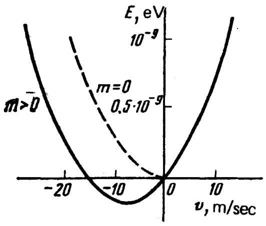

FIG. 1.

no longer the case (see the figure). At $m=0$, the plot of the electron energy against its velocity along the dislocation has a kink at the point $v=0$. On the other hand, if $m>0$ then the minimum of the energy is reached at the drift velocity $-a\omega$, and not at $v=0$. The numbers marked on the figure correspond to $a=0.5\times10^{-8}$ cm, $\omega=1.6\times10^{11}$ rad/sec($H=10^4$ G), and $\mu=9.1\times10^{-28}$ g. The depth of the bound state at $m>0$ is very small ($\sim1.8\times10^{-10}$ eV). If $\varkappa=0$, then the screw dislocation exerts no influence on the electron.

The subject touched upon here is related, for example, to Refs. 2 and 3. In Ref. 2 is considered the dynamics of an electron near a linear defect with axisymmetrical potential $V\sim1/r$. Such a potential offers no advantages to any of the two directions long the defect. In Ref. 3 is considered the case of a screw dislocation. Allowance for the anisotropy of the conductivity tensor leads here to spiral trajectories of the electron when moving along the dislocation, leading to the prediction that a weak magnetic moment appears, parallel to the dislocation when an electron is made to flow along the dislocation. This agrees with our results, since the states of the electron with $m>0$ and $m<0$ are not on a par at $v\neq0$.

I am grateful to Corresponding Member of the USSR Academy of Sciences Yu. A. Osip'yan for suggesting the problem and to E. P. Vol'skii, V. P. Gantmakher, and V. Ya. Kravchenko for interest in the work.

[1]L. D. Landau and E. M. Lifshitz, Kvantovaya Mekhanika (Quantum Mechanics, Nonrelativistic Theory), Nauka, 1974, p. 525 [Pergamon].
[2]Yu. P. Boglaev and E. P. Vol'skii, Fiz. Tverd. Tela (Leningrad) 18, 3288 (1976) [Sov. Phys. Solid State 18, 1916 (1976)]. (1976)].
[3]T. Kondo and K. Kuroda, Sol. State Comm. 26, 61 (1978).

Translated by J. G. Adashko

Calculation of critical exponents by quantum field theory methods

A. A. Vladimirov, D. I. Kazakov, and O. V. Tarasov

Joint Institute for Nuclear Research
(Submitted 26 February 1979)
Zh. Eksp. Teor. Fiz. 77, 1035–1045 (September 1979)

The Gell-Mann–Low function and the anomalous dimensionalities of the quantum-field model $\mathcal{L}_{\rm int}=-(4\pi)^2g(\varphi^2)^2/4!$ are calculated in a four-loop approximation in the dimensional renormalization formalism. They are used to determine the coefficients of the ϵ expansion for the critical exponents up to the degree ϵ^4 inclusive. To reduce the series of the ϵ expansion, a summation method is used that includes a modified Borel transformation and conformal mapping. The obtained critical exponents are in good agreement with experiment and with results of other theoretical approaches.

PACS numbers: 64.60.Fr, 05.70.Jk

1. INTRODUCTION

The far-reaching analogies between statistical physics and quantum field theory[1] can be used effectively to obtain quantitative predictions concerning the character of the behavior of statistical systems in the vicinity of the phase-transition point.[2] The decisive role in this approach is played by the renormalization-group[3] and ε-expansion[4] methods. On the basis of a calculation of the usual quantum-field Feynman diagram of the φ^4 model in a space of $4-2\varepsilon$ dimensions, and of the solution of the renormalization-group equations, the critical exponents of the phase transitions are presented in the form of series in powers of ε, with the physical (three-dimensional) case corresponding to a value $\varepsilon=1/2$.

The greatest progress in this direction was made by Gurevich and Firsov[5] and by Levinson[6] who succeeded in calculating the contributions of the three-loop and some of the four-loop diagrams. However, in view of the asymptotic character of the obtained series in ε, further progress in this direction presupposes not only inclusion of diagrams of ever increasing order, but also the use of methods for "improving" and summing the asymptotic series. The realization of this program is the purpose of the present paper.

Recently, a number of workers[7,8] have developed simple and quite effective methods of calculating contributions of Feynman diagrams to the renormalization-group functions. The use of this technique has enabled us to

 © 1980 American Institute of Physics

calculate in analytic form the contributions of all the necessary diagrams and to carry through to conclusion the four-loop calculations in the φ^4 model. This makes it possible to write down for all the critical exponents series in powers of ε up to terms ε^4 inclusive. We apply to these series a summation method developed by us previously,[9] which includes a modified Borel transformation and conformal mapping under the sign of the Borel integral. The critical exponents obtained in this manner are in good agreement both with the experimental data and with the results of other theoretical approaches.[10]

2. RENORMALIZATION-GROUP FUNCTIONS OF THE φ^4 MODEL IN THE FOUR-LOOP APPROXIMATION

This section is the purely quantum-field part of our article. It is devoted to the calculation of the renormalization-group functions (the anomalous dimensionalities in the Gell-Mann–Low function) of the φ^4 model in the dimensional renormalization scheme.

