

# The effects of the interaction between ion-sound solitons and resonance particles in a plasma

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We give a simple derivation of the perturbed Boussinesq and Korteweg–de Vries equations which take into account the effects of the interaction between ion-sound waves and resonance particles. We obtain for the perturbation terms explicit expressions which describe the resonance interaction of solitons with plasma particles in cases which are of real interest from the point of view of laboratory and numerical experiments. We obtain the equations which describe the change in the soliton parameters due to their interaction with resonance particles in a form which is suitable for comparisons with experiments.

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## 1. INTRODUCTION

From the time of the publication of Ott and Sudan's paper<sup>1</sup> there have appeared a rather large number of papers devoted to theoretical and experimental studies of the interaction between ion-sound solitons and resonance particles in a plasma (see, e.g., Refs. 2 to 7 and the literature cited there). However, when one compares the theoretical results with experiments or different theoretical papers with one another, one discovers a number of contradictions and paradoxes which are caused often by not very clear statements about the applicability domains of the various approaches and sometimes simply by errors. It becomes necessary in that connection to construct a relatively simple theory which considers from a single point of view different limiting cases, makes precise their domain of applicability, and contains a number of new results which allow an experimental check. The present paper is a step in that direction.

The paper is constructed as follows. We give in Sec. 3 a simple derivation of the Boussinesq equations<sup>8</sup> which are supplemented by terms which taken into account the resonance interaction between waves and particles. From them there follows, in particular, the perturbed Korteweg–de Vries (KdV) equation with a right-hand side describing the resonance interaction. In form it is the same as the equation obtained earlier<sup>3,4</sup> by the more cumbersome reduced perturbation method. In Sec. 4 we evaluate the term which describes the resonance interaction in various limiting cases. In particular, we find for short times after the switching on of the field ( $t \ll \tau_r$ , where  $\tau_r$  is the characteristic resonance interaction time which as to order of magnitude equals the time it takes a resonance particle to traverse a characteristic wavelength) the Ott–Sudan equation.<sup>1</sup> However, it turns out that for solitons  $\tau_r$  is small compared to the characteristic time during which the resonance interaction acts upon them. We therefore after that consider the case  $t \gg \tau_r$ , and for that case we obtain the appropriate equations which are the basis of the further study. We apply in Sec. 5 the perturbation theory developed earlier<sup>9–11</sup> using the perturbation terms evaluated in Sec. 4 to a study of the evolution of the

solitons. As a result we obtain equations describing the change in the amplitude of the solitons and the "growth of soliton tails" due to the resonance interaction with particles.

## 2. BASIC EQUATIONS

We shall start from the equations

$$\frac{\partial f_i}{\partial t} + v \frac{\partial f_i}{\partial r} - \frac{e}{m_i} \frac{\partial \varphi}{\partial r} \frac{\partial f_i}{\partial v} = 0, \quad (2.1)$$

$$\frac{\partial f_e}{\partial t} + v \frac{\partial f_e}{\partial r} + \frac{e}{m_e} \frac{\partial \varphi}{\partial r} \frac{\partial f_e}{\partial v} = 0, \quad (2.2)$$

$$\nabla^2 \varphi = 4\pi e \int [f_e(r, v, t) - f_i(r, v, t)] dv, \quad (2.3)$$

where  $f_i$  and  $f_e$  are the ion and electron distribution functions and  $\varphi$  is the electric field potential. We assume in what follows that the conditions

$$kD \ll 1, \quad T_e \gg T_i \quad (2.4)$$

are fulfilled, where  $T_e$  and  $T_i$  are the temperatures of the two components,  $k$  is the wavenumber or, in the case of non-periodic waves, a character inverse wavelength, and  $D$  is the Debye radius.

In the linear approximation we get from (2.1) to (2.4) the well known dispersion relations which in the first approximation in  $(kD)^2$  and  $T_i/T_e$  give, in particular,

$$\text{Re } \omega \approx c_0 k - \beta k^2, \quad (2.5)$$

$$c_0 = (T_e/m_i)^{1/2} (1 + 3T_i/2T_e), \quad \beta = 1/2 c_0 D^2 \approx c_0^2 / 2\omega_{pi}^2. \quad (2.6)$$

We shall call  $c_0$  the modified ion-sound speed (in contrast to the normal one  $c_s = (T_e/m_i)^{1/2}$ ), and  $\beta$  the dispersion parameter. We assume in what follows that

$$\text{Im } \omega / \text{Re } \omega \sim \varepsilon \ll 1, \quad \delta n/n_0 \sim D^2 k^2 \sim \varepsilon_1 \ll 1 \quad (2.7)$$

( $\delta n = n - n_0$ ) and we shall stick to the following principle: after the transition to quantities of order unity, i.e.,  $\delta \tilde{n} = \varepsilon_1^{-1} \delta n$ ,  $\tilde{x} = \varepsilon_1^{-1/2} x$ , and so on, we retain in the equations only terms containing  $\varepsilon$  and  $\varepsilon_1$  to first degree, neglecting higher order terms ( $\varepsilon_1^2$ ,  $\varepsilon \varepsilon_1$ ,  $\varepsilon_1 T_i/T_e$ , and so on). The characteristic scale for the velocity of the non-linear waves will then be of order  $c_0$ . Since by virtue of (2.4)

$$v_i \ll c_0 \ll v_e, \quad v_{i,e} = (2T_{i,e}/m_{i,e})^{1/2},$$

we can to a good accuracy separate hydrodynamics effects (including dispersive effects) which are determined by the motion of the bulk of the particles with velocities of the order of thermal velocities from effects due to the interaction of the waves with resonance particles which have velocities rather close to the wave velocity. The degree of closeness will be made more precise in what follows.

Bearing this in mind we define the "mass" velocity and "mass" density of the system through the expressions

$$\mathbf{v}(\mathbf{r}, t) = \frac{1}{n} \int \mathbf{v} f_i(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}, \quad n(\mathbf{r}, t) = \int f_i(\mathbf{r}, \mathbf{v}, t) d\mathbf{v}, \quad (2.8)$$

where  $f$  here indicates that the integration is over the non-resonance region of velocity space (in the linear approximation this corresponds to an integration in the sense of the principal value).

