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Effect of linear defects on the local magnetization of a plane Ising lattice

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An exact solution is obtained for the local magnetization of a plane Ising lattice with a line of defects of two types. It is shown that near the critical point, at distances to the defect much shorter than the correlation radius, the local magnetization has a universal behavior that manifests itself in the fact that its critical exponent is a continuous function of the microscopic parameters of the system.

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1. INTRODUCTION

The study of the influence of various defects on the critical behavior of systems that undergo phase transitions is of condiserable influence both theoretically and experimentally. The theoretical study of this influence is based on phenomenological theories,[1-5] renormalization-group calculations,[6,7] high-temperature expansions as well as on exactly solvable models.[1,11-14] Principal attention is being paid to the study of the influence of the defects on the global characteristics of the system, such as the transition temperature, the free energy, the specific heat, etc. No less interesting, however, is the study of the influence of the defects on local characteristics such as the local magnetization and the correlation functions. The reason is, first, that the influence of the defects on the local characteristics is stronger than on the global ones, and second, that in a number of experiments it is precisely the local characteristics that are measured.

This paper presents a rigorous analysis of the influence of linear defects of two different types on the local magnetization of a plane Ising lattice. Fragments of the lattices considered are shown in Fig. 1. The local magnetization is calculated exactly for these models, and a study is made of its dependence on the distance to the defect and on the size of the defect. The most interesting feature of the solution is the dependence of the critical exponent of the local magnetization on the interaction parameters.

The systems considered simulate real objects in which the defects are due to the presence of one dislocation. They constitute the linear approximation in the analysis of objects having a low dislocation density.

2. FORMULATION OF PROBLEM

The interaction energy of the considered lattices can be represented as a sum of two terms

$$E=E_0+\Delta E, \tag{1}$$

where E_0 is the energy of the interaction of the defect-free lattice:

$$E_0=-\sum_{m=1-M}^{M}\sum_{n=1-N}^{N}(J_1 s_{mn}s_{m,n+1}+J_2 s_{mn}s_{m+1,n}), \tag{2}$$

ΔE is the energy of the perturbation due to the presence of the defects. For the lattice of the first type

$$\Delta E=-(J_1'-J_1)\sum_{m=1-M}^{M} s_{m0}s_{m1}, \tag{3}$$

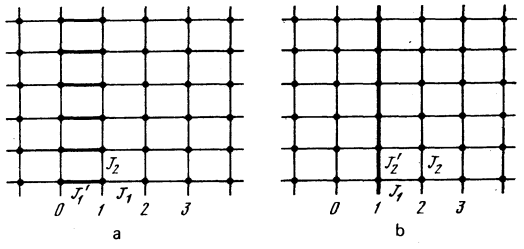

FIG. 1. Fragments of plane lattice with line of defects located between two neighboring columns (a) and inside one column (b).

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and for the lattice of the second type

$$\Delta E=-(J_2'-J_2)\sum_{m=1-M}^{M} s_{m1}s_{m+1,1}. \tag{4}$$

In these expressions s_{mn} is the spin variable connected with a lattice site having coordinates m and n, and taking on values ± 1; J_1, J_1', and J_2, J_2' are the energies of the interactions between the horizontal and vertical pairs of neighboring spins, respectively (see Fig. 1). Here and below it is assumed that the lattice is infinite in both directions ($M, N \to \infty$).

We define the local magnetization of a lattice with defects in analogy with the definition of the magnetization of a defect-free lattice,[15,16] on the basis of the limiting value of the spin-spin correlation function

$$\langle s_n\rangle = \lim_{\mathfrak{R}\to\infty}\frac{\langle s_{mn}s_{m,n+\mathfrak{R}}\rangle}{\langle s_{m,n+\mathfrak{R}}\rangle}. \tag{5}$$

We have omitted from the left-hand side of this expression the row index because of the translational invariance of the lattices in question in the vertical direction.

For spins with large column number the local magnetization does not differ from the magnetization $\langle s\rangle_0$ of a defect-free lattice[15-17]

$$\lim_{\mathfrak{R}\to\infty}\langle s_{m,n+\mathfrak{R}}\rangle = \langle s\rangle_0 = \begin{cases}\mathfrak{M}, & T<T_c\\ 0, & T>T_c\end{cases},$$
$$\mathfrak{M}=|1-(\operatorname{sh}2K_1\operatorname{sh}2K_2)^{-2}|^{1/4}=B\tau^{1/4}[1+O(\tau)],\quad \tau=|1-T/T_c|, \tag{6}$$
$$B=[4(K_{1c}\operatorname{cth}2K_{1c}+K_{2c}\operatorname{cth}2K_{2c})]^{1/4},\quad K_i=J_i/kT,$$

where k is Boltzmann's constant, T is the temperature, the subscript c marks the value of the function at the critical point T_c. Substituting (6) in (6) we get

$$\langle s_n\rangle = (\langle s\rangle_0)^{-1}\lim_{\mathfrak{R}\to\infty}\langle s_{mn}s_{m,n+\mathfrak{R}}\rangle. \tag{7}$$

It is known[16,18] that the two-dimensional Ising lattice can be represented as a model of noninteracting fermions. To realize such a representation, we introduce the fermion operators

$$a_{mn}^{+}=s_{m,n-1}\prod_{\alpha=n}^{N}g_{m\alpha},\quad ia_{mn}^{-}=s_{mn}\prod_{\alpha=n}^{N}g_{m\alpha}, \tag{8}$$

where

$$g_{m\alpha}=\exp(-2K_2 s_{m-1,\alpha}s_{m\alpha}). \tag{9}$$

For a lattice with defects of the second type it is necessary at $\alpha=1$ to replace K_2 by K'_2 in (9).

