omega_{\mu ab}^{(0)}$ are assumed to be summed and represented in the form $V_{a\mu}^{(1)}$ and $\Omega_{\mu ab}^{(1)}$. When no confusion is possible, we shall omit the superscript 0 of the background variables and denote $V_{a\mu}^{(1)}=v_{a\mu}$, and $\Omega_{\mu ab}^{(1)}=\omega_{\mu ab}$ and perform all operations of raising and lowering of indices by means of the background $V_{a\mu}$ and $g_{\mu\nu}$ and the operation of differentiation with respect to the background connections $\Omega_{\mu ab}$. Infinite expansions necessarily arise in $V^{a\mu}$ and V, and, as a consequence, in the other quantities:

$$V^{a\mu}\to V^{a\mu}-v^{\mu a}+v^{\mu c}v_c{}^a+\dots,$$
$$V\to V(1+v+{}^1/_2 v^2-{}^1/_2 v^{\mu a}v_{a\mu}+\dots) \tag{1.7}$$

(the symbol $\to$ means that in the expansion the quantity on the left goes over into the quantity on the right).

The Riemann and Ricci tensors go over into the expansions

$$R_{\mu\nu ab}=R_{\mu\nu ab}^{(0)}+R_{\mu\nu ab}^{(1)}+R_{\mu\nu ab}^{(2)}+\dots, \quad R_{\mu a}=R_{\mu a}^{(0)}+R_{\mu a}^{(1)}+R_{\mu a}^{(2)}+\dots,$$
$$R_{\mu\nu ab}^{(1)}=\omega_{\nu ab;\mu}-\omega_{\mu ab;\nu}+2S_{\mu\nu}^{(0)b}\omega_{\sigma ab}, \quad R_{\mu\nu ab}^{(2)}=\omega_{\mu a}{}^c\omega_{\nu cb}-\omega_{\nu a}{}^c\omega_{\mu cb},$$
$$R_{\mu a}^{(1)}=V^{b\nu}R_{\mu\nu ab}^{(1)}-v^{\nu b}R_{\mu\nu ab}^{(0)},$$
$$R_{\mu a}^{(2)}=V^{b\nu}R_{\mu\nu cb}^{(2)}-v^{\nu b}R_{\mu\nu ab}^{(1)}+v^{\mu c}v_c{}^a R_{\mu\nu ab}^{(0)}.$$

Substitution of the above series in the gravitational Lagrangian L_2 enables us to write it in the form of the series

$$L_2=L_2^{(0)}+L_2^{(1)}+L_2^{(2)}+\dots,$$

where in particular

$$L_2^{(2)}=\frac{V}{4k^2}\Big[V^{a\mu}R_{\mu a}^{(2)}+vR^{(1)}-v^{\mu a}R_{\mu a}^{(1)}$$
$$+v^{\mu c}v_c{}^a R_{\mu a}^{(0)}-vv^{\mu a}R_{\mu a}^{(0)}+\frac{1}{2}(v^2-v^{\mu a}v_{a\mu})R^{(0)}\Big]. \tag{1.8}$$

We are now justified in regarding $v_{\mu a}$ and $\omega_{\mu ab}$ as field variables, and therefore we can vary with respect to them. We then obtain

$$\frac{\delta L_2^{(1)}}{\delta v^{\mu a}}=-\frac{V}{2k^2}\left(R_{\mu a}^{(0)}-\frac{1}{2}V_{a\mu}R^{(0)}\right), \tag{1.9}$$

$$\frac{\delta L_2^{(1)}}{\delta\omega_{\mu ab}}=\frac{V}{2k^2}(S_\lambda{}^{(0)a\lambda}V^{b\mu}-S_\lambda{}^{(0)b\lambda}V^{a\mu}+S^{(0)ab\mu}), \tag{1.10}$$

$$\frac{\delta L_2^{(2)}}{\delta v^{\mu a}}=-\frac{V}{2k^2}\Big[R_{\mu a}^{(1)}-\frac{1}{2}V_{a\mu}R^{(1)}-\frac{1}{2}v_{a\mu}R^{(0)}$$
$$-v_a{}^b\left(R_{\mu b}^{(0)}-\frac{1}{2}V_{b\mu}R^{(0)}\right)-v_\mu{}^\nu\left(R_{\nu a}^{(0)}-\frac{1}{2}V_{a\nu}R^{(0)}\right)+v\left(R_{\mu a}^{(0)}-\frac{1}{2}V_{a\mu}R^{(0)}\right)\Big], \tag{1.11}$$

$$\frac{\delta L_2}{\delta\omega_{\mu ab}}=\frac{V}{4k^2}\{V^{a\mu}(v_;{}^b-v^{\nu b}{}_{;\nu})-V^{b\mu}(v_;{}^a-v^{\nu a}{}_{;\nu})+v^{\mu b;a}-v^{\mu a;b}+V^{a\mu}\omega^{\nu b}{}_\nu$$
$$-V^{b\mu}\omega_\nu{}^{\nu a}+\omega^{a\mu b}-\omega^{b\mu a}+2[S_\lambda^{(0)a\lambda}(vV^{b\mu}-v^{\mu b})-S_\lambda{}^{b\lambda}(vV^{a\mu}-v^{\mu a})$$
$$+S_{\lambda\nu}^{(0)\lambda}(v^{\nu b}V^{a\mu}-v^{\nu a}V^{b\mu})]\}. \tag{1.12}$$

Substitution of the decompositions (1.6) in the Lagrangian $L_{3/2}$ reduces it to a sum of three terms: $L_{3/2}^{(0)}+L_{3/2}^{(1)}+L_{3/2}^{(2)}$. For our purposes, it is sufficient to write out the first two terms, i.e., the zeroth and first order in the gravitational variables, since we shall in what follows regard the ψ_μ field itself as weak:

$$L_{3/2}^{(0)}=-{}^1/_2 e^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho, \tag{1.13}$$

$$L_{3/2}^{(1)}=-{}^1/_2 e^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_a v^a{}_\mu D_\nu\psi_\rho-{}^1/_4 e^{\lambda\mu\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu\omega_{\nu ab}\sigma^{ab}\psi_\rho. \tag{1.14}$$