We consider an $O(n)$-symmetry model of an n-component scalar field, specified by the Lagrangian

$$\mathscr{L}=\frac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a-\frac{m_B^2}{2}\varphi^a\varphi^a-\frac{16\pi^2}{4!}g_B(\varphi^a\varphi^a)^2,\qquad a=1,2,\ldots,n. \tag{1}$$

To calculate the Feynman diagrams we use dimensional regularization and the procedure of minimal substrations or, equivalently, the scheme of dimensional renormalization.[11] Namely, we subtract from each diverging integral only the singular terms of its expansion in a Laurent series in $\varepsilon=(4-d)/2$ where d is the dimensionality of space-time. In terms of the renormalization constants, this means that the latter are expanded in a series in reciprocal powers of ε:

$$Z\left(\frac{1}{\varepsilon},g\right)=1+\sum_{\nu=1}^{\infty}\frac{c_\nu(g)}{\varepsilon^\nu}. \tag{2}$$

The renormalized Green's functions Γ_R are obtained from the regularized ones by the following limiting transition:

$$\Gamma_R(\{p\},\mu^2,m^2,g)=\lim_{\varepsilon\to0}Z_\Gamma\left(\frac{1}{\varepsilon},g\right)\Gamma(\{p\},m_B^2,g_B,\varepsilon), \tag{3}$$

where

$$m_B^2=Z_m\left(\frac{1}{\varepsilon},g\right)m^2,\quad g_B=(\mu^2)^\varepsilon gZ_4\left(\frac{1}{\varepsilon},g\right)Z_2^{-2}\left(\frac{1}{\varepsilon},g\right). \tag{4}$$

Here $\{p\}$ are the momentum arguments of the Green's function Γ; μ is the renormalization parameter; Z_m, Z_4, and Z_2 are respectively the renormalization constants of the mass, of the four-point vertex Γ_4, and of the reciprocal propagator $D^{-1}\equiv\Gamma_2$. The quantities Z_m and g_B satisfy relations analogous to (2):

$$Z_m\left(\frac{1}{\varepsilon},g\right)=1+\sum_{\nu=1}^{\infty}\frac{b_\nu(g)}{\varepsilon^\nu}, \tag{5}$$

$$g_B=(\mu^2)^\varepsilon\left(g+\sum_{\nu=1}^{\infty}\frac{a_\nu(g)}{\varepsilon^\nu}\right). \tag{6}$$

The functions $a_\nu(g)$, $b_\nu(g)$ and $c_\nu(g)$ (their independence of the mass is proved in Ref. 12) are uniquely calculated by perturbation theory by the requirement that the limiting transition (3) be possible. The quantities $a_1(g)$, $b_1(g)$, and $c_1(g)$ are connected with the functions that enter in the differential equation of the renormalization group

$$\left[\mu^2\frac{\partial}{\partial\mu^2}+\beta(g)\frac{\partial}{\partial g}+\gamma_m(g)m^2\frac{\partial}{\partial m^2}-\gamma_\Gamma(g)\right]\Gamma_R(\{p\},\mu^2,m^2,g)=0, \tag{7}$$

by the relations[13]

$$\beta(g)=\frac{\partial g}{\partial\ln\mu^2}\bigg|_{g_B}+\varepsilon g=\left(g\frac{\partial}{\partial g}-1\right)a_1(g), \tag{8}$$

$$\gamma_m(g)=\frac{\partial\ln m^2}{\partial\ln\mu^2}\bigg|_{m_B^2,g_B}=g\frac{\partial}{\partial g}b_1(g), \tag{9}$$

$$\gamma_\Gamma(g)=\frac{\partial\ln Z_\Gamma}{\partial\ln\mu^2}\bigg|_{m_B^2,g_B}=-g\frac{\partial}{\partial g}c_1(g). \tag{10}$$

Consequently, to find the Gell-Mann–Low function $\beta(g)$ and the anomalous dimensionalities $\gamma(g)$ we must know the coefficients of $1/\varepsilon$ in the expansions of the renormalization constants Z in the reciprocal powers of ε. The technique of such calculations was developed in Refs. 7 and 8. It was used by us to carry out the corresponding calculations in the four-loop approximation of the theory (1). The functions $\beta(g)$, $\gamma_2(g)$, and $\gamma_4(g)$ were determined from formulas (8) and (10), and to find $\gamma_m(g)$ we found it convenient to use the equality $\gamma_m(g)=\gamma_2(g)-\gamma_{\varphi^2}(g)$, which following from the relation $Z_m=Z_{\varphi^2}Z_2^{-1}$, where

$$\gamma_{\varphi^2}=\frac{\partial\ln Z_{\varphi^2}(g)}{\partial\ln\mu^2}\bigg|_{m_B^2,g_B},$$