Bearing in mind the conditions stated above we get from (2.1) by standard methods the following hydrodynamic equations:

$$\frac{d\mathbf{v}}{dt} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{e}{m_i} \nabla\phi - \frac{3T_i}{m_i} \nabla \ln n, \quad (2.9)$$

$$\frac{\partial n}{\partial t} + \nabla(n\mathbf{v}) = 0. \quad (2.10)$$

We now turn to the kinetic equation for the electrons. Bearing in mind that the characteristic scale of their acceleration in the given potential  $\phi(\mathbf{r}, t)$  is larger than the ion acceleration by a factor  $m_i/m_e$  we can in (2.2) when analyzing the motion of the non-resonance electrons neglect the time derivative and this leads to  $f_e(\mathbf{v}, \mathbf{r}) = f_e(\frac{1}{2}v^2 + e\phi/m_e)$ , where  $f_e(\frac{1}{2}v^2)$  determines the electron distribution function when there is no field. We shall assume that the latter is Maxwellian. In that case

$$n_e(\phi) = \int f_e d\mathbf{v} = n_0 \exp\left(\frac{e\phi}{T_e}\right), \quad (2.11)$$

where  $n_0$  is the equilibrium density.

The Poisson equation (2.3) can now be written in the form

$$\nabla^2 \phi = 4\pi en_0 [\exp(e\phi/T_e) - n/n_0] + 4\pi e \int_{(\mathcal{r})} (f_e - f_i) d\mathbf{v}, \quad (2.12)$$

where  $(\mathcal{r})$  indicates integration over the resonance region of velocity space which is excluded from the quantities  $n$  and  $n_e$  by virtue of the definitions (2.8) and (2.11).

### 3. PERTURBED BOUSSINESQ AND KORTEWEG-de VRIES EQUATIONS

We introduce the notation:

$$\Phi = e\phi/m_i, \quad (3.1)$$

$$S = \frac{1}{n_0} \int_{(\mathcal{r})} (f_e - f_i) d\mathbf{v}. \quad (3.2)$$

We then have from (2.12)

$$\nabla \left( \frac{\Phi}{c_0^2} \right) = \left( \frac{n}{n_0} + \frac{\nabla^2 \Phi}{\omega_{pi}^2} + S \right)^{-1} \left( \frac{\nabla n}{n_0} + \frac{\nabla \nabla^2 \Phi}{\omega_{pi}^2} + \nabla S \right). \quad (3.3)$$

One verifies that  $S \ll \varepsilon$  (see (2.7)). Following the above formulated principle we must here retain only

those terms which after the transition to quantities of order unity do not contain powers of  $\varepsilon$  and  $\varepsilon_1$  higher than the first one. As a result we get

$$\nabla \frac{\Phi}{c_0^2} \approx \frac{\nabla n}{n} + D^2 \nabla \nabla^2 \frac{n}{n_0} + \nabla S. \quad (3.4)$$

Taking  $\nabla\phi$  from (3.4) and substituting it into (2.9) we get

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -c_0^2 \frac{\nabla n}{n} - 2\beta c_0 \nabla \nabla^2 \frac{n}{n_0} - c_0^2 \nabla S. \quad (3.5)$$

We shall call this equation the perturbed Boussinesq equation (the perturbation is given by the last term on the right-hand side). Without the last two terms on the right-hand side it corresponds to the hydrodynamic equation without dispersion with an adiabatic index  $\gamma = 1$ . The second term on the right-hand side describes the ion-sound dispersion and the third one corresponds to the interaction of a wave with the resonance particles.

In the linear approximation (3.5), (2.10) lead to the well known expressions for the frequency and growth rate of ion-sound waves [in particular, to (2.5)]. For the evaluation of the integral (3.2) we must then solve the linearized Eqs. (2.1), (2.2) with  $\phi \sim \exp(ikx - i\omega t)$  and substitute into (3.2) those parts of the functions  $f_e$  and  $f_i$  which are proportional to  $\delta(\omega - \mathbf{k} \cdot \mathbf{v})$ .

To obtain the generalized  $KdV$  equation corresponding to the set (3.5), (2.10) it is necessary to restrict ourselves to the one-dimensional flow of the type of a quasi-simple wave (see Ref. 8, Sec. 15), i.e., look for a solution of that set in the form

$$n(x, t) = n(V) + \psi(x, t), \quad V = V(x, t), \quad (3.6)$$

where  $n(V)$  is the same function as for a simple wave propagating in the positive direction and  $\psi$  is a quantity of the order of the two last terms in (3.5). We find the function  $n(V)$  by substituting (3.6) into the set (3.5), (2.10) where we put  $\beta = 0$ ,  $S = 0$ . As a result we get

$$dn(V)/dV = n(V)/c_0. \quad (3.7)$$

Substituting (3.6) into (3.5) and using the fact that with the accuracy used for  $\psi$  we can put  $\psi_t \approx c_0 \psi_x$ , we have

$$\frac{\partial \psi}{\partial x} = -\frac{n_0}{c_0^2} \left( \beta \frac{\partial^2 V}{\partial x^2} + \frac{c_0^2}{2} \frac{\partial S}{\partial x} \right), \quad (3.8)$$

$$\frac{\partial V}{\partial t} + (c_0 + V) \frac{\partial V}{\partial x} + \beta \frac{\partial^2 V}{\partial x^2} = -\frac{c_0^2}{2} \frac{\partial S}{\partial x}. \quad (3.9)$$

Equation (3.9) is the required perturbed  $KdV$  equation where the right-hand side describes the interaction between the ion-sound waves and the resonance particles in the plasma. This equation was first obtained by other means in Refs. 3 and 4.

Using Eqs. (3.4) and (3.6) to (3.8) we can express  $V$  in terms of  $n$  or  $\Phi$  and obtain similar equations for these two quantities. For instances,

$$\frac{\partial \Phi}{\partial t} + \left( c_0 + \frac{1}{c_0} \Phi \right) \frac{\partial \Phi}{\partial x} + \beta \frac{\partial^2 \Phi}{\partial x^2} = -\frac{c_0^2}{2} \frac{\partial S[\Phi]}{\partial x}, \quad (3.10)$$

where  $S[\Phi]$  is a functional of  $\Phi$  and is determined in principle by the solution of the kinetic Eqs. (2.1), (2.2) for a given profile of  $\Phi$  and by Eq. (3.2).