The variation of the sum $L_{3/2}^{(0)}+L_{3/2}^{(1)}$ with respect to ψ_μ contains terms that can be combined in a total divergence. Separating divergence terms in different ways, we can form two expressions for the variation of the Lagrangian, one of which is convenient for verifying the superinvariance of the total action and the other for deriving the equations of motion. In each of the expressions, the total divergence is omitted, since it affects neither the verification of the invariance of the action nor the derivation of the equations of motion. To verify the invariance, it is convenient to use the expression (here, we denote by $S_{\mu\nu}^{(1)a}=s_{\mu\nu}^a$ the small correction to the torsion)

$$\delta_\psi(L_{3/2}^{(0)}+L_{3/2}^{(1)})=e^{\lambda\mu\nu\rho}[(D_\nu\delta\bar{\psi}_\lambda)\gamma_5(\gamma_\mu+v^a{}_\mu\gamma_a)\psi_\rho-{}^1/_2\delta\bar{\psi}_\lambda\gamma_5\sigma^{ab}\omega_{\nu ab}\gamma_\mu\psi_\rho$$
$$-{}^1/_2\delta\bar{\psi}_\lambda\gamma_5(S_{\mu\nu}^{(0)\sigma}\gamma_\sigma+S_{\mu\nu}^{(0)\sigma}v_\sigma{}^a\gamma_a+s_{\mu\nu}{}^a\gamma_a)\psi_\rho], \tag{1.15}$$

and to derive the equations of motion the expression

$$\delta_\psi(L_{3/2}^{(0)}+L_{3/2}^{(1)})=-e^{\lambda\mu\nu\rho}\delta\bar{\psi}_\lambda\gamma_5[(\gamma_\mu+v^a{}_\mu\gamma_a)D_\nu\psi_\rho+{}^1/_2\omega_{\nu ab}\gamma_\mu\sigma^{ab}\psi_\rho$$
$$-{}^1/_2(S_{\mu\nu}^{(0)\sigma}\gamma_\sigma+S_{\mu\nu}^{(0)\sigma}v^a{}_\sigma\gamma_a+s_{\mu\nu}{}^a\gamma_a)\psi_\rho]. \tag{1.16}$$

In deriving the relations (1.15) and (1.16), we have made essential use of the identity

$$D_\nu(\gamma_a v^a{}_\mu)-D_\mu(\gamma_a v^a{}_\nu)-2S^{(0)\sigma}_{\nu\mu}v_a{}^\sigma\gamma_a-2s_{\nu\mu}{}^a\gamma_a+\omega_\nu{}^a{}_\mu\gamma_a-\omega_\mu{}^a{}_\nu\gamma_a=0,$$

which is the linear part in the expansion of the exact identity

$$D_\nu\gamma_\mu-D_\mu\gamma_\nu-2S_{\nu\mu}{}^\sigma\gamma_\sigma=0.$$

The variations of the approximate Lagrangian $L_{3/2}$ with respect to the gravitational variables $v_{a\mu}$ and $\omega_{\mu ab}$ can be readily obtained from (1.14).

Invariance of the approximate Lagrangian under supertransformations

If we were to use (1.6), assuming $v_{a\mu}$, $\omega_{\mu ab}$, and ψ_μ to be connected by supersymmetric transformations and were to take into account all the terms in the expansion of the Lagrangian $L_2+L_{3/2}$ and the transformations (1.2), we would recover a supersymmetric theory in the form of the successive approximations. Since we shall use an approximate Lagrangian, we write the transformation (1.2) in an approximate form as well;

$$\begin{gathered}\delta V^a{}_\mu=0,\quad \delta\Omega_{\mu ab}=0,\quad \delta v^a{}_\mu=ik(\varepsilon\gamma^a\psi_\mu),\\ \delta\psi_\mu=k^{-1}(\varepsilon_{;\mu}+{}^1/{}_2\omega_{\mu ab}\sigma^{ab}\varepsilon),\\ \delta\omega_{\mu ab}=-k\varepsilon(b_{\mu ab}-{}^1/{}_2V_{b\mu}b_{ca}{}^c+{}^1/{}_2V_{a\mu}b_{cb}{}^c),\end{gathered}\tag{1.17}$$

where $b_c^{\sigma\tau}=V^{-1}e^{\sigma\tau\nu\rho}\gamma_5\gamma_c D_\nu\psi_\rho$, and in the expression for $\delta\omega_{\mu ab}$ we have omitted small terms whose order can be written as $O(\psi v)$ and $O(\psi\omega)$.

Invariance of the sum $L_2^{(1)}+L_{3/2}^{(0)}$ (with divergence omitted) holds to terms of order $O(\psi^3)$, which must be taken into account in verifying the invariance of the Lagrangian $L_2^{(1)}+L_2^{(2)}+L_{3/2}^{(0)}+L_{3/2}^{(1)}$. To verify the super-invariance of this Lagrangian, we use the expressions (1.11), (1.12), and (1.15) and the identities

$$e^{\lambda\mu\nu\rho}(R^{(1)}_{\mu\nu ab}V^a{}_\lambda+R^{(0)}_{\mu\nu ab}v^a{}_\lambda-4S^{(0)\rho}_{\mu\lambda}S^{(0)}_{\nu\rho a}v_b{}^a+2s_{\mu\lambda b;\nu}+2S^{(0)a}_{\mu\lambda}\omega_{\nu ba})=0,$$

which are the linear part of the cyclic identities for the Riemann tensor in a space with torsion. The invariance of this Lagrangian is satisfied up to terms of order $O(\psi^5)$, which must be taken into account in the following step, etc.