and Z_{φ^2} is the renormalization constant of the two-point Green's function with the insert φ^2, i.e.,

$$\left\langle\varphi(x)\varphi(0)\int dy\,\varphi^2(y)\right\rangle.$$

For the renormalization-group functions we obtain the following expressions:

$$\gamma_2(g)=\frac{g^2}{36}(n+2)-\frac{g^3}{16\cdot27}(n+2)(n+8)+\frac{5g^4}{64\cdot81}(n+2)(-n^2+18n+100)+O(g^5), \tag{11}$$

$$\begin{aligned}\gamma_m(g)={}&\frac{g}{6}(n+2)-\frac{5g^2}{36}(n+2)+\frac{g^3}{72}(n+2)(5n+37)\\&-\frac{g^4(n+2)}{64\cdot243}\left[-n^2+7578n+31060+288\zeta(4)(5n+22)\right.\\&\left.+48\zeta(3)(3n^2+10n+68)\right]+O(g^5),\end{aligned} \tag{12}$$

$$\begin{aligned}\gamma_4(g)={}&-\frac{g}{6}(n+8)+\frac{g^2}{9}(5n+22)-\frac{g^3}{16\cdot27}(35n^2+942n+2992\\&+96\zeta(3)(5n+22))+\frac{g^4}{8\cdot243}[-5n^3+1640n^2+20624n+49912\\&+24\zeta(3)(63n^2+764n+2332)-72\zeta(4)(5n^2+62n+176)\\&+480\zeta(5)(2n^2+55n+186)]+O(g^5),\end{aligned} \tag{13}$$

$$\begin{aligned}\beta(g)={}&-g(\gamma_4(g)-2\gamma_2(g))\\={}&\frac{g^2}{6}(n+8)-\frac{g^3}{6}(3n+14)+\frac{g^4}{16\cdot27}(33n^2+922n+2960+96\zeta(3)(5n+22))\\&-\frac{g^5}{32\cdot243}[-5n^3+6320n^2+80456n+196648+96\zeta(3)(63n^2+764n+2332)\\&-288\zeta(4)(5n^2+62n+176)+1920\zeta(5)(2n^2+55n+186)]+O(g^6).\end{aligned} \tag{14}$$

In particular, at $n=1$

$$\gamma_2(g)=\frac{g^2}{12}-\frac{g^3}{16}+\frac{65}{192}g^4, \tag{15}$$

$$\gamma_m(g)=\frac{g}{2}-\frac{5g^2}{12}+\frac{7g^3}{4}-\frac{3g^4}{4}\left(\frac{159}{16}+\zeta(3)+2\zeta(4)\right), \quad (16)$$

$$\gamma_4(g)=-\frac{3}{2}g+3g^2-g^3\left(\frac{147}{16}+6\zeta(3)\right)$$
$$+g^4\left(\frac{297}{8}+39\zeta(3)-9\zeta(4)+60\zeta(5)\right), \quad (17)$$

$$\beta(g)=\frac{3}{2}g^2-\frac{17}{6}g^3+g^4\left(\frac{145}{16}+6\zeta(3)\right)$$
$$-g^5\left(\frac{3499}{96}+39\zeta(3)-9\zeta(4)+60\zeta(5)\right). \quad (18)$$

The presented expressions for the β functions and the anomalous dimensionalities can be used in quantum field theory (in four-dimensional space) to study the ultraviolet asymptotic Green's functions. They can also be used in various departures from perturbation theory and from the continuation of its results into the region of coupling-constant values $g \gtrsim 1$, as was done, for example, in Refs. 9 and 14. In the present paper we use the results to determine the critical exponents of phase transitions within the framework of a field-theoretical approach based on the ε expansion, to critical phenomena.

3. CRITICAL EXPONENTS AND THE ε EXPANSION

The renormalization group method, transferred from quantum field theory to statistical physics, was used successfully to describe the behavior of various systems near a second-order phase transition point. The scale invariance that manifests itself in the critical phenomena finds a natural interpretation in terms of the renormalization group: the scaling behavior in the vicinity of the critical point, due to the appearance in the system of long-range order, can be described in language of Euclidean quantum field theory, which has an infrared-stable point. The fixed point g_0 of the renormalization-group equation

$$p^2\frac{\partial}{\partial p^2}\bar{g}\left(\frac{p^2}{\mu^2},g\right)=\beta\left(\bar{g}\left(\frac{p^2}{\mu^2},g\right)\right) \quad (19)$$

is determined from the condition $\beta(g_0)=0$ and is called infrared-stable if $\beta'(g_0)>0$. In this case the effective charge $\bar{g}(p^2/\mu^2,g)$ tends to g_0 when its momentum argument p^2 tends to zero, i.e., at large distances. If the theory contains an infrared-stable point, the nondimensionalized Green's functions demonstrate at small p^2 a power-law behavior[3]:

$$\Gamma_R \underset{p^2\to 0}{\sim} (p^2)^{-\gamma_\Gamma(g_0)} \quad (20)$$

with an exponent equal to the value of the corresponding anomalous dimensionality at the point $g=g_0$.