#### 4. EVALUATION OF $S[\Phi]$

In this section we obtain asymptotic expressions for  $S[\Phi]$  for a number of limiting cases, assuming that  $\Phi$  has the form of a quasi-stationary wave, i.e.,

$$\Phi = \Phi \left( x - \int V_{ph}(t) dt; \varepsilon x, \varepsilon t \right),$$

where  $V_{ph}$  is the phase velocity of the wave,  $dV_{ph}/dt \sim \varepsilon \ll 1$ . Let  $\delta$  be the characteristic spatial dimension of the wave and let the time of variation of the parameters of the potential  $\Phi$  be appreciably larger than  $\delta/V_{ph}$ . We can then change to a local system of reference moving with the velocity  $V_{ph}$  and assume that in the adiabatic approximation the profile of the wave does not change. We shall denote the velocity of the resonance particles in that system by  $v$  and the coordinate by  $\xi$ . We introduce the "energy" which in the adiabatic approximation is an integral of motion, i.e.,  $dE/dt \sim \varepsilon$ ,

$$E_{i,e} = v^2/2 + U_{i,e}(\xi), \quad (4.1)$$

$$U_i(\xi) = \Phi(\xi), \quad U_e(\xi) = -\frac{m_i}{m_e} \Phi(\xi). \quad (4.2)$$

The kinetic Eqs. (2.1) and (2.2) become in the frame of reference considered (neglecting terms  $\sim \varepsilon$ )

$$\left( \frac{\partial f}{\partial t} \right)_{E,\xi} + v_{\pm}(E, \xi) \left( \frac{\partial f}{\partial \xi} \right)_{E,t} = 0, \quad (4.3)$$

where  $f = f_i, f_e$  (we drop the indices  $i$  and  $e$  where they are inessential) and

$$v_{\pm}(E, \xi) = \pm \{2[E - U(\xi)]\}^{1/2}. \quad (4.4)$$

Here and henceforth the upper and lower signs refer to particles moving, respectively, in the positive and negative directions. If  $\Phi_0$  is a characteristic wave amplitude we can define the widths of the resonance regions in the ion and electron velocity spaces by

$$(\Delta v)_{i,e} = (2\Phi_0)^{1/2}, \quad (\Delta v)_{r,e} = (m_i/m_e)^{1/2} (\Delta v)_{r,i}. \quad (4.5)$$

The resonance interaction time  $\tau_r$  is defined as the time it takes a resonance particle to traverse a distance  $\delta$  in the wave system, i.e.,

$$\tau_{r,i} = \delta (2\Phi_0)^{-1/2}, \quad \tau_{r,e} = (m_i/m_e)^{1/2} \tau_{r,i}. \quad (4.6)$$

We assume now that the wave field was switched on at time  $t = 0$  and we rewrite (3.2) in the form

$$S = N_i' - N_e', \quad N^r = \frac{1}{n_0} \int_{(r)} (f - f_0) dv. \quad (4.7)$$

Changing to integration over the energy, we get

$$N^r = \frac{1}{n_0} \sum_{(\pm)} \left\{ \int_{v(t)}^{v_0} \frac{dE}{(2[E - U(\xi)])^{1/2}} [f_0(v_{\pm}(E, \xi_0^{\pm})) - f_0(v_{\pm}(E, \xi))] \right. \\ \left. + \int_{v_0}^{\infty} \frac{f_0(v_{\pm}(E, \xi_0^{\pm})) - f_0(v_{\pm}(E, \xi))}{(2[E - U(\xi)])^{1/2}} dE \right\}. \quad (4.8)$$

We assume here that  $U_0 = \max U(\xi)$  while  $E_r$  denotes the effective energy width of the resonance region  $E_{r,i} \sim 2\Phi_0$ ,  $E_{r,e} \sim 2(m_i/m_e)\Phi_0$  (the basic results obtained below are independent of  $E_r$ ). In deriving Eq. (4.8) we used the Liouville theorem

$$f(v, \xi, t) = f_0(v_{\pm}(E, \xi_0^{\pm})),$$

where  $f_0(v)$  is the unperturbed distribution function and

$\xi_0^{\pm} = \xi_0^{\pm}(\xi, E, t)$  is the initial coordinate of a particle which at time  $t$  is at the point  $\xi$  and has a velocity  $v_{\pm}(E, \xi)$ .

Expression (4.8) is general. It can, for instance, be applied both to periodic and to solitary waves. In this paper we restrict ourselves to the latter. In the case where the solitary wave is a potential well it is convenient for us to change the notation, bearing in mind that  $U_0 = \min U(\xi) < 0$  and  $\max U(\xi) = 0$ . In that case we must replace in (4.8)  $U_0$  by zero and the first integral will refer to trapped particles and the second one to untrapped particles (see figure).

We evaluate Eq. (4.7) for two limiting cases:  $t \ll \tau_r$  and  $t \gg \tau_r$ , where  $\tau_r$  is defined in (4.6).

a) *Short time interval* ( $t \ll \tau_r$ ). In that case we may assume that

$$|v(E, \xi) - v(E, \xi_0)| \ll (2U_0)^{1/2}, \quad \xi_0^{\pm} \approx \xi \pm (2E)^{1/2} t. \quad (4.9)$$

We shall also assume that in the resonance region, with good accuracy,

$$f_0(v) = f_0(v_r) + (v - v_r) f_0'(v_r), \quad (4.10)$$

where  $v_r$  corresponds to exact resonance (in the reference frame considered  $v_r = 0$ , in the frame in which the plasma is at rest  $v_r = v_{ph}$ ). In that case

$$N^r(\xi) \approx \frac{f_0'(v_r)}{n_0} \int_{\max U}^{\infty} dE \{ [E - U(\xi - (2E)^{1/2} t)]^{1/2} - [E - U(\xi + (2E)^{1/2} t)]^{1/2} \} \\ \approx \frac{f_0'(v_r)}{2n_0} \int_{\max U}^{\infty} \frac{dE}{E} [U(\xi + (2E)^{1/2} t) - U(\xi - (2E)^{1/2} t)]. \quad (4.11)$$