Approximate equations of motion

To verify the invariance of the action, we naturally do not use the equations of motion. We now derive the equations of motion themselves, assuming that the ψ_μ field is not the source for the background geometry (we actually assume that $L_{3/2}^{(0)}$, which is quadratic in ψ_μ, is a quantity of the same order as the term $L_2^{(1)}$, which is linear in $v_{a\mu}$ and $\omega_{\mu ab}$, which we write formally as $\psi^2\sim v$, $\psi^2\sim\omega$). Then from $L_2^{(1)}+L_{3/2}^{(0)}$ we obtain the equations for the background geometry and the free ψ_μ field (see (1.9) and (1.10)):

$$R^{(0)}_{\mu a}-{}^1/{}_2V_{a\mu}R^{(0)}=0,\quad S^{(0)}_{\mu\nu\rho}=0,\quad e^{\lambda\mu\nu\rho}\gamma_\mu\psi_{\rho;\nu}=0.$$

In the following approximation, which is what interests us, we obtain from (1.11), (1.12), (1.16), and the variations of $L_{3/2}^{(0)}+L_{3/2}^{(1)}$ with respect to $v_{a\mu}$ and $\omega_{\mu ab}$ the required equations of motion

$$e^{\lambda\mu\nu\rho}\{(\gamma_\mu+v^a{}_\mu\gamma_a)\psi_{\rho;\nu}+{}^1/{}_2\omega_{\nu ab}\gamma_\mu\sigma^{ab}\psi_\rho-{}^1/{}_2s_{\mu\nu}{}^\sigma\gamma_\sigma\psi_\rho\}=0,\tag{1.18}$$

$$R^{(1)}_{\mu a}-{}^1/{}_2V_{a\mu}R^{(1)}=-ik^2V^{-1}e^{\lambda}{}_a{}^{\nu\rho}\bar\psi_\lambda\gamma_5\gamma_\mu\psi_{\rho;\nu},\tag{1.19}$$

$$s_{\mu\nu\rho}={}^1/{}_2ik^2(\bar\psi_\mu\gamma_\rho\psi_\nu).\tag{1.20}$$

The connection between $s_{\mu\nu\rho}$ and $\omega_{\mu ab}$ is determined by the equations

$$\begin{gathered}\omega_{\nu ab}={}^1/{}_2[(v_{ba}-v_{ab})_{;\nu}+(v_{a\nu}+v_{\nu a})_{;b}-(v_{b\nu}+v_{\nu b})_{;a}]+k_{\nu ab},\\ k_{\mu\nu\rho}=-s_{\mu\nu\rho}+s_{\nu\rho\mu}-s_{\rho\mu\nu}.\end{gathered}\tag{1.21}$$

The last term in (1.18) vanishes after the substitution of (1.20) and allowance for the identity (1.4). Elimination of $\omega_{\nu ab}$ from (1.18) reduces it to a nonlinear equation for ψ_μ.

2. GLOBALLY SUPERSYMMETRIC THEORY ON A FLAT BACKGROUND

We require invariance of the action constructed from the Lagrangian truncated at $L_2^{(2)}$ and $L_{3/2}^{(0)}$ (this is analogous to the assumption $\psi\sim v$, $\psi\sim\omega$). We shall assume that the background equations of motion $R^{(0)}_{\mu a}=0$, and $S^{(0)}_{\mu\nu\sigma}=0$ are satisfied; then the total Lagrangian can be written in the form

$$L=\frac{V}{4k^2}[vR^{(1)}+V^{a\mu}R^{(2)}_{\mu a}-v^{\mu a}R^{(1)}_{\mu a}]-\frac{1}{2}e^{\lambda\mu\nu\rho}\bar\psi_\lambda\gamma_5\gamma_\mu\psi_{\rho;\nu}.\tag{2.1}$$

To solve the posed problem, it is convenient to use the so-called second-order formalism,[2] in which $\Omega_{\nu ab}$ are not regarded as independent variables but expressed in terms of $V_{a\mu}$. In our approximation, we assume

$$\omega_{\nu ab}={}^1/{}_2[(v_{ba}-v_{ab})_{;\nu}+(v_{a\nu}+v_{\nu a})_{;b}-(v_{\nu b}+v_{b\nu})_{;a}].$$

In addition, we present our exposition in terms of the metric theory and do not use tetrad vectors. We use the fact that the matrix of a small rotation of a tetrad is antisymmetric[10]:

$$V_{a\mu}'=V_{a\mu}+\beta_{a\mu},\quad \beta_{ab}=-\beta_{ba}.$$

Using a transformation of this kind, we can make the antisymmetric part of $v_{\mu\nu}$ vanish; $v_{[\mu\nu]}=0$. Then the first correction to the metric tensor, which, as usual, we denote by $h_{\mu\nu}$, has the form

$$h_{\mu\nu}=v_{\mu\nu}+v_{\nu\mu}=2v_{\mu\nu}.$$

Thus, the Lagrangian (2.1) can be written in the metric form

$$L=\frac{1}{8k^2}(-g)^{1/2}\left(\frac{h}{2}R^{(1)}-h^{\mu\nu}R^{(1)}_{\mu\nu}\right)-\frac{1}{2}e^{\lambda\mu\nu\rho}\bar\psi_\lambda\gamma_5\gamma_\mu\psi_{\rho;\nu},\tag{2.2}$$

where

$$\begin{gathered}R^{(1)}_{\mu\nu}={}^1/{}_2(-h_{\mu\nu;\lambda}{}^\lambda+h_\mu{}^\lambda{}_{;\nu\lambda}+h_\nu{}^\lambda{}_{;\mu\lambda}-h_{;\mu\nu}),\\ R^{(1)}=-h_{;\lambda}{}^\lambda+h^{\lambda\sigma}{}_{;\lambda\sigma}.\end{gathered}$$

The supertransformations are written in the form

$$\begin{gathered}\delta h_{\mu\nu}=ik\varepsilon(\gamma_\mu\psi_\nu+\gamma_\nu\psi_\mu),\\ \delta\psi_\mu=k^{-1}[\varepsilon_{;\mu}+{}^1/{}_4(h_{\mu\nu;\lambda}-h_{\mu\lambda;\nu})\sigma^{\lambda\nu}\varepsilon].\end{gathered}\tag{2.3}$$

Note that this approach is equivalent to linearization of the first imperfect variant of supergravity without contact interaction and with absent invariance in the terms cubic in the ψ_μ field (Ref. 2).