The operators (8) under the mean-value sign satisfy the Fermi anticommutation relations [if they are defined, at identical first coordinates, in accordance with (A.7)]:

$$\{a_{mn}^{\Delta}, a_{mn'}^{\Delta'}\}=2\delta_{nn'}\delta_{\Delta\Delta'}, \tag{10}$$

where $\Delta=\pm$, $\delta_{nn'}$ and $\delta_{\Delta\Delta'}$ are Kronecker symbols. The inverse transition from the spin operators to the fermion operators is based on the following relations:

$$s_{m,n-1}s_{mn}=ia_{mn}^{+}a_{mn}^{-},\quad s_{m-1,n}s_{mn}=C_2^{\cdot}-iS_2^{\cdot}a_{mn}^{-}a_{m,n+1}^{+};$$
$$C_i=\operatorname{ch}2K_i,\quad S_i=\operatorname{sh}2K_i,\quad C_i^{\cdot}=C_i/S_i,\quad S_i^{\cdot}=1/S_i. \tag{11}$$

Using (11) and the fact that $(s_{mn})^2=1$, we can represent an arbitrary function of the spin operators in the form of a product of the fermion operators (8). In the subsequent calculations it is more convenient, however, to use linear combinations of the operators (8):

$$b_{mn}^{\pm}=\operatorname{ch}K_1 a_{mn}^{\pm}\mp i\operatorname{sh}K_1 a_{mn}^{\mp}. \tag{12}$$

It is easy to verify that the transformation (12) is canonical, i. e., that the operators $b_{mn}^{\pm}$ also satisfy the Fermi anticommutation relations.

Using (11) and (12), we can represent the expression in the right-hand side of (7) in the form of a product of fermion operators:

$$\langle s_n\rangle = \frac{1}{\langle s\rangle_0}\lim_{\mathfrak{R}\to\infty}\left\langle P\prod_{l=1}^{\mathfrak{R}} ib_{m,n+l}^{+}b_{m,n+l}^{-}\right\rangle, \tag{13}$$

where P is the Dyson chronological-ordering operator, which arranges the Fermion operators in increasing order of the coordinate m. The operator P in (13) can be replaced by the Wick chronological-ordering operator. We then obtain in the right-hand side of (13) the many-point Green's function

$$\langle s_n\rangle = \frac{1}{\langle s\rangle_0}\lim_{\mathfrak{R}\to\infty}\left\langle\!\!\left\langle\prod_{l=1}^{\mathfrak{R}} ib_{m,n+l}^{+}b_{m,n+l}^{-}\right\rangle\!\!\right\rangle, \tag{14}$$

where $\langle\langle\ldots\rangle\rangle=\langle T\ldots\rangle$, and T is the Wick chronological-ordering operator.

The many-point Green's function satisfies the Wick theorem, and as a result the right-hand side can be represented in the form of a Pfaffian whose elements are two-point Green's functions. The solutions for these functions are given in the Appendix. Recognizing that

$$\langle\!\langle b_{mn}^{+}b_{mn'}^{+}\rangle\!\rangle=\langle\!\langle b_{mn}^{-}b_{mn'}^{-}\rangle\!\rangle=\delta_{nn'},$$

we obtain

$$\langle s_n\rangle=(\langle s\rangle_0)^{-1}\lim_{\mathfrak{R}\to\infty}\det(A_n^{(\mathfrak{R})}), \tag{15}$$

where $A_n^{(\mathfrak{R})}$ is a matrix with elements

$$\langle\!\langle ib_{m,n+j+1}^{+}b_{m,n+k+1}^{-}\rangle\!\rangle=\langle\!\langle ib_{m,n+j+1}^{+}b_{m,n+k+1}^{-}\rangle\!\rangle_0+\langle\!\langle b_{m,n+j+1}^{+}b_{m,n+k+1}^{-}\rangle\!\rangle_{\text{per}}; \tag{16}$$
$$j,k=0,1,2,\ldots,\mathfrak{R}-1,$$

$$\langle\!\langle ib_{m,n+j+1}^{+}b_{m,n+k+1}^{-}\rangle\!\rangle_0=(2\pi)^{-1}\int_{-\pi}^{\pi}d\varphi\exp[i(j-k)\varphi]\Phi_0(\varphi), \tag{17}$$

$$\langle\!\langle ib_{m,n+j+1}^{+}b_{m,n+k+1}^{-}\rangle\!\rangle_{\text{per}}=-(2\pi)^{-1}\int_{-\pi}^{\pi}d\varphi\exp[-(2n+j+k)\eta]\Phi_{1,2}(\varphi). \tag{18}$$

Expression (18) is written for lattices with defects of the first and second type, respectively. The subscript 0 means that the function pertains to the defect-free lattice, and the subscript "per" refers to the perturbation due to the defects. The expressions for $\Phi_0(\varphi)$, $\Phi_{1,2}(\varphi)$ and $\eta(\varphi)$ are given in the Appendix [(A.13), (A.16), (A.20), and (A.18), respectively].