We have here taken into account the fact that at  $t \ll \tau_r$  the main contribution to  $N^r(\xi)$  comes from the high energy region ( $E \gg U_0$ ), and have expanded the integrand in (4.11) in powers of  $U/E$ . Using the relation

$$\int_{\pm(2\alpha)^{1/2}}^{\infty} \frac{dE}{E} U(\xi \pm (2E)^{1/2} t) = 2 \int_{\xi' - \xi}^{\xi' + \xi} \frac{d\xi'}{\xi' - \xi} U(\xi'),$$

we find that for  $t \ll \tau_r$

$$N^r = \frac{f_0'(v_r)}{n_0} \int_{-\infty}^{\infty} \frac{d\xi'}{\xi' - \xi} U(\xi'). \quad (4.12)$$

If now  $t \ll \tau_{r,e}$  (and, according to (4.6),  $\tau_{r,e} \ll \tau_{r,i}$ ) we can write the quantity (4.7) in the form

$$S[\Phi] = \frac{1}{n_0} \left[ f_{0i}'(v_{r,i}) + \frac{m_i}{m_e} f_{0e}'(v_{r,e}) \right] \int_{-\infty}^{\infty} \frac{dx' \Phi(x', t)}{x' - x}. \quad (4.13)$$

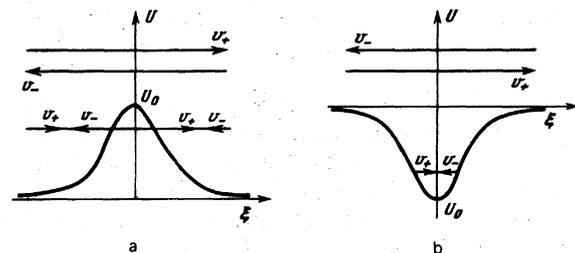


FIG. 1. Free and reflected particles in the field of a potential hump (a), free and trapped particles in the field of a potential well (b). For ion-sound solitons case a is realized for ions ( $U = U_i = \Phi$ ), case b for electrons ( $U = U_e = -(m_i/m_e)\Phi$ ).

This expression is the same as the one obtained by Ott and Sudan<sup>1</sup> from other considerations. Our approach shows that the domain of its applicability is restricted by the condition  $t \ll (m_e/m_i)^{1/2} \tau_r$ , which is a very rigid one as will become clear from what follows.

b) "Long" time interval ( $t \gg \tau_r$ ). It is in that case convenient to split the unperturbed distribution function (in the wave frame) into an even and an odd part:

$$f_1(v) = 1/2 [f_0(v) - f_0(-v)], \quad f_2(v) = 1/2 [f_0(v) + f_0(-v)]. \quad (4.14)$$

Similarly we put

$$N^r = N_1^r + N_2^r, \quad N_1^r = N^r \{f_1\}, \quad N_2^r = N^r \{f_2\}. \quad (4.15)$$

The functionals  $N^r \{f_n\}$  are defined here by Eq. (4.8) with  $f_n$  ( $n=1, 2$ ) replacing  $f_0$ .

We assume now that  $U(\xi)$  has the shape of a solitary positive pulse ( $\max U(\xi) = U_0 > 0$ ,  $U(\pm\infty) = 0$ ; see curve *a* in the figure). We consider  $N_1^r$ . The contribution from the free particles (i.e.,  $E > U_0$ ) is vanishingly small in that case. Indeed, at  $t \gg \tau_r$  we can put for such particles

$$v_{\pm}(E, \xi_0^{\pm}) \approx v_{\pm}(E, \mp\infty) = \pm(2E)^{1/2}, \quad (4.16)$$

so that

$$\sum_{(\pm)} f_1(v_{\pm}(E, \xi_0^{\pm})) = \sum_{(\pm)} f_1(v_{\pm}(E, \xi)) = 0. \quad (4.17)$$

The quantity  $N_1^r$  thus contains only a contribution from the reflected particles and can be written in the form

$$N_1^r(\xi) \approx -\frac{1}{n_0} \sum_{(\pm)} \int_{v(\xi)}^{v_0} \frac{dE I_{\pm}(\xi, E)}{(2[E-U(\xi)])^{1/2}}, \quad (4.18)$$

$$I_{\pm}(\xi, E) = f_1(v_{\pm}(E, \xi)) - f_1(v_{\pm}(E, \xi_0^{\pm})) = \int_{\xi_0^{\pm}}^{\xi} \frac{\partial f_1(v_{\pm}(E, \xi'))}{\partial \xi'} d\xi'. \quad (4.19)$$

Let now  $\xi < \xi_m$ , where  $U(\xi_m) = U_0$  (curve *a* in the figure). In that case only those particles can have positive velocities at the point  $\xi$  which have not been reflected. For such particles we can for  $t \gg \tau_r$  put in (4.18)  $\xi_0^+ = -\infty$ , i.e.,

$$I_+(\xi, E) = f_1(\{2[E-U(\xi)]\}^{1/2}) - f_1((2E)^{1/2}). \quad (4.20)$$

Particles with negative velocities at the point  $\xi$  must necessarily be reflected at sufficiently large  $t$ , i.e., we can write

$$I_-(\xi, E) = \int_{\xi_0^-}^{\xi} \frac{\partial f_1(v_-(E, \xi'))}{\partial \xi'} d\xi' + \int_{\xi_0^-}^{\xi} \frac{\partial f_1(v_-(E, \xi'))}{\partial \xi'} d\xi', \quad (4.21)$$

where  $\xi(E)$  is the point of reflection, i.e.,  $U(\xi(E)) = E$ . Putting  $\xi_0^- = -\infty$  in (4.21) and using the fact that  $v_-(E, \xi(E)) = 0$ , we get

$$I_-(\xi, E) = -f_1((2E)^{1/2}) - f_1(\{2[E-U(\xi)]\}^{1/2}).$$

As a result for  $\xi < \xi_m$  (4.18) becomes

$$N_1^r(\xi) = \frac{2^{1/2}}{n_0} \int_{v(\xi)}^{v_0} \frac{dE f_1((2E)^{1/2})}{[E-U(\xi)]^{1/2}}. \quad (4.22)$$

Similar considerations for  $\xi > \xi_m$  lead to the conclusion that  $N_1^r(\xi)$  in that case differs from (4.22) by its sign. The general result for a potential hump has thus the form

$$N_1^r(\xi) = -\frac{2^{1/2}}{n_0} \int_{v(\xi)}^{v_0} \frac{dE f_1((2E)^{1/2})}{[E-U(\xi)]^{1/2}} \text{sign}(\xi - \xi_m). \quad (4.23)$$