Substituting the transformations (2.3) in the variation of the Lagrangian (2.2), we see that there remain terms that prevent the invariance:

$$\frac{1}{4k}e^{\lambda\mu\nu\rho}\left[\varepsilon_{;\nu}(h_{\lambda\varkappa;\sigma}-h_{\lambda\sigma;\varkappa})\sigma^{\varkappa\sigma}\gamma_5\gamma_\mu\psi_\rho+\frac{\varepsilon}{2}\gamma_5 R^{(0)\varkappa}_{\mu\nu\sigma}h_{\lambda\varkappa}\gamma^\sigma\psi_\rho\right]. \tag{2.4}$$

We shall require the vanishing of these terms without the imposition of any restrictions on the field variables ψ_μ and $h_{\mu\nu}$; for this, the two terms in (2.4) must vanish separately. For the vanishing of the first term, we must require $\varepsilon_{;\nu}=0$. These differential equations have the integrability conditions $R^{(0)}_{\mu\nu ab}\sigma^{ab}\varepsilon=0$. Using collective indices $\mu\nu\to A$ (Ref. 11) and writing down these equations in the canonical orthogonal frame for each of the Petrov types, we find that for $\varepsilon\neq 0$ they are satisfied only on a flat background, i.e., for $R^{(0)}_{\mu\nu ab}=0$. In this case, the second term in (2.4) also vanishes, and we therefore obtain an action on a flat background that is invariant under global supersymmetry transformations:

$$\delta h_{\mu\nu}=ik\varepsilon(\gamma_\mu\psi_\nu+\gamma_\nu\psi_\mu),\quad \delta\psi_\mu=\frac{1}{4k}(h_{\mu\nu,\lambda}-h_{\mu\lambda,\nu})\sigma^{\lambda\nu}\varepsilon. \tag{2.5}$$

We note that the invariance will be exact in the sense that terms of higher order which need to be omitted do not appear. From the Lagrangian (2.2) there follow the field equations

$$h_{\mu\nu,\lambda}{}^{\lambda}-h_{\mu}{}^{\lambda}{}_{,\lambda\nu}-h_{\nu}{}^{\lambda}{}_{,\lambda\mu}-h_{,\mu\nu}=0, \tag{2.6}$$

$$\gamma^\mu(\psi_{\nu,\mu}-\psi_{\mu,\nu})=0. \tag{2.7}$$

It is well known that in flat space, using the arbitrariness in the definition of $h_{\mu\nu}$,

$$h_{\mu\nu}'=h_{\mu\nu}+\xi_{\mu,\nu}+\xi_{\nu,\mu},$$

we can reduce the number of degrees of freedom by imposing the gauge conditions[12]

$$h_\mu{}^\nu{}_{,\nu}=0,\quad h=0, \tag{2.8}$$

$$h_{\mu\nu}u^\nu=0, \tag{2.9}$$

of which only eight are independent. Thus, the conditions (2.8) and (2.9) leave only two independent components of $h_{\mu\nu}$, these corresponding to the two independent polarization states.

We show that similar conditions can be imposed on the spin 3/2 field. From the fact that on a flat background the quantities ψ_μ can be subjected to the gauge (not small) transformations[13]

$$\psi_\mu'=\psi_\mu+\alpha_{,\mu}$$

(α is an arbitrary spinor), which leave the equations invariant, it follows that we can achieve fulfillment of the condition

$$\psi^\mu{}_{,\mu}=0. \tag{2.10}$$

After this, transformations which do not violate (2.10) must satisfy the differential equations $\alpha^{,\mu}{}_{,\mu}=0$. The remaining gauge freedom consists of specifying the eight functions

$$\alpha|_{t=0}=\alpha(x^1,x^2,x^3),\ \alpha_{,0}|_{t=0}=\alpha_{,0}(x^1,x^2,x^3)$$

on the initial hypersurface, and these can be made to satisfy the eight conditions

$$(\gamma^\mu\psi_\mu)|_{t=0}=0,\quad (u^\mu\psi_\mu)|_{t=0}=0,$$

where u^μ is some vector. Equations (2.7) with these initial data ensure fulfillment of these conditions for each instant of time, i.e.,

$$\gamma_\mu\psi^\mu=0, \tag{2.11}$$

$$u^\mu\psi_\mu=0, \tag{2.12}$$

if u^μ satisfies the system of differential equations

$$u^{\mu,\nu}=u^\mu a^\nu+bg^{\mu\nu}.$$

Such a vector always exists on a flat background (see Sec. 4).

The conditions (2.10), (2.11), and (2.12) leave one four-component quantity (for example, ψ_3) independent. As in the case of the neutrino field, one can reduce the description to a two-component quantity having two independent solutions corresponding to two independent helicity states.[14]

We consider how the gauge conditions for the fields $h_{\mu\nu}$ and ψ_μ transform under the supertransformations (2.5). We show that the set of conditions (2.8), (2.10), and (2.11) is preserved. Indeed,

$$\delta h_\mu{}^\nu{}_{,\nu}=ik\varepsilon(\gamma_\mu\psi^\nu{}_{,\nu}+\gamma^\nu\psi_{\mu,\nu})=0,$$

$$\delta h=2ik\varepsilon(\gamma^\mu\psi_\mu)=0,$$

$$\delta\psi^\mu{}_{,\mu}=\frac{1}{4k}(h^\mu{}_{\lambda,\mu\nu}-h^\mu{}_{\nu,\mu\lambda})\sigma^{\lambda\nu}\varepsilon=0,$$

$$\delta(\gamma^\mu\psi_\mu)=\frac{1}{4k}(h_{\mu\lambda,\nu}-h_{\mu\nu,\lambda})\gamma^\mu\sigma^{\lambda\nu}\varepsilon=\frac{1}{4k}(h_{,\nu}-h^\mu{}_{\nu,\mu})\gamma^\nu\varepsilon=0.$$

Thus, the supertransformations do not violate them.