A consistent approach to the description of critical phenomena in terms of the φ^4 quantum-field model was developed in Refs. 15 and 16, where explicit connections were obtained between the exponents of the power-law asymptotic relations (20) and the critical exponents that characterize the scaling behavior of the statistical-physics quantities near the critical temperature T_c. Thus, for example, at $T=T_c$ the asymptotic form of the correlation function $\Gamma(\mathbf{x})$ as $|\mathbf{x}|\to\infty$ is determined by the exponent η:

$$\Gamma(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\sim} 1/|\mathbf{x}|^{d-2+\eta}. \quad (21)$$

The correlation length ξ at $t=T-T_c\to 0$ satisfies the following scaling law:

$$\xi \underset{t\to 0}{\sim} t^{-\nu}(1+\mathrm{const}\cdot t^{\omega\nu}+\ldots), \quad (22)$$

where ω characterizes the degree of deviation from scaling. All the remaining critical exponents are expressed in terms of η and ν.[16]

On the basis of the analogy between the partition function and the quantum-field generating functional, we can establish the following parallels between the statistical-physics and field quantities[16]: the quantized field φ is identified with the order parameter, the temperature difference $T-T_c$ with the square of the bare mass m_B^2, the correlation function $\Gamma(\mathbf{x})$ with the propagator $\langle\varphi(x)\varphi(0)\rangle$, and the reciprocal correlation length ξ^{-1} corresponds to the physical mass m_{ph} which determines the position of the pole of the Fourier propagator $D(p^2)$: $D^{-1}(p^2=-m_{ph}^2)=0$. To find the critical exponents on the basis of the φ^4 theory, we must now investigate quantitatively the character of the power-law behavior of m_{ph} as $m_B^2\to 0$, and also the asymptotic behavior of $D(p^2)$ in the limit as $p^2\to 0$ for $m_B^2=0$. The renormalization-group equations make it possible to reduce this problem to a determination of the values of the anomalous didensionalities $\gamma_m(g)$ and $\gamma_2(g)$ at the infrared-stable point $g=g_0$.

As seen from (14), the function $\beta(g)$ vanishes at $g=0$. Thus, at $d=4$ we have $g_0=0$ and all the anomalous dimensionalities $\gamma(g_0)$ also vanish. This means that in the limit as $p_0^2\to 0$ the interaction vanishes, the φ^4 theory is at $d=4$ in fact a free theory in the infrared region.

The idea of the ε expansion is to take the case $d=4$ as the zeroth approximation, construct a perturbation theory in powers of $2\varepsilon=4-d$, and then, putting $2\varepsilon=1$, proceed to the case $d=3$ which is of physical interest. In fact, when carrying out in the φ^4 model the renormalizations at finite $\varepsilon\neq 0$, we must use for the Gell-Mann–Low function the quantity

$$\beta_\varepsilon(g)=-\varepsilon g+\beta(g), \quad (23)$$

which vanishes at a certain point $g_0(\varepsilon)$ and has at this point a positive derivative. At small ε the infrared-stable point $g_0(\varepsilon)$ is close to zero, $g_0(\varepsilon)\sim\varepsilon$, and this makes it possible, by reexpanding $\gamma[g_0(\varepsilon)]$ in powers of ε, to express all the critical exponents in the form of expansions in powers of ε.

We present now explicit relations that connect the critical exponents with the anomalous dimensionalities $\gamma(g_0)$. The corresponding formulas were obtained in Refs. 16, but we find it useful to repeat the derivation of these relations within the framework of the dimensional-renormalization scheme employed by us.

Taking the Fourier transform of (21) and comparing the result with the asymptotic expression for the propagator:

$$D(p^2) \underset{p^2\to 0}{\sim} (p^2)^{-1+\gamma_2(g_0)}, \quad (24)$$

we get

$$\eta=2\gamma_2(g_0). \tag{25}$$

The second exponent ν is connected with the physical mass m_{ph}. It is a renormalization-invariant quantity and satisfies the renormalization-group equation without anomalous dimensionalities:

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\beta_\varepsilon(g)\frac{\partial}{\partial g}+\gamma_m(g)m^2\frac{\partial}{\partial m^2}\right)m_{ph}^2(m^2,\mu^2,g)=0. \tag{26}$$

From the theorem on homogeneous functions we have

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+m^2\frac{\partial}{\partial m^2}\right)m_{ph}^2=m_{ph}^2. \tag{27}$$

Then, eliminating $\mu^2\partial/\partial\mu^2$ from (26) with the aid of (27), we get

$$\left[m^2\frac{\partial}{\partial m^2}-\frac{\beta_\varepsilon(g)}{1-\gamma_m(g)}\frac{\partial}{\partial g}-\frac{1}{1-\gamma_m(g)}\right]m_{ph}^2=0. \tag{28}$$