We now assume that  $U(\xi)$  has the form of a potential well (curve *b* in the figure) and  $t \gg \tau_r$ . In that case Eqs. (4.16) and (4.17) remain the same for free particles as these particles do not contribute to  $N_1^r$ , as before. As regards the trapped particles which are multiply reflected from the walls of the well, for them  $f_1(v_{\pm}(E, \xi_0^{\pm}))$  is a fast-oscillating function of  $\xi$  (the period of the oscillations tends to zero with  $\tau_r/t$ ). It is clear that in that case one must replace the quantity  $f_1(v_{\pm}(E, \xi_0^{\pm}))$  by its average which equals zero. The contribution from the trapped particles is thus also equal to zero. Hence, in the case of a well

$$N_1^r(\xi) = 0, \quad t \gg \tau_r. \quad (4.24)$$

Finally, using (4.8), (4.14), and (4.15) we get the following expression for  $N_2^r(\xi)$  (for a potential hump)

$$N_2^r[U] = \frac{2^{1/2}}{n_0} \int_U^{E_r} \frac{dE}{(E-U)^{1/2}} [f_2((2E)^{1/2}) - f_2(\{2(E-U)\}^{1/2})] \\ = \frac{2^{1/2}}{n_0} \int_U^{\infty} \frac{dE f_2((2E)^{1/2})}{(E-U)^{1/2}} - 1 - N^{nr}[U], \quad (4.25a)$$

$$N^{nr}(U) = \frac{2^{1/2}}{n_0} \int_{E_r}^{\infty} \frac{f_2((2E)^{1/2}) - f_2(\{2(E-U)\}^{1/2})}{(E-U)^{1/2}} dE. \quad (4.25b)$$

One checks easily that  $N^{nr}$  is nothing but the negative change in the density of non-resonance particles.

In the case of a potential well we have

$$N_2^r[U] = \frac{n^{tr}}{n_0} + \frac{2^{1/2}}{n_0} \int_U^{\infty} \frac{f_2((2E)^{1/2})}{(E-U)^{1/2}} dE - N^{nr}[U] - 1. \quad (4.25c)$$

Here  $n^{tr}$  is the average density of the particles trapped in the potential well and  $N^{nr}$  is determined by the same Eq. (4.25b). We shall use Eqs. (4.25) below.

We emphasize the substantial difference between  $N_1^r$  and  $N_2^r$ . If  $U$  is an even function of  $\xi - \xi_m$  (for instance, for a soliton),  $N_1^r(\xi)$  is an odd function and  $N_2^r(\xi)$  an even one. This fact plays an important role in what follows leading to the fact that  $N_2^r(\xi)$  does not play a role in a number of important effects in the evolution of solitons. We note also that  $N_2^r$  vanishes if we use Eq. (4.10) as only even powers of  $v$  contribute to  $N_2^r$ .

In concluding this section we transform Eq. (4.23) into a more convenient form, using the unperturbed distribution function in the laboratory frame of reference and introducing dimensionless variables. We put

$$f_0(v) = (n_0/v_T) F((v+V_{ph})/v_T), \quad (4.26)$$

where  $F(\sigma)$  is a dimensionless function and  $v_T$  the thermal velocity. Changing to the dimensionless variables

$$v = (2U_0/v_T^2)^{1/2} \mu, \quad \mu = V_{ph}/v_T, \quad (4.27)$$

$$\chi(\xi) = U(\xi)/U_0, \quad (4.28)$$

we have for ions in the field of a potential hump

$$N_{1i}^r[\chi(\xi)] = -v_i \int_0^{(1-\chi)^{1/2}} dw [F_i(\mu_i + v_i(w^2 + \chi)^{1/2}) \\ - F_i(\mu_i - v_i(w^2 + \chi)^{1/2})] \text{sign}(\xi - \xi_m). \quad (4.29)$$

For sufficiently small  $v_i$  the expression in the square

brackets can be expanded in a series which, restricting ourselves to the first term, gives

$$N_{1e} \approx -v_i^2 F_1'(\mu_i) \left\{ (1-\chi)^{\mu_i} + \chi \ln \left[ \frac{1+(1-\chi)^{\mu_i}}{\chi^{\mu_i}} \right] \right\} \text{sign}(\xi - \xi_m). \quad (4.30)$$

One must then take into account that the definitions (4.27) lead to

$$\mu_i \approx \frac{c_0}{v_i} = \left( \frac{T_e}{2T_i} \right)^{1/2} > 1, \quad v_i = \left( \frac{2\Phi_0}{v_i^2} \right)^{1/2} = \left( \frac{\Phi_0}{c_0^2} \right)^{1/2} \left( \frac{T_e}{T_i} \right)^{1/2}. \quad (4.31)$$

If  $F_1(\sigma)$  is an analytical function of  $\sigma^2$  (for instance, Maxwellian), the validity of the expansion of  $F_1$  in powers of  $v_i$  in (4.30) then has the form  $v_i \ll 1/\mu_i$ , i.e.,

$$v_i \mu_i = (T_e/T_i) (\Phi_0/c_0^2)^{1/2} \ll 1. \quad (4.32)$$

We see that this is, in general, a rather rigid condition. As to  $N_{1e}^r$  in the same field, it is a potential well for the electrons so that according to (4.24)  $N_{1e}^r = 0$  ( $t \gg \tau_{re}$ ).

## 5. EVOLUTION OF AN ION-SOUND SOLITON FOLLOWING ITS INTERACTION WITH RESONANCE PARTICLES

We assume that at  $t=0$  the ion-sound wave had the shape of a soliton satisfying Eq. (3.10) without the right-hand side:

$$\Phi_s = \Phi_0 \text{sech}^2[(x-x_0)/\delta], \quad (5.1)$$

$$\delta = (c_0/\omega_{pi}) (\delta c_0^2/\Phi_0)^{1/2}. \quad (5.2)$$

For  $t > 0$  the solution of Eq. (3.10) with a small right-hand side can then be written in the form<sup>9-11</sup>

$$\Phi = \Phi_s(x-x_0(t), \varepsilon t) + \psi(x, t), \quad (5.3)$$

where  $\Phi_s$  has the form (5.1) with a slowly changing amplitude  $\Phi_0(t)$  and a width connected with  $\Phi_0$  by the same relation as for an unperturbed soliton, i.e., (5.2);  $\psi(x, t)$  describes the distortion of the shape of the soliton caused by the perturbation.