The other set of gauge conditions (2.9) and (2.12) can also be preserved under supertransformations. We have

$$\delta(u^\mu\psi_\mu)=\frac{1}{4k}u^\mu(h_{\mu\lambda,\nu}-h_{\mu\nu,\lambda})\sigma^{\lambda\nu}\varepsilon=0,$$

although

$$\delta(h_{\mu\nu}u^\mu)=ik\varepsilon[\gamma_\mu\psi_\nu u^\mu+\gamma_\nu(\psi_\mu u^\mu)]\neq 0.$$

But if we require that after the supertransformations there be preserved the condition $v_{[\mu\nu]}=0$, i.e., $\delta(v_{[\mu\nu]})=0$, which can be achieved by a new twisting of the tetrad V^a_μ, then

$$\delta(h_{\mu\nu}u^\nu)=2ik\varepsilon\gamma_\nu(\psi_\mu u^\mu)=0.$$

The physical meaning of the gauge conditions given above is that they distinguish pure spin states (in the sense of eliminating lower spins); the supersymmetry does not disturb these states.

We note finally that it is possible to localize global supertransformations on a flat background, for which the introduction of additional fields is required. Invariance of the action with minimal number of fields that close the algebra of supertransformations irrespective of fulfillment of the equations of motion has been established.[15]

3. WEAK SUPERGRAVITY ON NONVACUUM BACKGROUND

We begin by considering the matter Lagrangian L_m

in general form. We shall assume that among the gravitational variables only V^a_μ occurs in it. As an example of such a Lagrangian we can take the Lagrangian of a hydrodynamic fluid in tetrad variables:

$$L_m=-\frac{\varkappa}{2k^2}V\Lambda_m, \tag{3.1}$$

$$\Lambda_m={}^1/_2(p+\rho)u_a u_b V^{a\mu}V^b{}_\mu-{}^1/_2(\rho-p). \tag{3.2}$$

By the definition of the energy-momentum tensor,

$$T_{\mu a}=\frac{1}{V}\frac{\partial(V\Lambda_m)}{\partial V^{a\mu}}. \tag{3.3}$$

Using (3.3), we obtain for $T_{\mu a}$ the usual expression

$$T_{\mu a}=(p+\rho)u_a u_\mu-pV_{a\mu}. \tag{3.4}$$

We can add L_m to the exact Lagrangians for the spin 2 and 3/2 fields, assuming that only V^a_μ in this Lagrangian are subjected to supertransformations. In the exact theory, the addition of L_m to $L_2+L_{3/2}$ leads to noninvariance of the action. The nonvanishing variation of the Lagrangian is expressed by the term

$$\delta L_m=\frac{\varkappa V}{2k}\bar{\varepsilon}T_{\mu a}\gamma^\mu\psi^a,$$

where $T_{\mu a}$ is defined in accordance with (3.3).

Leaving aside the question of the fulfillment of the conditions $T_{\mu a}\gamma^\mu\psi^a=0$ in the general case, we consider approximate supergravity with allowance for the terms that follow from L_m. The fulfillment of the conditions $T_{\mu a}\gamma^\mu\psi^a=0$ will be considered in Sec. 5 in the special case of an isotropic background metric.

We expand Λ_m up to quadratic terms:

$$\Lambda_m\to\Lambda_m-\frac{\partial\Lambda_m}{\partial V^{a\mu}}v^{\mu a}+\frac{\partial\Lambda_m}{\partial V^{a\mu}}v^\mu{}_c v^{ca}-\frac{\partial^2\Lambda_m}{\partial V^{a\mu}\partial V^{b\nu}}v^{\mu a}v^{\nu b}. \tag{3.5}$$

We substitute the expansion (1.7) and (3.5) in (3.1). The first-order terms $L_m^{(1)}$ in $v_{\mu a}$ are combined with the first-order terms $L_2^{(1)}$ from the gravitational Lagrangian, and as coefficients of them we have the background equations

$$R^{(0)}_{\mu a}-{}^1/_2V_{a\mu}R^{(0)}=\varkappa T^{(0)}_{\mu a}.$$

The quadratic terms have the form

$$L_m^{(2)}=-\frac{\varkappa V}{2k^2}\Big[\frac{\partial\Lambda_m}{\partial V^{a\mu}}v^{\mu c}v_c{}^a+\frac{\partial^2\Lambda_m}{\partial V^{a\mu}\partial V^{b\nu}}v^{\mu a}v^{\nu b}-\frac{\partial\Lambda_m}{\partial V^{a\mu}}vv^{\mu a}+\frac{1}{2}\Lambda_m(v^2-v^{\mu a}v_{a\mu})\Big].$$

The variation with respect to the gravitational variables is

$$\frac{\delta L_m^{(2)}}{\delta v^{\mu a}}=-\frac{\varkappa V}{2k^2}[T^{(0)}_{\mu c}v_a{}^c+T^{(0)}_{ca}v^c{}_\mu-vT^{(0)}_{\mu a}-T^{(1G)}_{\mu a}].$$

Here, $T^{(1G)}_{\mu a}$ is defined as

$$T^{(1G)}_{\mu a}=V_{a\mu}\frac{\partial\Lambda_m}{\partial V^{b\nu}}v^{\nu b}-v_{a\mu}\Lambda_m-\frac{\partial^2\Lambda_m}{\partial V^{a\mu}\partial V^{b\nu}}v^{\nu b} \tag{3.6}$$

and is equal to the linear term in the expansion of $T_{\mu a}$ in the gravitational variables $v_{a\mu}$.

Vanishing of the first variation of the action, i.e., invariance of the total Lagrangian, is achieved when the following additional conditions are satisfied:

$$T^{(0)}_{\mu a}\gamma^\mu\psi^a=0, \tag{3.7}$$

$$(T^{(1G)}_{\mu a}-T^{(0)}_{ca}v^c{}_\mu-T^{(0)}_{\mu c}v^c{}_a)\gamma^\mu\psi^a=0. \tag{3.8}$$

The first of these equations arises from the condition of invariance of $L_2^{(1)}+L_{3/2}^{(0)}+L_m^{(1)}$, and the second from the invariance of the Lagrangian of the next approximation, i.e., including $L_2^{(2)}$, $L_{3/2}^{(1)}$, and $L_m^{(2)}$. Naturally, (3.7) is the zeroth and (3.8) the linear term in the expansion of the equation $T_{\mu a}\gamma^\mu\psi^a=0$ with respect to the gravitational variables.