The calculation of the magnetization of lattices with defect lines has thus been reduced to the calculation of determinants of semi-infinite matrices whose elements are represented as sums of two terms, the first dependent on the difference of the indices, and the second on the sum.

3. CALCULATION OF THE MAGNETIZATION

To calculate the determinants (15) we use the method of an earlier paper.[19] We note for this purpose that the terms of the expansion of the determinant (15) or of the

Pfaffian (14) can be broken up into separate groups, depending on the number of Green's functions that correspond to the perturbation:

$$\langle s_n\rangle=\langle s\rangle_0\Big[1+\sum_{k=1}^{\infty}F_k\Big], \tag{19}$$

where F_k is the sum of terms, each of which contains k Green's functions corresponding to the perturbation:

$$F_k=(\langle s\rangle_0)^{-2}\sum_{(\alpha,\beta)=0}^{\infty}{}'\Big\langle\!\Big\langle\prod_{l=1}^{k}(ib^+_{m,n+\alpha_l+1}b^-_{m,n+\beta_l+1})\Big\rangle\!\Big\rangle_{\text{per}}A^{(\mathfrak{N})}(\alpha_1\alpha_2\dots\alpha_k|\beta_1\beta_2\dots\beta_k), \tag{20}$$

where the summation is carried out over values of α and β that satisfy the conditions

$$A^{(\mathfrak{N})}(\alpha_1\alpha_2\dots\alpha_k|\beta_1\beta_2\dots\beta_k)=(-1)^Q\Big\langle\!\Big\langle\prod_{j=1}^{\mathfrak{N}-k}(ib^+_{m,n+\alpha_j'}b^-_{m,n+\beta_j'})\Big\rangle\!\Big\rangle_0 \tag{21}$$

$$\Big(Q=\sum_{i=1}^{k}\alpha_i+\beta_i\Big)$$

is the aggregate of the Green's functions of the defect-free lattice at the given choice of the functions corresponding to the perturbation; $\alpha_1',\dots,\alpha_{\mathfrak{N}-k}$ $(\beta_1',\dots,\beta_{\mathfrak{N}-k})$ are the indices that remain after eliminating from the set 0, 1, 2, ..., $\mathfrak{N}-1$ the indices $\alpha_1,\dots,\alpha_k$ $(\beta_1,\dots,\beta_k)$.

As shown in Ref. 19, expression (21) can be represented initially in terms of the minors of a Toeplitz matrix with element (17), and then in terms of the elements of the inverse matrix, which are obtained as $\mathfrak{N}\to\infty$ by the Wiener-Hopf method. As a result we have as $\mathfrak{N}\to\infty$

$$A^{(\mathfrak{N})}(\alpha_1\alpha_2\dots\alpha_k|\beta_1\beta_2\dots\beta_k)=\langle s\rangle_0^2\sum_{(\text{all }\alpha)}(-1)^P\Gamma_{\beta_1|\alpha_1}\Gamma_{\beta_2|\alpha_2}\dots\Gamma_{\beta_k|\alpha_k}, \tag{22}$$

where

$$\Gamma_{\alpha|\beta}=\sum_{l=0}^{\min(\alpha,\beta)}\gamma^{(1)}_{\alpha-l}\gamma^{(2)}_{\beta-l},\qquad \gamma^{(1,2)}_{\alpha}=(2\pi)^{-1}\int_{-\pi}^{\pi}d\varphi\,e^{\mp i\alpha\varphi}g_{\pm}(e^{i\varphi}).$$

Substituting (22) in (20), we obtain after some algebraic simplifications

$$F_k=\frac{1}{k!}\sum_{(\alpha,\beta)=0}^{\infty}\Big\langle\!\Big\langle\prod_{l=1}^{k}(ib^+_{m,n+\alpha_l+1}b^-_{m,n+\beta_l+1})\Big\rangle\!\Big\rangle_0\prod_{l=1}^{k}\Gamma_{\beta_l|\alpha_l}, \tag{23}$$

where in contrast to (20) the summation over α and β is free of restrictions.

To analyze expression (23) it is convenient to use a diagram technique. To this end we set in correspondence to the operators b^+ and b^- two different points, and set in correspondence with the Green's functions and the elements Γ two different lines that join these points. Then the expansion (19) becomes a sum over the connected and unconnected diagrams, with the unconnected diagram corresponding to the product of its connected parts. It is possible to apply to this expansion the connectivity theorem,[20] as a result of which the magnetization can be represented in the form

$$\langle s_n\rangle=\langle s\rangle_0\exp\Big(\sum_{k=1}^{\infty}\frac{1}{k}\bar F_k\Big), \tag{24}$$

where $k^{-1}\bar F_k$ is that part of (23) which correspond to the connected diagrams; its graphic representation is shown in Fig. 2. To write down the analytic expression it remains only to sum over α and β, as a result of which we get