After the obvious transformations of Eq. (3.5) of Ref. 9, we obtain for the amplitude of the perturbed soliton the formula:

$$\frac{d\Phi_0}{dt} = -\frac{\gamma\bar{\delta}}{12} c_0 \omega_{pi} \Phi_0^{3/2} \int_{-\infty}^{\infty} \frac{\partial S[\Phi_s]}{\partial z} \text{sech}^2 z dz, \quad (5.4)$$

and the position of its center (Ref. 10, Eq. (2.46))

$$\frac{dx_0}{dt} = c_0 + \frac{\Phi_0}{3c_0} - \frac{c_0^2}{4\Phi_0} \int_{-\infty}^{\infty} \frac{\partial S(\Phi_s)}{\partial z} \left( \frac{z}{\text{ch}^2 z} + \text{th} z + \text{th}^2 z \right) dz. \quad (5.5)$$

Here  $S$  is defined in (4.7) and  $z = (x-x_0)/\delta$ .

The general expression for the variation of the shape of the soliton  $\psi(x, t)$  was given in an earlier paper.<sup>10</sup> Here we limit ourselves to its asymptotic behavior:

$$\begin{aligned} \frac{\psi(z, t)}{\Phi_0} &\approx -\frac{3c_0^4}{2\Phi_0^2} \left\{ \text{th}^2 z \int_{-\infty}^{\infty} \frac{\partial S[\Phi_s]}{\partial z'} \text{th}^2 z' dz' \right. \\ &\quad \left. - z^2 e^{2z} \int_{-\infty}^{\infty} \frac{\partial S[\Phi_s]}{\partial z'} \text{sech}^2 z' dz' \right\}, \quad z \rightarrow -\infty, \end{aligned} \quad (5.6)$$

$$\frac{\psi(z, t)}{\Phi_0} \approx -\frac{3c_0^4}{2\Phi_0^2} z^2 e^{-2z} \int_{-\infty}^{\infty} \frac{\partial S[\Phi_s]}{\partial z'} \text{sech}^2 z' dz', \quad z \rightarrow \infty. \quad (5.7)$$

It is clear from (5.6) that as  $z \rightarrow -\infty$  the expression for  $\psi(z, t)$  has a finite limit  $\psi(z = -\infty) \equiv \psi_-$ ,

$$\frac{\psi_-}{\Phi_0} = -\frac{3c_0^4}{2\Phi_0^2} \int_{-\infty}^{\infty} \frac{\partial S[\Phi_s(z)]}{\partial z} \text{th}^2 z dz. \quad (5.8)$$

This expression determines the so-called tail which is formed behind the soliton as a result of the action of the perturbation. Already at a distance of four to five times the length of the soliton the tail becomes independent of the soliton and the velocity of its elements is practically equal to zero, while its length grows proportionally to the path traversed by the soliton from the moment the perturbation was switched on. Since the amplitude  $\psi_-$  of the tail is, as can be seen from (5.8), determined by the soliton amplitude  $\Phi_0$  at the moment of emission and as the latter changes slowly with time according to Eq. (5.4),  $\psi_-$  changes correspondingly slowly in space. See Refs. 10 to 12 for more details about soliton tails.

We find from (5.4) that the characteristic time during which the soliton parameters change appreciably due to the action of the perturbation—the perturbation time  $\tau_p$ —is as to order of magnitude given by the relation

$$\tau_p^{-1} \sim \omega_{pi} \left( \frac{c_0^2}{\Phi_0} \right)^{1/2} \int_{-\infty}^{\infty} dz \frac{\partial S[\Phi_s]}{\partial z} \text{sech}^2 z. \quad (5.9)$$

In order that perturbation theory is applicable it is necessary that this time is large compared to the soliton time  $\tau_s$ .<sup>9,10</sup>

$$\tau_s = \delta / (\Phi_0/3c_0) = (3\sqrt{6}/\omega_{pi}) (c_0^2/\Phi_0)^{1/2} \quad (5.9a)$$

( $\tau_s$  is the time the unperturbed soliton traverses a distance  $\delta$  in the frame of reference moving with velocity  $c_0$ ). The ratio  $\varepsilon = \tau_s/\tau_p$  is the basic small parameter of the perturbation theory. It then turns out that the domain of applicability of the perturbation theory is in time limited by the condition<sup>10</sup>  $t \ll \tau_p(\tau_p/\tau_s)^{1/2}$ .

In this connection one should note that the characteristic time for the interaction of a soliton with resonance particles is appreciably shorter than  $\tau_s$ . Indeed, according to (4.6)

$$\tau_r/\tau_s \sim (\Phi_0/c_0^2)^{1/2} \ll 1 \quad (5.10)$$

( $\Phi_0/c_0^2 \sim \delta n/n_0$  is the basic parameter guaranteeing the validity of the unperturbed  $KdV$  equation). Hence it follows, in particular, that the Ott-Sudan Eq. (4.13), which as we showed above is valid for  $t \ll \tau_{re}$ , does not, strictly speaking, have a domain of applicability in the problem considered, as even after a time  $\tau_s$  the perturbation does not manage to manifest itself.

We must thus use the expressions for  $t \gg \tau_r$  which we found above. Hence it follows, in particular, that the contributions from the electrons to (5.4), (5.8) vanish. Indeed, the ion-sound soliton is a potential well for the electrons ( $U_e = -(m_i/m_e)\Phi$ ) so that according to (4.24)  $N_{1e}^r = 0$ ,  $N_{2e}^r[\Phi_s] = N_{2e}^r[\Phi_s]$ . However,  $N_{2e}^r[\Phi_s(z)]$  is an even function of  $z$  and, hence, does not contribute to the corresponding integral.<sup>11</sup> Correspondingly it is convenient to change in (5.4) to (5.8) to the variables (4.27), (4.28) and to take into account that  $U_0 = \Phi_0$ ,  $v_T = v_i$  (i.e.,  $\nu = \nu_i$ ,  $\mu = \mu_i$ ). Equations (5.4) and (5.8) then become

$$\frac{dv_i}{dt} = \frac{\omega_{pi}}{4\sqrt{3}} \mu_i A(\mu_i, v_i), \quad (5.11)$$

$$\frac{\psi_-}{\Phi_0} = \frac{6\mu_i^4}{v_i^4} [B(\mu_i, v_i) - A(\mu_i, v_i)], \quad (5.12)$$

where

$$A(\mu, v) = \int_{-\infty}^{\infty} dz N_1^r[\chi_s(z)] \frac{\partial \chi_s}{\partial z} \\ = \frac{4}{v^2} \left[ \int_{\mu}^{+\infty} d\sigma (\sigma - \mu)^2 F(\sigma) + \int_{\mu}^{-\infty} d\sigma (\sigma - \mu)^2 F(\sigma) \right], \quad (5.13)$$

$$B(\mu, v) = - \int_{-\infty}^{\infty} dz N_1^r[\chi_s(z)] = 2 \left[ \int_{\mu}^{+\infty} d\sigma F(\sigma) + \int_{\mu}^{-\infty} d\sigma F(\sigma) \right]. \quad (5.14)$$

Here  $N_1^r[\chi]$  is defined in (4.29) and  $\chi_s(z) = \text{sech}^2 z$  is the dimensionless soliton profile.