The linearized Einstein equation now takes the form

$$R^{(1)}_{\mu a}-{}^1/_2V_{a\mu}R^{(1)}-{}^1/_2v_{a\mu}R^{(0)}=\varkappa T^{(1G)}_{\mu a}-ik^2V^{-1}e_a{}^{\lambda\nu\rho}\bar{\psi}_\lambda\gamma_5\gamma_\mu\psi_{\nu;\rho}. \tag{3.9}$$

Equations (1.8), (1.20), and (1.21) for $\omega_{\mu ab}$ and ψ_μ remain formally unchanged. We shall investigate the fulfillment of (3.7) and (3.8) and the equations of motion in Sec. 5, in which we use the Friedmann solutions as background metric.

4. CONFORMAL INVARIANCE OF THE EQUATIONS FOR THE GRAVITINO

The equations for the ψ_μ field on a flat background were derived in Sec. 2. The generalization of these equations in curved space entails difficulties, and one of the achievements of supergravity is their elimination. We consider first the properties of the equations of the ψ_μ field in ordinary gravitational theory.

The simplest generalization of Eqs. (2.7) to a nonflat background (without torsion) is obtained by the replacement in them of the ordinary divergence by the covariant divergence:

$$\gamma^\mu(\psi_{\nu;\mu}-\psi_{\mu;\nu})=0. \tag{4.1}$$

However, Eqs. (4.1) now in general become inconsistent. Their conditions of integrability have the form

$$(R_{\mu\nu}-{}^1/_2g_{\mu\nu}R)\gamma^\mu\psi^\nu=0 \tag{4.2}$$

and are satisfied identically only for $R_{\mu\nu}=0$.

In the general case, Eqs. (4.1) are also not invariant under the gauge transformation

$$\psi_\mu'=\psi_\mu+\alpha_{;\mu}. \tag{4.3}$$

Invariance holds if $R_{\mu\nu}\gamma^\nu\alpha=0$, which for $\alpha\neq0$ is satisfied only in vacuum ($R_{\mu\nu}=0$). Thus, it is only in vacuum that one can use (4.3) to achieve fulfillment of the equations $\psi^\mu_{;\mu}=0$, and by means of the remaining arbitrariness ensure also fulfillment of

$$\gamma^\mu\psi_\mu=0. \tag{4.4}$$

The wave equation that follows from (4.1) with allowance for the condition (4.4) has the form

$$\psi_{\lambda;\rho}{}^\rho+R_{\nu\rho\lambda}{}^\mu\sigma^{\nu\rho}\psi_\mu-{}^1/_4R\psi_\lambda=0.$$

Note that in the presence of (4.4) the consistency conditions (4.2) are satisfied identically in Einstein spaces ($R_{\mu\nu}=\lambda g_{\mu\nu}$) (Ref. 16).

Although the system (4.1) is not completely integrable on a nonvacuum background, it does admit a restricted class of solutions for the ψ_μ field [solutions satisfying the conditions (4.2)]. It is therefore meaningful to investigate the property of conformal invariance of Eqs. (4.1) [or rather of the solutions satisfying (4.2)] without assuming that we necessarily have a vacuum background.

Under conformal transformations of the metric, we have by definition $\tilde{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}$, and $\tilde{V}_{a\mu}=\Omega V_{a\mu}$. The spinor fields must transform in accordance with the spin of the field. In spinor form, the transformation law has the form[17]

$$\tilde{\varphi}_{AB\ldots C}=\Omega^{-1}\varphi_{AB\ldots C}$$

(here, the capital letters are spinor indices). If we combine wherever possible spinor indices into vector indices, we obtain for integral spins

$$\tilde{\varphi}_{\alpha\beta\ldots\gamma}=\Omega^{s-1}\varphi_{\alpha\beta\ldots\gamma},\quad s=0,1,2,\ldots,$$

and for half-integral spins

$$\tilde{\varphi}_{\alpha\beta\ldots\gamma A}=\Omega^{s-3/2}\varphi_{\alpha\beta\ldots\gamma A},\quad s={}^1/_2,{}^3/_2,\ldots,$$

where s is the spin of the field. In accordance with this law, we must set

$$\tilde{\psi}_\mu=\psi_\mu. \tag{4.5}$$

Nontrivial is the question of the conformal transformation of the spinorial connection. The general expression for the connection, which satisfies the condition $\gamma_{\mu;\nu}=0$, is[18]

$$\Gamma_\nu=-{}^1/_2\Omega_{\nu ab}\sigma^{ab}+A_{,\nu}I, \tag{4.6}$$

where I is the identity matrix and A is an arbitrary scalar. The addition of a gradient term to the spinorial connection does not affect the alternation of the second derivatives of the spinors.

In a space without torsion, the transformation law for $\Omega_{\mu ab}$ is determined by the transformation law for $V_{a\mu}$. Under a conformal transformation of A we can assume that $A=A+\frac{1}{2}\ln\Omega$, and then

$$\Gamma_\nu=\Gamma_\nu+{}^1/_2\gamma^\mu(\ln\Omega)_{,\mu}\gamma_\nu.$$

If Eqs. (4.1) are to be conformally invariant, it is necessary and sufficient for the conditions (4.4) to be satisfied and

$$(\ln\Omega)_{,\mu}\psi^\mu=0. \tag{4.7}$$

The condition (4.7) ensures the conformal invariance of the equation $\psi^\mu_{;\mu}=0$. For given ψ_μ, (4.7) restricts the the choice of the conformal transformations under which the solutions of the equations are conformally transformed.