The use of standard methods of analysis of equations of this type[3] leads to the conclusion that

$$m_{ph}^2 \underset{m^2\to 0}{\sim} (m^2)^{1/[1-\gamma_m(g_0)]}. \tag{29}$$

Assuming μ and g to be fixed, we have

$$m^2\sim m_B^2\equiv t. \tag{30}$$

Consequently

$$\xi^{-1}=m_{ph}\underset{t\to 0}{\sim} t^{1/2[1-\gamma_m(g_0)]}. \tag{31}$$

Hence, comparing (31) with (22), we get

$$\nu={}^1/_2[1-\gamma_m(g_0)]. \tag{32}$$

Allowance for the second term in the expansion of the effective charge in the vicinity of the fixed point

$$\bar g\left(\frac{p^2}{\mu^2},g\right)=g_0+\text{const}\left(\frac{p^2}{\mu^2}\right)^{\beta'(g_0)}+\ldots \tag{33}$$

makes it possible to find the correction to the solution (29) and by the same token determine the exponent ω:

$$\omega=2\beta_\varepsilon'(g_0). \tag{34}$$

On the basis of formulas (25), (32), and (34) and also on the basis of the functions $\gamma_2(g)$, $\gamma_m(g)$, and $\beta(g)$, calculated in the four-loop approximation, we can write down the expansions in powers of ε up to terms $\sim\varepsilon^4$ for $g_0(\varepsilon)$ and for the exponents η, ν, and ω. We note that the coefficients of these expansions do not depend on the employed renormalization scheme. Our results, which are given below, agree with Ref. 6, in which the corresponding calculations were extended up to ε^3 for g_0, ν, and ω and up to ε^4 for η. To recast the results in the standard notation employed in the ε-expansion formalism, we choose the expansion quantity to be 2ε:

$$\begin{aligned}g_0(\varepsilon)=&\frac{3}{n+8}(2\varepsilon)+\frac{9(3n+14)}{(n+8)^3}(2\varepsilon)^2+\frac{3(2\varepsilon)^3}{8(n+8)^5}[-33n^3+110n^2\\&+1760n+4544-96\zeta(3)(5n+22)(n+8)]+\frac{(2\varepsilon)^4}{16(n+8)^7}[-5n^5-2670n^4\\&-5584n^3+52784n^2+309312n+529792+1920\zeta(5)(n+8)^2(2n^2+55n+186)\\&-288\zeta(4)(n+8)^3(5n+22)-96\zeta(3)(-63n^4-422n^3+4452n^2\\&+39432n+72512)]+O(\varepsilon^5),\end{aligned} \tag{35}$$

$$\begin{aligned}\eta=&\frac{(n+2)}{2(n+8)^2}(2\varepsilon)^2\Big[1+\frac{2\varepsilon}{4(n+8)^2}(-n^2+56n+272)+\frac{(2\varepsilon)^2}{16(n+8)^4}\cdot\\&\cdot(-5n^4-230n^3+1124n^2+17920n+46144-384\zeta(3)(5n+22)(n+8))\Big]+O(\varepsilon^5),\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{1}{\nu}=&2-\frac{n+2}{n+8}(2\varepsilon)-\frac{n+2}{2(n+8)^3}(13n+44)(2\varepsilon)^2-\frac{(n+2)(2\varepsilon)^3}{8(n+8)^5}\cdot\\&\cdot[-3n^3+452n^2+2672n+5312-96\zeta(3)(5n+22)(n+8)]\\&-\frac{(n+2)(2\varepsilon)^4}{32(n+8)^7}[-3n^5-398n^4+12900n^3+81552n^2+219968n+357120\\&+1280\zeta(5)(n+8)^2(2n^2+55n+186)-288\zeta(4)(n+8)^3(5n+22)\\&-16\zeta(3)(n+8)(3n^4-194n^3+148n^2+9472n+19488)]+O(\varepsilon^5),\end{aligned} \tag{37}$$

$$\begin{aligned}\omega=&2\varepsilon-\frac{3(3n+14)}{(n+8)^2}(2\varepsilon)^2+\frac{(2\varepsilon)^3}{4(n+8)^4}[33n^3+538n^2+4288n+9568\\&+96\zeta(3)(n+8)(5n+22)]-\frac{(2\varepsilon)^4}{16(n+8)^6}[-5n^5+1488n^4+46616n^3+419528n^2\\&+1750080n+2599552+32\zeta(3)(n+8)(189n^3+1644n^2+5748n+11616)\\&+1920\zeta(5)(n+8)^2(2n^2+55n+186)-288\zeta(4)(n+8)^3(5n+22)]+O(\varepsilon^5).\end{aligned} \tag{38}$$