$$\bar F_k=-(4\pi)^{-k}\int_{-\pi}^{\pi}d\varphi_1\dots\int_{-\pi}^{\pi}d\varphi_k\prod_{l=1}^{k}\exp[-(2n-1)\eta_l]\Phi_{1,2}(\varphi)\Big[\operatorname{sh}\frac{1}{2}(\eta_l+\eta_{l+1})\Big]^{-1}; \tag{25}$$

$$\varphi_{k+1}\equiv\varphi_1,\qquad \eta_l=\eta(\varphi_l).$$

4. ANALYSIS IN THE SCALING REGION

We consider the behavior of the local magnetization in the most interesting–scaling–region, defined by the following relations:

$$\xi_1^{-1}=(\xi S_1^{1/2})\tau^{-1}\to 0,\qquad n\to\infty,\qquad n/\xi_1-\text{arbitrary}, \tag{26}$$

where ξ is the correlation radius of the defect-free lattice[18]:

$$\xi^{-1}=[2(C_1C_2-S_1-S_2)]^{1/2}=\mathscr{D}\tau+O(\tau^2),\quad \mathscr{D}=1/2(C_1C_2)^{-1/2}B^2. \tag{27}$$

An analysis of expression (25) in the region (26) shows that the largest contribution to the integration is made by the region of small values of φ, in which the quantities

$$\eta(\varphi)=(\xi^{-2}+S_2\varphi^2)S_1^{-1}+O(\varphi^4).$$

are also small. Then, using the change of variables $\varphi=(\xi S_2^{1/2})^{-1}\sinh u$ and extending the limits of integration with respect to u to $\pm\infty$, we get

$$\langle s_n\rangle=\langle s\rangle_0 f(x,\varkappa). \tag{28}$$

The scaling function $f(x,\varkappa)$, which describes the deviation of the magnetization of a lattice with defects from the magnetization of a defect-free lattice is represented in the form

$$f(x,\varkappa)=\exp\Big(\sum_{k=1}^{\infty}\frac{1}{k}\bar f_k\Big), \tag{29}$$

$$\bar f_k=-\Big(\frac{\varkappa}{\pi}\Big)^k\int_0^{\infty}du_1\dots\int_0^{\infty}du_k\prod_{l=1}^{k}\exp(-x\operatorname{ch}u_l)\frac{\operatorname{sh}^2u_l}{(\operatorname{ch}u_l-\varkappa)(\operatorname{ch}u_l+\operatorname{ch}u_{l+1})}. \tag{30}$$

$$u_{k+1}\equiv u_1.$$

For a lattice with defects of the first type

$$x=(2n-1)\xi_1^{-1},\quad \varkappa=(C_1-C_1')(C_1C_1'-1)^{-1},\quad -C_1^{-1}\leqslant\varkappa<1, \tag{31}$$

and for a lattice with defects of the second type[1)]

$$x=2(n-1)\xi_1^{-1},\quad \varkappa=\operatorname{th}2(K_2-K_2'),\quad -1<\varkappa<1. \tag{32}$$

Expanding the exponenetial of (29) in a series and recognizing that

$$(\operatorname{ch}u_l+\operatorname{ch}u_{l+1})^{-1}=\int_0^{\infty}dy_l\exp[-y_l(\operatorname{ch}u_l+\operatorname{ch}u_{l+1})],$$

we can represent the scaling function $f(x,\varkappa)$ in the form

$$f(x,\varkappa)=1+\sum_{k=1}^{\infty}f_k; \tag{33}$$

$$f_k=\frac{1}{k!}\Big(\frac{-\varkappa}{\pi}\Big)^k\int_0^{\infty}dy_1\dots\int_0^{\infty}dy_k\mathscr{K}_{\varkappa}\begin{pmatrix}y_1y_2\dots y_k\\ y_1y_2\dots y_k\end{pmatrix};$$

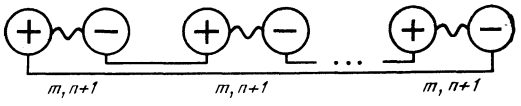

FIG. 2. Diagram corresponding to expression (25).

$$\mathcal{K}_\varkappa\begin{pmatrix} y_1 y_2 \dots y_k \\ y_1' y_2' \dots y_k' \end{pmatrix} = \begin{vmatrix} \mathcal{K}_\varkappa(y_1+y_1'+x) \dots \mathcal{K}_\varkappa(y_1+y_k'+x) \\ \dots\dots\dots\dots\dots\dots \\ \mathcal{K}_\varkappa(y_k+y_1'+x) \dots \mathcal{K}_\varkappa(y_k+y_k'+x) \end{vmatrix}, \quad (34)$$

$$\mathcal{K}_\varkappa(y) = \int_0^\infty du \exp(-y \operatorname{ch} u) \frac{\operatorname{sh}^2 u}{\operatorname{ch} u - \varkappa}.$$

It is then easy to see that $f(x, \varkappa)$ is the Fredholm determinant of the integral equation

$$\Psi(y) - \frac{\varkappa}{\pi} \int_0^\infty \mathcal{K}_\varkappa(y+y'+x)\Psi(y')dy' = \chi(y).$$

From the general theory of integral equations[21] follows the regular convergence of the series (33) for the scaling function $f(x, \varkappa)$ at all values $x > 0$.