The question of the class of allowed conformal factors Ω arises. Solutions satisfying the conditions

$$\gamma^\mu\psi_\mu=0,\quad w^\mu\psi_\mu=0, \tag{4.8}$$

where $w_\mu=(\ln\Omega)_{,\mu}$, transform conformally. A sufficient condition for Eqs. (4.1) to admit solutions satisfying (4.8) is the existence of a vector w^μ satisfying the system of equations

$$w^{\mu;\nu}=w^\mu a^\nu+bg^{\mu\nu},$$

where a^ν and b are, respectively, an arbitrary vector and an arbitrary scalar. Spaces in which the vector w^μ exists have been investigated earlier.[19] Briefly, the result is as follows: 1) if w^μ is isotropic, then the metric of space belongs to the Petrov types O, N, and III, the vector w^μ in N and III spaces coinciding with a multiple principal null direction; 2) if w^μ is not isotropic, then it exists in type O and in the class of metrics called equidistant.[20]

Applying these conclusions to the solutions ψ_μ of Eqs. (4.1) in conformally flat universes [$\Omega=a(\eta)$]

$$ds^2=a^2(\eta)(d\eta^2-dx^2-dy^2-dz^2) \tag{4.9}$$

we see that the solutions in which we are interested satisfying the conditions (4.8) are identical with the solutions in a flat universe.

Note that we have chosen the simplest law of transformation of the spinorial connection; however, it follows from the fact that it is not defined uniquely that its transformation law may also not be unique. If we assume $\tilde{A}=A+(n+1/2)\ln\Omega$, where n is any real number, then

$$\Gamma_\nu=\Gamma_\nu+{}^1/_2\gamma^\mu(\ln\Omega)_{,\mu}\gamma_\nu+n(\ln\Omega)_{,\nu},$$

and this affects the transformation law of fields of half-integral spins. It is necessary to assume

$$\tilde{\varphi}_{\alpha\beta\ldots\gamma A}=\Omega^{n+s-3/2}\varphi_{\alpha\beta\ldots\gamma A}$$

and, in particular, $\tilde{\psi}_\mu=\Omega^n\psi_\mu$ for $s=3/2$, for (4.1) to be conformally invariant if (4.4) and (4.7) are satisfied. In the special case $n=-1/2$, one can avoid introducing the additional scalar field A.

We now turn to the investigation of the conformal invariance of Eqs. (1.5) for the ψ_μ field in supergravity. In supergravity the simplest expression is chosen for the spinorial connection Γ_ν with gradient term omitted. However, we intend to retain the expression (4.6) and assume that the scalar A does not participate in the supertransformations ($\delta A=0$); its presence will then have no influence on the superinvariance of the action (1.1).

We can now assume that the conformal image of the spinorial connection has the form

$$\Gamma_\nu=\Gamma_\nu+{}^1/_2\gamma^\mu(\ln\Omega)_{,\mu}\gamma_\nu+\Xi_\nu,$$

where $\Xi_\nu=1/2(\tilde{K}_{\nu ab}-K_{\nu ab})\sigma^{ab}$ are the differences between the torsions of the original and transformed spaces; in Eq. (1.5), because of (1.3b), these terms give a cubic nonlinearity. Since we have adopted the transformation law (4.5) for the ψ_μ field, $\Xi_\nu=0$, and in this case torsion does not prevent conformal invariance of the equations. Thus, the equations of the completely integrable system (1.5) are conformally invariant if (4.4) and (4.7) are satisfied.

Of course, the conformal invariance of the equations achieved by the introduction of the additional field is not unconditional (in the sense that the conformal invariance of the equations of electrodynamics is), but the possibility of achieving it in the present case through the freedom in the definition of the spinorial connection appears to us interesting. If we had not introduced the scalar A and had transformed only $\Omega_{\mu ab}$, to have compensated the terms linear in the ψ_μ field in (1.5) we should have had to have taken $\tilde{\psi}_\mu = \Omega^{-1/2}\psi_\mu$, and then the cubic nonlinearity would have made a contribution violating the conformal invariance of the equations.

5. SUPERGRAVITY ON THE BACKGROUND OF HOMOGENEOUS ISOTROPIC COSMOLOGICAL MODELS

We now consider the theory constructed in the first section on the background of Friedmannian models. The system of equations (3.9) can be simplified by substituting the expression for small torsion from (1.20). Then the antisymmetric parts of the linearized Einstein equations (3.9) cancel each other, as in the exact theory. The symmetric parts give equations for the gravitational perturbations with source on the right-hand side containing a term corresponding to the gravitational perturbation of the matter, and also the energy-momentum tensor of the ψ_μ field and terms transferred from the left-hand side deriving from the torsion. Taken all together, the terms corresponding to the ψ_μ field form the energy-momentum tensor in the second-order formalism.

We go over to the metric representation (setting $v_{[\mu\nu]} = 0$), and then the symmetric part of Eq. (3.9) takes the form

$$R_\mu^{(1)\nu} - {}^1/_2\delta_\mu^\nu R^{(1)} = \varkappa T_\mu^{(1G)\nu} + \mathcal{T}_\mu^{\ \nu}, \tag{5.1}$$

where

$$\mathcal{T}_\mu^{\ \nu} = {}^1/_2 ik^2\bar{\psi}^\lambda(\gamma^\nu\psi_{\lambda;\mu} + \gamma_\mu\psi_{\lambda;}{}^\nu) - ik^2\bar{\psi}^\lambda(\gamma^\nu\psi_{\mu;\lambda} + \gamma_\mu\psi^\nu{}_{;\lambda}).$$

We shall solve Eq. (1.18) for the ψ_μ field by successive approximation. We set

$$\psi_\mu = \psi_\mu^{(1/2)} + \psi_\mu^{(3/2)};\quad \psi^{(1/2)} \sim h^{1/2},\ \psi^{(3/2)} \sim h^{3/2}.$$

Then in the principal approximation

$$\gamma^\mu(\psi_{\nu;\mu} - \psi_{\mu;\nu}) = 0. \tag{5.2}$$

Equations (5.2) on a Friedmannian background admit solutions satisfying the conditions (4.8) if as w^μ we take the vector $u^\mu = (1/a, 0, 0, 0)$. Since the vector u_μ is collinear to the vector $(\ln\Omega)_{,\mu}$, the equations are conformally invariant under transformations with $\Omega = a(\eta)$, and the corresponding solutions are identical with the solutions in a flat universe, in which Eqs. (5.2) admit solutions in the form of plane waves. We emphasize that the conformal invariance of Eqs. (5.2) also holds without introduction of the field A if the transformation law $\tilde{\psi}_\mu = \Omega^{-1/2}\psi_\mu$ is adopted.