4. SUMMATION OF THE ε-EXPANSION SERIES

It is well known that the perturbation-theory series in the coupling constant g are asymptotic. In recent years, a technique was developed for determining the asymptotic forms of the coefficients of such series in high orders in g.[17] The ε-expansion series that appear in the solution of the equation $\beta_\varepsilon[g_0(\varepsilon)]=0$ also turn out to be asymptotic. It was shown in Ref. 18 that asymptotic estimates of the coefficients of the perturbation-theory series in g lead to the following estimates of the coefficients of higher orders of the ε expansion $f(2\varepsilon)=\sum_k(-2\varepsilon)^k f_k$:

$$f_k \underset{k\to\infty}{\sim} k!a^k k^b c, \tag{39}$$

where f stands for g_0, η, $1/\nu$, or ω, and the coefficients a and b are given respectively by

$$a=\frac{3}{n+8},\quad b=\begin{cases}4+n/2 & \text{for } g_0\\ 3+n/2 & \text{for } \eta\\ 4+n/2 & \text{for } 1/\nu\\ 5+n/2 & \text{for } \omega\end{cases}. \tag{40}$$

It follows from (39) that the ε-expansion series have a zero convergence radius. Therefore the direct substitution $\varepsilon=1/2$ in (36)–(38) cannot lead to any reliable conclusions concerning the critical exponents in the physical point $d=3$.

For the transition to the value $\varepsilon=1/2$ we shall use a method developed by us[9] for summing asymptotic series, in which account is taken, on the one hand, of the exact values of the coefficients of lower orders (35)–(38), and on the other hand, of information obtained from the asymptotic estimates (39) and (40). The method is based on the use of a modified Borel transformation

$$f(2\varepsilon)=\int_0^\infty\frac{dx}{2\varepsilon a}e^{-x/2\varepsilon a}\left(\frac{x}{2\varepsilon a}\right)^{b+3/2}B(x). \tag{41}$$

Then

$$B(x)=\sum_k(-x)^k B_k,\quad B_k=\frac{f_k}{a^k\Gamma(k+b+5/2)}\underset{k\to\infty}{\sim}\frac{c}{k^{3/2}}. \tag{42}$$

The series (42) for the function $B(x)$ has a unity convergence circle. The function $B(x)$, as follows from (39), is free of singularities on the integration integral $[0,\infty)$ and has a square-root branch point at $x=-1$. To continue the function analytically beyond the limits of the unit circle, we use the conformal mapping $x\to w$:

$$w(x)=[(1+x)^{1/2}-1]/[(1+x)^{1/2}+1]. \tag{43}$$

The integration $[0,\infty)$ then goes over into the segment $[0,1]$, while the cut $(-\infty,-1]$ goes over into the convergence unit circle inside of the series in w obtained by re-expanding the function $B[x(w)]$. The coefficient of w^N is determined on the basis of the coefficients f_k of the initial ε expansion up to $k=N$, inclusive. Therefore, terminating the series in w with the N-th term:

$$B(x)\approx\left(\frac{x}{w}\right)^{\lambda}\sum_{k}^{N}B_k^{(\lambda)}w^k, \tag{44}$$

we obtain for the function $f(2\varepsilon)$ and approximate expression $f_N(2\varepsilon)$ that corresponds to taking N orders of pertubation theory into account.

The choice of the concrete value of the parameter λ introduced here is very important. It determines the exponents of the power-law asymptotic form of the function $f_N(2\varepsilon)$ at large ε:

$$f_N(2\varepsilon)\underset{\varepsilon\to\infty}{\sim}(2\varepsilon)^{\lambda}. \tag{45}$$

With solvable models with known asymptotic form in the coupling constant g as an example, we can see that good convergence of the sequence of the approximants $f_N(g)$ to the true function $f(g)$ is ensured only if the choice of λ is correct, i.e., matched to the asymptotic form of $f(g)$ as $g\to\infty$ (Ref. 9). Since the asymptotic form of the critical exponents relative to ε is unknown, we fix λ just to satisfy the requirement of fastest convergence of our approximation procedure.

We introduce the set of quantities

$$\Delta_N=1-f_N(1)/f_{N-1}(1), \tag{46}$$

which characterize the relative change of f_N when account is taken of the next order of perturbation theory. For a correct guess of the aysmptotic form of the function $f(2\varepsilon)$, the relative errors Δ_n should decrease rapidly. A numerical analysis shows the presence of a sharp minimum of the values of $|\Delta_N|$ at a definite value of λ, separate for each critical exponent. This method was verified for the solvable model and yielded good results.[9] In our case, the use of this method leads to the following values of the parameter λ:

$$\lambda=\begin{cases}1.6-1.7 & \text{for } g_0\\ 2.8-2.9 & \text{for } \eta\\ 1.2-1.3 & \text{for } 1/\nu\\ 0.5-0.6 & \text{for } \omega\end{cases}. \tag{47}$$

To determine the parameter λ we used also another method, proposed in Ref. 20. It was based on the fact that the asymptotic form of $f_N(2\varepsilon)$ is determined in the region of large ε by the numerically estimated asymptotic values of the coefficients $B_k^{(\lambda)}$ as $k\to\infty$. The values of λ obtained in this manner are close to those indicated in (47).