At sufficiently large values of x, the scaling function $f(x, \varkappa)$ can be approximated by the first terms of the expansion (33)

$$f(x,\varkappa) = 1 - \frac{\varkappa}{2\pi} \int_0^\infty du\, e^{-x \operatorname{ch} u} \left(\frac{\operatorname{sh}^2 u}{\operatorname{ch} u - \varkappa}\right) \frac{1}{\operatorname{ch} u} + O(e^{-2x}). \quad (35)$$

At $x \gg 1$, the asymptotic form of (35) is

$$f(x,\varkappa) = 1 - \frac{\varkappa}{1-\varkappa}\left(\frac{1}{8\pi}\right)^{1/2} \frac{e^{-x}}{x^{3/2}}, \quad x \gg (1-\varkappa)^{-1}; \quad (36)$$

$$f(x,\varkappa) = 1 - \frac{\varkappa}{(2\pi)^{1/2}} \frac{e^{-x}}{x^{1/2}}, \quad (1-\varkappa)^{-1} \gg x. \quad (37)$$

It should be noted that expression (35) approximates well the scaling function also at $x \sim 1$. The maximum value of the correction term is reached at $\varkappa = 1$ and does not exceed 1% for $x > 0.3$

To find the asymptotic form of $f(x, \varkappa)$ at small x, we consider the representation (29), (30). The asymptotic form of the multiple integral in (30) at $x \ll 1$ can be obtained in the same manner as in Ref. 22:

$$f_k(x, \varkappa) = \sigma_k(\varkappa) \ln x^{-1} + E_k(\varkappa) + O(1), \quad (38)$$

where

$$\sigma_k(\varkappa) = -\frac{1}{\pi} \int_0^\infty dp \left(\frac{\varkappa}{\operatorname{ch} \pi p}\right)^k.$$

Substituting (38) in (29) we get

$$f(x, \varkappa) = E(\varkappa) x^{\sigma(\varkappa)} [1 + O(x)], \quad (39)$$

where

$$\sigma(\varkappa) = \frac{1}{2\pi^2} [\arccos(-\varkappa)]^2 - \frac{1}{8}, \quad (40)$$

$E(\varkappa)$ is a constant that does not depend on x. The numerical value of this constant lies in the interval

$$0.875810\ldots = B(-1) \leqslant B(\varkappa) \leqslant B(1) = 1.097665\ldots.$$

Recognizing that $B(0) = 1$, we can conclude that $B(\varkappa)$ is practically a linear function of $\varkappa$, and obtain the numerical value of this function with approximate accuracy 1%. Plots of the function $f(x, \varkappa)$ for several values of $\varkappa$, based on the asymptotic expression (35) and (39), are shown in Fig. 3.

The most interesting feature of the obtained solution is the non-universal behavior of the local magnetization at distances n smaller than half the correlation radius ξ_1 of the defect-free model. This non-universality consists in the fact that the critical exponent β_{1oc} of the magnetization is a continuous function of the

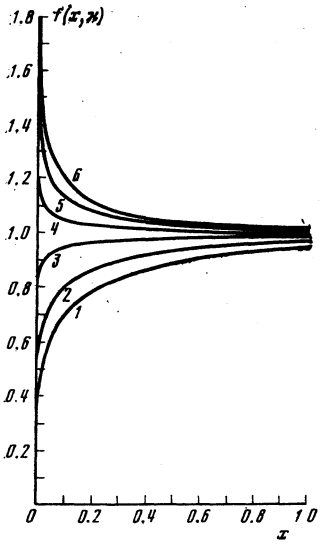

FIG. 3. Plot of the scaling function $f(x, \varkappa)$ for different values of $\varkappa$: 0.8 (1); 0.6 (2); 0.2 (3); −0.2 (4); −0.6 (5); −1 (6).

interaction parameters

$$\langle s_n \rangle \sim \tau^{\beta_{1oc}(\varkappa)} n^{\sigma(\varkappa)} \quad \text{at} \quad n \ll \xi_1/2, \quad (41)$$

where

$$\beta_{1oc} = \frac{1}{2\pi^2} [\arccos(-\varkappa)]^2, \quad (42)$$

and $\varkappa$ is determined by expressions (31) and (32). A plot of the function β_{1oc} is given in Fig. 4. For the model with defects of the first type, β_{1oc} varies in the range

$$\frac{1}{2\pi^2} [\arccos(C_1^{-1})]^2 \leqslant \beta_{1oc} < 0.5,$$

and for the model with defects of the second type in the range $0 < \beta_{1oc} < 0.5$.

Thus, the local magnetization of a lattice with a line of defects, at distances larger than $\xi_1/2$ to the defect, hardly differs from the magnetization of a defect-free lattice. The correction term necessitated by the perturbation is exponentially small in this region compared with the magnetization of a defect-free lattice [see (36) and (37)]. However, with further decrease of the distance the local magnetization deviates strongly from that of defect-free lattice, increasing or decreasing when the parameter $\varkappa$ takes on negative and positive values, respectively (see Fig. 3). The deviation becomes so substantial, that it leads to a change of the critical exponent β_{1oc} that characterizes the dependence of the local magnetization on the relative temperature τ[see (41)]. In this region the local magnetization ex-

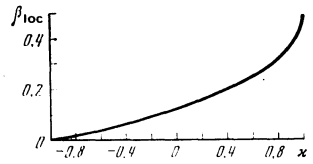

FIG. 4. Dependence of the critical exponent β_{1oc} of the local magnetization on the microscopic parameter $\varkappa$.

hibits a non-universal behavior—its critical exponent is a continuous function of $\varkappa$ [see (42) and Fig. 4]; this behavior is connected in turn with the parameters of the interactions of the considered lattices [see (31) and (32)].