We now take into account the fact that the ψ_μ variables are operators and not c-number functions. Then

$$\psi_\mu = \sum_{\mathbf{k}} (a_{\mathbf{k}}\varphi_\mu(\mathbf{k})\exp(-ik_\lambda x^\lambda) + a_{\mathbf{k}}^+\varphi_\mu(\mathbf{k})\exp(ik_\lambda x^\lambda)),$$

where, as usual, $a_{\mathbf{k}}^+$ and $a_{\mathbf{k}}$ are the operators of creation and annihilation of a particle with momentum $\mathbf{k}$ (we have here taken into account the Majorana property), these operators satisfying anticommutation relations; φ_μ is a normalized spinor. From the gauge conditions (4.8), there follow

$$\gamma^\mu\varphi_\mu = 0,\quad u^\mu\varphi_\mu = 0. \tag{5.3}$$

We can now calculate the operator $\mathcal{T}_\mu^\nu$ on the right-hand side of Eqs. (5.1). The left-hand side of the equations can be interpreted either as an operator expression, and then the operators for the gravitons are coupled to the operators for the gravitinos, or as a c-number expression, and it is then assumed that on the right-hand side we have the expectation value of the energy-momentum tensor of the ψ_μ field with respect to some state.

Since we have established that the equations for the ψ_μ field are conformally invariant, gravitinos are not created, and if they were absent initially they do not appear. Then the expectation value of $\mathcal{T}_\mu^\nu$ with respect to the vacuum state vanishes[5]:

$$\langle 0|\mathcal{T}_\mu^{\ \nu}|0\rangle = 0,$$

and therefore the source associated with the ψ_μ field vanishes in Eqs. (5.1). They reduce to the ordinary equations for gravitational perturbations on a Friedmannian background:

$$\begin{aligned}&{}^1/_2(-h_\mu{}^\nu{}_{;\lambda}{}^\lambda + h_\mu{}^\lambda{}_{;\lambda}{}^\nu + h^{\nu\lambda}{}_{;\lambda\mu} - h_{;\mu}{}^\nu + h^{\nu\lambda}R^{(0)}_{\lambda\mu} + h_\mu{}^\lambda R_\lambda{}^{(0)\nu} + 2R^{(0)}_{\mu\lambda\sigma}{}^\nu h^{\lambda\sigma})\\ &\quad - {}^1/_2\delta_\mu^\nu(h^{\lambda\sigma}{}_{;\lambda\sigma} - h_{;\lambda}{}^\lambda - h^{\lambda\sigma}R^{(0)}_{\lambda\sigma}) = \varkappa T_\mu^{(1G)\nu}.\end{aligned}$$

An expression for $T_\mu^{(1G)\nu}$ can be obtained either from (3.6) or directly from (3.4) under the assumption $\delta p = \delta\rho = \delta u^i = 0$. On a Friedmannian background, we can, because of the gauge freedom, make $h_{\mu\nu}$ satisfy the gauge conditions

$$h_\mu{}^\nu{}_{;\nu} = 0,\quad h_{\mu\nu}u^\nu = 0,\quad h = 0.$$

Then $T_\mu^{(1G)\nu} = 0$ (with mixed indices), and we arrive at the ordinary equations for free gravitational waves. Therefore, all the conclusions about superadiabatic amplification of gravitational waves and the production of gravitons remain valid on a Friedmannian background (Ref. 6).

We now verify whether the obtained solutions are consistent with the requirements of superinvariance. For the hydrodynamic tensor, the condition $T_{\mu\nu}\gamma^\mu\psi^\nu = 0$ is formulated as

$$(\rho + p)u_\mu\gamma^\mu(u^\nu\psi_\nu) - p\gamma^\mu\psi_\mu = 0. \tag{5.4}$$

The conditions (3.7) and (3.8) are, respectively, the principal and following terms in the expansion of this equation. Equations (5.4) are satisfied if $\gamma^\mu\psi_\mu = 0$ and $u^\mu\psi_\mu = 0$. We assume that the condition $\gamma^\mu\psi_\mu = 0$ is necessary for the selection of solutions describing pure $s = 3/2$ spin states, and we assume that it is satisfied. The condition $u^\mu\psi_\mu = 0$ has the form $(u^{(0)\mu} + \delta u^\mu)\psi_\mu = 0$ and is also satisfied since $u^\mu\psi_\mu = 0$ on a Friedmannian background, and also $\delta u^\mu = 0$ for gravitational-wave perturbations. Thus, in the considered approximation (5.4) is satisfied.

In conclusion, it should be noted that in (5.1) we have used the ψ_μ field of the lowest approximation, i.e., we have taken into account terms $\psi_\mu \sim \psi_\mu^{(1/2)} \sim h^{1/2}$. Terms of the following approximation ($\psi_\mu^{(3/2)} \sim h^{3/2}$) can be found from Eq. (1.5), in which the geometry is represented in the form of a sum of the Friedmannian metric and small gravitational-wave corrections, and allowance is also made for a small torsion ($s \sim \psi^2 \sim h$). In this approximation, the metric is not conformally flat. In Eq. (5.1), there appears the source $\mathscr{T}^\nu_\mu \sim h^2$, but it is of the same order as the terms describing the nonlinear interaction of the gravitational waves themselves. These effects require a special treatment.

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1) A review of papers devoted to supersymmetries, the first attempts to include gravitation in the supersymmetry scheme, and the further development of this theory can be found in Ref. 1.

2) It is interesting to note that attempts have been made to find generalized (non/Einstein) theories of gravitation giving conformally invariant equations for weak gravitational waves. It is found that under the not particularly restrictive and fairly reasonable requirements that are imposed on these theories such theories do not exist.[8]

3) The semicolon as a subscript denotes the total covariant derivative with allowance for coordinate, tetrad, and spinor indices. If the differentiated quantity does not have indices of a particular kind, the corresponding connection is absent in the derivative.

4) Such constructions are called the "background-field method."[9]

5) A possible conformal anomaly of the trace of the energy-momentum transfer has a dynamical influence on the background metric but not on transverse gravitational perturbations.

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