Now, starting from the ε-expansions (36)–(38) and from the values (47) of λ, we can use formulas (41) and (42) at $N=4$ to calculate the critical exponents at the point $2\varepsilon=1$. The results of the summation are given in Table I. The error interval was assumed by us to be $\pm\Delta_4 f_4(1)$. We present also the values of the charge at the fixed point $g_0(2\varepsilon)$ at $\varepsilon=\frac{1}{2}$: $g_0(1)=0.488\pm0.006(n=1)$; $0.436\pm0.006(n=2)$; $0.393\pm0.006(n=3)$.

TABLE I

	1	2	3	4
		$n=1$		
η	0.0333±0.0001	0.0315±0.0025	–	0.016±0.014
ν	0.628±0.002	0.6300±0.0008	$0.638^{+0.002}_{-0.008}$	0.625±0.005
ω	0.781±0.015	0.782±0.010	–	–
		$n=2$		
η	0.0352±0.0001	0.0335±0.0025	–	–
ν	0.666±0.004	0.6693±0.0010	–	0.675±0.001
ω	0.779±0.015	0.778±0.008	–	–
		$n=3$		
η	0.0354±0.0001	0.0340±0.0025	0.043±0.014	–
ν	0.700±0.007	0.7054±0.0011	$0.715^{+0.025}_{-0.015}$	–
ω	0.779±0.007	0.779±0.006	–	

Note. Comparison of our results (column 1) with calculations within the framework of the φ^4 model for $d=3$ (column 2), the high-temperature expansion in the three-dimensional models of Ising ($n=1$) and Heisenberg ($n=3$) (column 3) and with the experiment (column 4). The numbers in columns 2–4 were taken from Ref. 10.

The dependence of the obtained critical exponents on the concrete choice of the parameter λ is illustrated in Table II (for the case $n=1$), from which it is seen that fixing the values of λ [in our approach–by means of relations (47)] is one of the most important aspects of the method described above for summing the asymptotic series.

It is of interest to trace also the variation of the critical exponents as a function of the number of the included terms of the perturbation-theory series. In Table III we give (for $n=1$) the values of the exponents obtained both by directly substituting $2\varepsilon=1$ in (36) and (37), and by using the summation methods described in the present section [respectively $\eta_N^{pt}(1), \nu_N^{pt}(1)$ and $\eta_N(1), \nu_N(1)$]. As seen from the table, the use of special summation methods improved radically the approximating properties of perturbation theory.

The summation technique developed in the present paper was used by us to calculate the critical exponents at $2\varepsilon=2$, which corresponds[21] (at $n=1$) to the two-dimensional Ising model, which admits of an exact solution. For the exponents η and ν, the exact values of which in this model are respectively $\frac{1}{4}$ and 1, we obtained $\eta_4(2)=0.18, \nu_4(2)=0.92$. We note in conclusion that the corrections $\sim\varepsilon^4$ obtained by us for the critical exponents when used alone (i.e., without employing the summation methods) make the agreement with the published results only worse. The procedure described by us has made it possible to improve substantially this agreement, this being both an additional proof of its effectiveness, and a direct confirmation of the applicability of the quantum-field approach, based on the ε expansion, to the calculation of critical exponents.

TABLE II

λ	η	ν	ω	λ	η	ν	ω
0	0.0259	0.617	0.764	2	0.0334	0.630	0.679
1	0.0312	0.627	0.777	3	0.0334	0.638	−0.150

TABLE III

	N					N			
	1	2	3	4		1	2	3	4
$\eta_N^{pt}(1)$	–	0.0185	0.0372	0.0289	$\nu_N^{pt}(1)$	0.6	0.645	0.597	0.731
$\eta_N(1)$	–	0.0320	0.0332	0.0333	$\nu_N(1)$	0.620	0.626	0.625	0.628

We are grateful to V. A. Zagrebnov, A. A. Midgdal, K. G. Chetyrkin, and D. V. Shirkov for useful discussions.