5. CONCLUSION

We discuss now the reason for the non-universality of the critical behavior of the local magnetization from the point of view of the general theory of phase transitions. The first example of a system with non-universal behavior was the Baxter model.[23] The non-universality of this model consisted in the fact that its critical exponents are continuous function of the parameter of four-particle interaction, and this interaction can be regarded in this case as a perturbation. It was shown on the basis of a phenomenological approach that the non-universality of the Baxter model is due to the fact that the physical dimensionality of the system d coincides with the scale dimensionality Δ_{per} of the perturbation ($d=\Delta_{per}$).[24,25] It was assumed in Refs. 24 and 25 that the physical dimensionality d_{per} of the perturbation coincides with the dimensionality of the system. In real objects, however, besides the case $d_{per}=d$, the situation $d_{per}<d$ is frequently encountered.

Application of the phenomenological approach[25,26] to this case shows that if $\Delta_{per}=d_{per}$, then the critical behavior of the system can become non-universal, with a continuous dependence of the critical exponents on the microscopic parameter of the perturbation. Here, however, in contrast to the case $d_{per}=d$, the functions of the perturbation parameters can become only the exponents that characterize the local quantities near the perturbation.

It is precisely this situation, $\Delta_{per}=d_{per}<d$, which is realized in the plane models with linear defects considered by us, and the local critical exponents connected with the spin operators s are continuous function of the microscopic perturbation parameter. In particular, for the exponent of the local magnetization we have in first-order perturbation theory

$$\beta_x=\beta_0+\lambda a+O(\lambda^2), \tag{43}$$

where $\lambda=K_1-K_1'$, $a=1/\pi S_1$ and $\lambda=K_2-K_2'$, $a=1/\pi$ respectively for the defects of the first and second type (a is the coefficient of the operator algebra,[27,28] and its exact value was obtained in Ref. 29). It is easily seen that expression (42) also takes on the form (43) in first order in λ. Thus, the results of the phenomenological and rigorous approaches are in agreement.

Application of the phenomenological approach[25,26] to the general situation when $d_{per}<d$ leads to the following conclusion: if the physical and the scale dimensionalities of the perturbation coincides, then application of the perturbation brings about a situation wherein either the correlation functions near the perturbation take on a scale-invariant form, or else the local critical exponents become continuous functions of the microscopic parameters of the system.

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APPENDIX

We consider first a lattice with defects of the first kind. The equations of motion with respect to m, for the fermion operators (8) of this lattice, can be obtained either on the basis of the transition matrix[18] or by using the method of Ref. 30. If we regard a_{mn}^{+} and a_{mn}^{-} as components of the volumn vector

$$a_{mn}=\begin{pmatrix} a_{mn}^{+} \\ a_{mn}^{-} \end{pmatrix} \tag{A.1}$$

these equations can be written in matrix form

$$a_{m+1,n}-\sum_{n'}P_{nn'}a_{m,n'}=0, \tag{A.2}$$

where

$$P_{nn'}=\mathfrak{c}\delta_{n-1,\,n'}+\mathfrak{a}\delta_{nn'}+\mathfrak{b}\delta_{n+1,\,n'}+(\delta_{n0}\mathfrak{b}'+\delta_{n1}\mathfrak{a}'+\delta_{n2}\mathfrak{c}')\delta_{1,\,n'}, \tag{A.3}$$

$$\mathfrak{a}=\begin{pmatrix} C_1C_2^* & -iS_1C_2^* \\ iS_1C_2^* & C_1C_2^* \end{pmatrix},\quad \mathfrak{b}=\begin{pmatrix} 0 & 0 \\ -iC_1S_2^* & -S_1S_2^* \end{pmatrix},\quad \mathfrak{c}=\begin{pmatrix} -S_1S_2^* & iC_1S_2^* \\ 0 & 0 \end{pmatrix}, \tag{A.4}$$

$\mathfrak{a}'$, $\mathfrak{b}'$, $\mathfrak{c}'$ are obtained from $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ by replacing C_1 and S_1:

$$\overline{C}_1=\begin{cases} 0 \\ C_1'-C_1, \end{cases}\qquad \overline{S}_1=\begin{cases} 0 \\ S_1'-S_1 \end{cases}, \tag{A.5}$$

$$C_i'=\operatorname{ch}2K_i',\ S_i'=\operatorname{sh}2K_i',\ K_i'=J_i'/kT.$$

The first line of (A.5) corresponds to the defect-free lattice, and the second to the lattice with defects.