For small soliton amplitudes when condition (4.32) is satisfied we can restrict ourselves to the first two terms in the expansion of (5.13) and (5.14) in powers of  $v$  which leads to<sup>2)</sup>

$$\frac{dv_i}{dt} = \frac{\omega_{pi}}{2\sqrt{3}} \mu_i \left[ v_i^2 F_i'(\mu_i) + \frac{v_i^4}{9} F_i'''(\mu_i) \right], \quad (5.15)$$

$$\psi_-/\Phi_0 = -\frac{1}{3} \mu_i^4 F_i'''(\mu_i). \quad (5.16)$$

If for the sake of simplicity we neglect the last (small) term in (5.15) we obtain the following law for the evolution of the soliton amplitude, which is valid when  $\mu_i v_i \ll 1$ :

$$v_i(t) = v_i(0) [1 - v_i(0) \alpha t]^{-1}, \quad \alpha = \omega_{pi} \mu_i F_i'(\mu_i) / 2\sqrt{3}. \quad (5.17)$$

When  $F_i'(\mu) < 0$  the wave is damped with a characteristic time  $\tau_p = [v_i(0) |\alpha|]^{-1}$ . When  $F_i'(\mu) > 0$  there occurs an explosive type instability (besides, Eq. (5.17) loses its validity when  $v_i > 1/\mu_i$ ). Equation (5.17) has been obtained by other means (from energy conservation considerations).<sup>7</sup>

It is interesting to note that if we had used Ott and Sudan's Eq. (4.13) for  $S[\Phi]$  we would have obtained for  $v_i(t)$  the same form as (5.17) but with another  $\alpha$ :

$$\alpha_{(os)} = \frac{0.73}{2\sqrt{3}} \omega_{pi} \mu_i \left[ F_i'(\mu_i) + \frac{m_i}{m_e} F_e'(\mu_i) \right]. \quad (5.18)$$

The difference in the values of  $\alpha$  is qualitatively important here, as for  $\mu_i \geq 4$  the dominant role in (5.18) is played by the term with  $F_e$ . This term may be positive when there is a current present and this would lead to an instability, provided this equation were valid.

In actual (laboratory and numerical) experiments it is difficult to realize condition (4.32) under which Eqs. (5.15) to (5.17) are valid. We give results which are obtained in the case of large soliton amplitudes when  $v_i \mu_i \gg 1$ , but, of course,  $2\Phi_0/c_0^2 = (v_i/\mu_i)^2 \ll 1$ . Assuming to fix our ideas that the distribution function is Maxwellian, i.e.,  $F(\sigma) = \pi^{-1/2} \exp(-\sigma^2)$  we get in this case

$$\frac{dv_i}{dt} \approx \frac{\omega_{pi} \exp[-(\mu_i - v_i)^2]}{(12\pi)^{1/2} (\mu_i - v_i)} \left[ 1 + \frac{1}{v_i (\mu_i - v_i)} + \frac{v_i^2 - 1}{2v_i^2 (\mu_i - v_i)^2} \right], \\ \frac{\psi_-}{\Phi_0} \approx \frac{18\mu_i^4 \exp[-(\mu_i - v_i)^2]}{\pi^{1/2} v_i^4 (\mu_i - v_i)} \left[ 1 + \frac{2}{3v_i (\mu_i - v_i)} + \frac{v_i^2 - 2}{6v_i^2 (\mu_i - v_i)^2} \right].$$

We turn finally to the change in the soliton velocity due to its resonance interaction with particles. Using (5.5), (4.7) and also the fact that  $N_1^r(z)$  is odd and  $N_2^r(z)$  even we get after simple transformations

$$\frac{1}{c_0} \frac{dx_0}{dt} = 1 + \frac{\Phi_0}{3c_0^2} + \frac{\psi_-}{6c_0^2} + \frac{c_0^2}{2\Phi_0} \int_{-\infty}^{\infty} dz S_2[\Phi_s] \frac{\text{ch } z - z \text{ sh } z}{\text{ch}^2 z}, \quad (5.19)$$

where  $S_2 = N_2^r - N_2^e$ .

Starting from Eqs. (4.25) and assuming that the unperturbed distribution functions are Maxwellian we get after some calculations

$$S_2[\Phi] = (1 - 2\Phi/V_{ph}^2)^{-1/2} \exp(-\Phi/c_0^2) + R_i[\Phi] - R_e[\Phi] \\ + N_{2e}^{tr}[\Phi] - N_{2i}^{tr}[\Phi]. \quad (5.20)$$

Here  $R_i[\Phi]$  and  $R_e[\Phi]$  determine the contributions from the resonance ions and electrons:

$$R_i[\Phi] = \frac{1}{2} \left( \frac{2\Phi}{c_0^2} \right)^{1/2} \left\{ \exp \left[ - \left( \frac{V_{ph} + (2\Phi)^{1/2}}{v_i} \right)^2 \right] \right. \\ \left. - \exp \left[ - \left( \frac{V_{ph} - (2\Phi)^{1/2}}{v_i} \right)^2 \right] \right\}, \quad (5.21)$$

$$R_e[\Phi] = \{n_e^{tr}[\Phi] - n_{e0}^{tr}[\Phi]\} / n_0, \quad (5.22)$$

$V_{ph} = c_0 + \Phi_0/3c_0$  is the unperturbed soliton speed,  $n_e^{tr}[\Phi]$  the trapped electron density,  $n_{e0}^{tr}$  the "equilibrium" trapped electron density, i.e., the density which would occur, if the trapped electrons were described by a Boltzman distribution with temperature  $T_e$ . We gave Eq. (5.21) under the condition  $\Phi_0/c_0^2 \gg T_i^2/T_e^2$  (i.e.,  $\mu_i v_i > 1$ ).