[1]A. M. Polyakov, Zh. Eksp. Teor. Fiz. 55, 1026 (1968). [Sov. Phys. JETP 28, 533 (1969)]. A. A. Migdal, Zh. Eksp. Teor. Fiz. 55, 1964 (1968) [Sov. Phys. JETP 28, 1036 (1968)].
[2]K. G. Wilson and J. Kogut, Phys. Repts. C 12, 75 (1974). A. Z. Patashinskiĭ and V. L. Pokrovskii, Usp. Fiz. Nauk 121, 55 (1977) [Sov. Phys. Usp. 20, 31 (1977)].
[3]N. N. Bogolyubov and D. V. Shirkov, Vvedenie v teoriyu kvantovannykh polei (Introduction to the Theory of Quantized Fields), Nauka, 1976. [Wiley]
[4]K. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).
[5]E. Brezin, J. C. LeGuillou, J. Zinn-Justin, and B. G. Nickel, Phys. Lett. 44A, 227 (1973).
[6]E. Brezin, J. C. LeGuillou, and J. Zinn-Justin, Phys. Rev. D9, 1121, 1974.
[7]A. A. Vladimirov, Preprint JINR E2-12248, Dubna, 1979.
[8]K. G. Chetyrkin and F. V. Tkachev, Preprint INR II-125, Moscow, 1979.
[9]D. I. Kazakov, O. V. Tarasov, and D. V. Shirkov, Teor. Mat. Fiz. 38, 15 (1979).
[10]J. C. LeGuillou and J. Zinn-Justin, Phys. Rev. Lett. 39, 95 (1977).
[11]G. t'Hooft, Nucl. Phys. B61, 455 (1973).
[12]J. C. Collins, Nucl. Phys. B80, 314 (1974).
[13]J. C. Collins and A. J. Macfarlane, Phys. Rev. D10, 1201 (1974).
[14]V. S. Popov, V. L. Eletskiĭ, and A. V. Turbiner, Zh. Eksp. Teor. Fiz. 74, 445 (1978) [Sov. Phys. JETP 47, 232 (1978)]. V. L. Eletskii and V. S. Popov, Yad. Fiz. 28, 1109 (1978) [Sov. J. Nucl. Phys. 28, 570 (1978)].
[15]K. Wilson, Phys. Rev. Lett. 28, 548 (1972).
[16]E. Brezin, J. C. LeGuillou, and J. Zinn-Justin, Phys. Rev. D8, 434, 2418 (1973).
[17]L. N. Lipatov, Zh. Eksp. Teor. Fiz. 72, 411 (1977) [Sov. Phys. JETP 45, 216 (1977)]. E. B. Bogomolny, Phys. Lett. 67B, 193 (1977).
[18]E. Brezin, J. C. LeGuillou, and J. Zinn-Justin, Phys. Rev. D15, 1544 (1977).
[19]C. A. Truesdell, Ann. Math. 46, 114 (1945).
[20]O. V. Tarasov, JINR Preprint R2-11879, Dubna, 1978; Lett. Math. Phys. 3, 143 (1979).
[21]J. Hubbard, Phys. Lett. 39A, 365 (1972).

Translated by J. G. Adashko

Oscillation of the conductivity of carriers with anistropic energy spectrum in a quantizing magnetic field

S. P. Andreev

Moscow Engineering Physics Institute
(Submitted 15 March 1979)
Zh. Eksp. Teor. Fiz. 77, 1046–1057 (September 1979)

A system of carriers is considered, having an anisotropic energy spectrum and scattered by randomly disposed attraction centers of finite radius $a \lesssim l (l = (c\hbar)/|e|H)^{1/2}$ is the magnetic length) in a quantizing magnetic field **H**. A new type of oscillations of the kinetic coefficients as a function of the magnetic field is observed. The oscillations are due to the dependence of the one-dimensional (on account of the magnetic field) scattering potential of each individual center on H and to the anisotropy of the effective mass of the carriers. The longitudinal and transverse conductivities of a gas of interacting electrons with soroidal equal surfaces ($m_1 = m_2 = m_\perp$, $m_3 = m_\parallel > m_\perp$) in a weak electric field $\mathbf{E}_0$ and in a quantizing magnetic field parallel to the spheroid axis $\mathbf{H} \| m_\parallel \| z$ are calculated. It is shown that both the longitudinal and transverse conductivities oscillate with changing magnetic fields, and the period is mainly $\propto H^{1/2}$. For definite values of the magnetic field intensity, this effect leads to a negative longitudinal magnetoresistance [L. S. Dubinskaya, Sov. Phys. JETP 29, 436 (1969); M. M. Aksel'rod et al., Phys. Stat. Sol. 9, k91, 1965)]. The possibility of experimentally observing the oscillations is discussed.

PACS numbers: 72.10.Di

1. We consider the spectrum of the states of an electron whose mass is highly anisotropic, $m_1 = m_2 = m_\perp \ll m_\parallel = m_3$, in a quantizing magnetic field $\mathbf{H} \| m_\parallel \| z$ and in a spherically symmetrical attraction field $U(r) < 0$ of finite radius $a \leqslant l$. In the absence of a center, the motion of the electron in the (x, y) plane perpendicular to **H** is quantized, and the energy difference between any two neighboring levels is $\hbar\omega_\perp$ ($\omega_\perp = eH/m_\perp c$). By virtue of the axial symmetry of the problem, at $U \neq 0$, the projection m of the orbital angular momentum of the electron on the direction of **H** is conserved. If the mixing of the levels $N = n + \frac{1}{2}(|m| + m)$ and $K = k + \frac{1}{2}(|m| + m)$ (Ref. 3) by the center is small[4]:

$$U_m^{kn} = \int_0^\infty \rho\, d\rho R_{km}(\rho) |U(\rho, 0)| R_{nm}(\rho) \ll \hbar\omega_\perp, \tag{1}$$

 0038-5646/79/090526-07$02.40 © 1980 American Institute of Physics