We introduce the casual Green's functions defined at $m_1\neq m_2$:

$$\langle\!\langle a_{m_1,n_1}a_{m_2,n_2}\rangle\!\rangle=\begin{bmatrix} \langle Ta_{m_1,n_1}^{+}a_{m_2,n_2}^{+}\rangle & \langle Ta_{m_1,n_1}^{+}a_{m_2,n_2}^{-}\rangle \\ \langle Ta_{m_1,n_1}^{-}a_{m_2,n_2}^{+}\rangle & \langle Ta_{m_1,n_1}^{-}a_{m_2,n_2}^{-}\rangle \end{bmatrix} \tag{A.6}$$

and at $m_1=m_2=m$:

$$\langle\!\langle a_{m,n_1}^{+}a_{m,n_2}^{+}\rangle\!\rangle=\Big\langle s_{m,n_1-1}s_{m,n_2-1}\prod_{\alpha=n_1}^{n_2-1}g_{m\alpha}\Big\rangle,\quad n_2\geqslant n_1.$$

$$\langle\!\langle a_{m,n_1}^{-}a_{m,n_2}^{-}\rangle\!\rangle=\Big\langle s_{m,n_1}s_{m,n_2}\prod_{\alpha=n_1}^{n_2-1}g_{m\alpha}\Big\rangle,\quad n_2\geqslant n_1,$$

$$\langle\!\langle a_{m,n_1}^{+}a_{m,n_2}^{-}\rangle\!\rangle=-i\Big\langle s_{m,n_1-1}s_{m,n_2}\prod_{\alpha=n_1}^{n_2-1}g_{m\alpha}\Big\rangle,\quad n_2\geqslant n_1, \tag{A.7}$$

$$\langle\!\langle a_{m,n_1}^{-}a_{m,n_2}^{+}\rangle\!\rangle=-i\Big\langle s_{m,n_1}s_{m,n_2-1}\prod_{\alpha=n_1}^{n_2-1}g_{m\alpha}\Big\rangle,\quad n_2>n_1.$$

At $n_1>n_2$ we define the Green's function in such a way that the anticommutation relations (10) are satisfied. By the same token we make the problem of the two-dimensional Ising lattice equivalent to the problem of noninteracting fermions.[16]

The equations of motion for the Green's functions are obtained from (A.2) with allowance for the anticommutation relations (10) and (A.7):

$$\langle\!\langle a_{m_1+1,n_1}a_{m_2,n_2}\rangle\!\rangle-\sum_{n}P_{n_1,n}\langle\!\langle a_{m_1,n}a_{m_2,n_2}\rangle\!\rangle=-2\delta_{m_1,m_2}P_{n_1,n_2}. \tag{A.8}$$

Taking into account the translational invariance with

respect to m, we change over to the equations of motion for the Fourier transforms of the Green's functions

$$e^{i\varphi}G_{n_1,n_2}-\sum_n P_{n_1,n}G_{n,n_2}=-2P_{n_1,n_2},$$

$$\langle\langle a^{\Delta_1}_{m_1,n_1}a^{\Delta_2}_{m_2,n_2}\rangle\rangle=(2\pi)^{-1}\int_{-\pi}^{\pi}\exp[i(m_1-m_2)\varphi]G^{\Delta_1\Delta_2}_{n_1,n_2}(\varphi)\,d\varphi. \tag{A.9}$$

The solution (A.9) is written in the form

$$G=2[I+e^{i\varphi}A^{-1}], \tag{A.10}$$

$$A=P-e^{i\varphi}I. \tag{A.11}$$

Thus, the determination of the Green's function reduces to finding the matrix inverse to (A.11). For a defect-free lattice P, and consequentely also A, is cyclic, and therefore its inverse matrix can be easily obtained with the aid of the Fourier transformation. As a result we get the following expressions for the Green's functions of a defect-free lattice[18]:

$$\begin{pmatrix}\langle\langle a^{+}_{m_1,n_1}a^{+}_{m_2,n_2}\rangle\rangle_0 & \langle\langle a^{+}_{m_1,n_1}a^{-}_{m_2,n_2}\rangle\rangle_0\\ \langle\langle a^{-}_{m_1,n_1}a^{+}_{m_2,n_2}\rangle\rangle_0 & \langle\langle a^{-}_{m_1,n_1}a^{-}_{m_2,n_2}\rangle\rangle_0\end{pmatrix}$$

$$=(2\pi)^{-1}\int_{-\pi}^{\pi}d\varphi\exp[-|m_1-m_2|\gamma+i(n_1-n_2)\varphi] \tag{A.12}$$

$$\times\begin{pmatrix}\operatorname{sign}(m_2-m_1)+iS_1S_2^{*}\sin\varphi/\operatorname{sh}\gamma & -i\Phi_0(\varphi)+(C_1-1)S_2^{*}\sin\varphi/\operatorname{sh}\gamma\\ i[\Phi_0(\varphi)]^{-1}-(C_1-1)S_2^{*}\sin\varphi/\operatorname{sh}\gamma & \operatorname{sign}(m_2-m_1)-iS_1S_2^{*}\sin\varphi/\operatorname{sh}\gamma\end{pmatrix},$$

where

$$\operatorname{ch}\gamma=C_1C_2^{*}-S_1S_2^{*}\cos\varphi,\quad \gamma>0,$$
$$\Phi_0(\varphi)=g_+(e^{i\varphi})g_-(\epsilon^{i\varphi}),\quad g_\pm(e^{i\varphi})=[(1-\alpha_{1,2}e^{\pm i\varphi})/(1-\alpha_{2,1}e^{\pm i\varphi})]^{1/2}, \tag{A.13}$$
$$\alpha_{1,2}=z_1^{\pm1}(1-z_2)/(1+z_2),\quad z_{1,2}=\operatorname{th}K_{1,2}.$$