Finally, the difference between the last two terms in (5.20) is, according to the definition (4.25b) nothing but the difference between the relative densities of the non-resonance electrons and ions. Using the Poisson equation for the unperturbed soliton profile we can write

$$N_{2e}^{tr}[\Phi_s] - N_{2i}^{tr}[\Phi_s] = \frac{\Phi_s}{6c_0^4} \frac{\partial^2 \Phi_s}{\partial z^2}. \quad (5.23)$$

Substituting (5.23) into (5.20) we get an expression which does not contain the arbitrariness connected with the choice of  $E_r$  in (4.25).

We now expand the first and second terms in (5.20) in powers of  $\Phi_s/c_0^2$  restricting ourselves to terms of second order.<sup>3)</sup> In that case

$$S_2[\Phi_s] \approx \frac{\Phi_s}{6c_0^4} \left( \frac{\partial^2 \Phi_s}{\partial z^2} + \frac{6\Phi_s^2}{\Phi_s} - 4\Phi_s \right) \\ + R_i[\Phi_s] - R_e[\Phi_s] = R_i[\Phi_s] - R_e[\Phi_s]. \quad (5.24)$$

We did here take into account the fact that the expression in brackets vanishes by virtue of the equation for the soliton ( $\Phi_s = \Phi_0 \text{sech}^2 z$ ). Thus,  $S_2[\Phi]$  is expressed solely in terms of the contribution from the trapped particles, as one should expect. It is then at once clear that the quantity  $R_i[\Phi_s]$  is exponentially small when  $T_i/T_e \ll 1$ . As to  $R_e[\Phi_s]$  we must assume that that quantity is sufficiently small so that the last term on the right-hand side of (5.19) would be small compared to the second term, i.e.,

$$\int_{-\infty}^{\infty} \frac{n_e^{tr} - n_{e0}^{tr}}{n_0} \frac{\text{ch } z - z \text{ sh } z}{\text{ch}^2 z} dz \ll \frac{\Phi_s^2}{c_0^4}. \quad (5.25)$$

If this condition is not satisfied, it is no longer possible to consider the resonance interaction between the soliton and the electrons to be a small perturbation; it makes a contribution comparable to the non-linear and the dispersion terms in Eq. (3.10). A detailed consideration shows, however, that when (5.25) is not satisfied, there occurs an instability. The oscillations

which are then excited lead to an exchange between trapped and free electrons until  $|n_e^{tr} - n_{e0}^{tr}|$  is decreased to a value satisfying condition (5.25). The characteristic time for this process is  $\leq \omega_{pi}^{-1}$ . Finally, we note that if we use model Boltzmann distribution functions for the trapped particles with an arbitrary temperature and neglect the contribution from the resonance ions, Eqs. (5.24), (5.19) lead to expressions which are equivalent to those obtained by Schamel.<sup>5</sup>

<sup>1</sup>In the Ott-Sudan Eq. (4.13) the electrons, on the other hand, give the main contribution, at least when  $T_e/T_i \geq 4$ .

<sup>2</sup>If we used (5.30) to evaluate  $A$  and  $B$ , which is equivalent to taking only terms  $\propto \nu^2$  into account in (5.13) and (5.14), we would find  $A = B$ , i.e.,  $\Psi_- = 0$ .

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## Contribution to the theory of nonlinear photoelasticity of solids

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Nonlinear modulation of the dielectric constant of a crystal by a sound wave is considered theoretically. An expression quadratic in the sound-wave amplitude is obtained for the amplitude of the change of the dielectric constant of the crystal; this expression is analyzed for various optical frequency bands. It is shown that the effect of nonlinear modulation of the dielectric constant of a crystal by a sound wave is particularly pronounced in the resonant case  $\hbar\omega \approx E_g$  for dielectrics, or else at  $\hbar\omega \approx E_g + F$  for degenerate semiconductors, when relatively low sound power is needed for the onset of nonlinear effects (here  $\omega$  is the frequency of the light wave,  $E_g$  is the width of the forbidden band of the crystal,  $F = m_c \epsilon_F / \mu$ , where  $m_c$ ,  $\mu$ , and  $\epsilon_F$  are respectively the effective mass of the conduction electrons, the reduced effective mass of the conduction and valence electrons, and the Fermi energy of the conduction electrons).

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The diffraction of light by sound (the acousto-optical (AO) interaction) has by now been the subject of a large number of studies, owing to the extensive practical use of this phenomenon in physical research and in modern technology (see the reviews<sup>1,2</sup>). In all the theoretical studies of the AO interaction it is assumed that one of its basic mechanisms is the change of the dielectric constant of the crystal upon propagation of the sound wave. It is customarily proposed that in view of the smallness of the relative deformation of the crystal by the passage of the sound, its dielectric constant  $\epsilon_{ik}$  in the presence of the sound wave can be expressed in the following form (see, e.g., Refs. 1-3)

$$\epsilon_{ik} = \epsilon_{ik}^0 + \Delta^1 \epsilon_{ik} = \epsilon_{ik}^0 - \epsilon_{il}^0 \epsilon_{km}^0 \rho_{lmnj} u_{nj}, \quad (1)$$

where  $\epsilon_{ik}^0$  is the dielectric constant of the crystal in the

absence of sound,  $u_{nj}$  is the crystal strain tensor in the sound wave, and  $\rho_{lmnj}$  is the crystal photoelasticity tensor (summation over repeated indices is understood from now on). Terms of higher order in  $u_{ik}$  are as a rule neglected in (1).

It should be noted, however, that in some cases a sound wave of even relatively small power is capable producing in the dielectric constant of the crystal a modulation that is essentially nonlinear in the parameter  $u_{ik}$ . In fact, in the simplest case at  $\hbar\omega < E_g$  there is no absorption of light in the crystal (neglecting the absorption "tail"). On the other hand, in the presence of a sound wave modulation of the quantity  $E_g$  sets in, and a situation becomes possible wherein the relation  $\hbar\omega > E_g - \Delta E_g$  is satisfied in a region of a decreased