It is known[20] that Green's functions of a perturbed system are expressed in terms of the Green's functions of the unperturbed system and the matrix elements of the perturbation operator. As a result we obtain for a lattice with defects of the first kind, after setting up the linear combinations (12),

$$\langle\langle b^{\Delta_1}_{m_1,n_1}b^{\Delta_2}_{m_2,n_2}\rangle\rangle=\langle\langle b^{\Delta_1}_{m_1,n_1}b^{\Delta_2}_{m_2,n_2}\rangle\rangle_0+\langle\langle b^{\Delta_1}_{m_1,n_1}b^{\Delta_2}_{m_2,n_2}\rangle\rangle_s, \tag{A.14}$$

$$\begin{pmatrix}\langle\langle b^{+}_{m_1,n_1}b^{+}_{m_2,n_2}\rangle\rangle_s & \langle\langle b^{+}_{m_1,n_1}b^{-}_{m_2,n_2}\rangle\rangle_s\\ \langle\langle b^{-}_{m_1,n_1}b^{+}_{m_2,n_2}\rangle\rangle_s & \langle\langle b^{-}_{m_1,n_1}b^{-}_{m_2,n_2}\rangle\rangle_s\end{pmatrix}$$

$$=(2\pi)^{-1}\int_{-\pi}^{\pi}d\varphi\exp[i(m_1-m_2)\varphi+(2-n_1-n_2)\eta]\begin{pmatrix}T_1^{+}(\varphi) & i\Phi_1(\varphi)\\ -i\Phi_1(\varphi) & T_1^{-}(\varphi)\end{pmatrix};$$

$$T_1^{\pm}(\varphi)=-i\sin\varphi[(S_2^{*}\operatorname{sh}\eta)^{-1}-(1\mp\varkappa_1)(S_2^{*}\operatorname{sh}\eta-\varkappa_1t_1)^{-1}]S_1^{-1},$$
$$t_1=S_1C_2^{*}-C_1S_2^{*}\operatorname{ch}\eta, \tag{A.15}$$

$$\Phi_1(\varphi)=[t_1(S_2^{*}\operatorname{sh}\eta)^{-1}+(\varkappa_1S_2^{*}\operatorname{sh}\eta-t_1)(S_2^{*}\operatorname{sh}\eta-\varkappa_1t_1)^{-1}]S_1^{-1}, \tag{A.16}$$

$$\varkappa_1=(C_1-C_1')(C_1C_1'-1)^{-1}, \tag{A.17}$$

$\eta=\eta(\varphi)$ is defined as the positive solution of the equation

$$\operatorname{ch}\eta=(C_1C_2^{*}-\cos\varphi)(S_1S_2^{*})^{-1}. \tag{A.18}$$

The Green's functions for the lattice with defects of the second kind; it is simpler, however to use for this purpose dual transformations.[28] As a result we obtain expressions (A.14), where the functions $T^{\pm}(\varphi)$ and $\Phi_1(\varphi)$ are replaced respectively by

$$T_2^{\pm}(\varphi)=-i\sin\varphi\{(\operatorname{ch}\eta\pm C_1\operatorname{sh}\eta)(S_1\operatorname{sh}\eta)^{-1}+[\varkappa_2(C_1C_2^{*}\operatorname{ch}\eta-S_1S_2^{*}\pm C_2^{*}\operatorname{sh}\eta)-(\operatorname{ch}\eta\pm C_1\operatorname{sh}\eta)\times(S_1\operatorname{sh}\eta-\varkappa_2t_2)^{-1}](S_2^{*})^{-1}e^{\eta},\quad t_2=S_1C_2^{*}\operatorname{ch}\eta-C_1S_2^{*}, \tag{A.19}$$

$$\Phi_2(\varphi)=[t_2(S_1\operatorname{sh}\eta)^{-1}+(\varkappa_2S_1\operatorname{sh}\eta-t_2)(S_1\operatorname{sh}\eta-\varkappa_2t_2)^{-1}](S_2^{*})^{-1}e^{\eta}, \tag{A.20}$$

$$\varkappa_2=\operatorname{th}2(K_2-K_2'). \tag{A.21}$$

It must be noted that the solution (A.14) represents the Green's functions of the right-hand side of the lattice, i.e., the functions made up of the fermion operators

$$a_{mn}^{+}=\operatorname{ch}K_1b_{mn}^{+}+i\operatorname{sh}K_1b_{mn}^{-}\quad(n\geqslant2)$$

and

$$a_{mn}^{-}=\operatorname{ch}K_1b_{mn}^{-}-i\operatorname{sh}K_1b_{mn}^{+}\quad(n\geqslant1).$$

[1)]The case $\varkappa=\pm1$ corresponds to a semi-infinite model and will be the subject of a separate study.

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