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Thermal runaway and convective heat transport by fast electrons in a plasma

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The distribution of the electron velocities in an inhomogeneously heated fully ionized plasma is investigated. It is shown that the fast growth of the mean free path with increasing energy gives rise to thermal runaway of the electrons—an abrupt growth of the number of fast electrons in the region of the cold plasma. The electron distribution function has a double Maxwellian character, i.e., it is characterized by two electron temperatures. The higher temperature is possessed by the fast electrons. Critical discontinuities appear, namely unusual types of discontinuities of the distribution function. The structure of the kinetic discontinuity is investigated. It is shown that besides the usual heat flux there appears a convective energy flux carried by the fast electrons. At not too small temperature gradients, the convective transport plays the principal part.

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1. INTRODUCTION

The mean free path of electrons in a plasma increases rapidly with increasing electron energy. Consequently the coupling between the energetic electrons and the plasma is very weak and even small forces cause a considerable deviation of their distribution from equilibrium. For example, even in a weak constant electric field the electron distribution function becomes strongly distorted in the region of high velocities, and a flux of runaway electrons is produced.[1]

Similar distortions of the distribution functions can arise also in the presence of electron-temperature gradients. Indeed, assume that in a given direction x there is present in the plasma a rather weak electron-temperature gradient

$$\gamma=\frac{l_T}{T_e}\left|\frac{dT_e}{dx}\right|\ll 1. \tag{1}$$

Here l_T is the mean free path of the thermal electrons. The temperature gradient (1) produces, naturally, only a small perturbation of the equilibrium distribution of the electrons in the principal (thermal) velocity region. For fast particles, however, the situation is substantially different. Their mean free path increases: $l_\varepsilon = l_T(\varepsilon/T_e)^2$, so that electrons with sufficiently high energy

$$\varepsilon\gtrsim\varepsilon_0=T_e/\gamma^{1/2} \tag{2}$$

move freely between the regions with substantially different temperatures. This leads to a strong distortion of the distribution function. In particular, the number of high-energy electrons in the region of the cold plasma increases sharply. This phenomenon can naturally be called thermal runaway of the electrons. The present paper is devoted to its investigation.

It is important that electrons with energy $\varepsilon\gtrsim\varepsilon_0$ carry heat by convection. Thus, so to speak, two heat fluxes are produced. One is by usual thermal conductivity and is due mainly to the electrons with low energies $\varepsilon \lesssim 10T_e$. The second flux is convective and due to fast electrons $\varepsilon\gtrsim\varepsilon_0$. At a sufficiently small electron-temperature gradient $\gamma<\gamma_k\approx 10^{-2}$ the principal role is played by the thermal conductivity. At $\gamma>\gamma_k$, on the contrary, the convective transport is more important. In this case the deformation of the distribution function exerts a decisive influence on the heat transport in the plasma, which proceeds mainly via convection by the fast electrons. It has a kinetic character and cannot be described within the framework of ordinary transport theory.

In the present paper we confine ourselves to plasma electrons; effects of similar type are typical, however, also of ions. We note also that, as shown in Ref. 2, perfectly analogous phenomena arise in transverse transport of supertrapped electrons and ions in toroidal magnetic traps.

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2. SIMPLIFICATION OF THE KINETIC EQUATION

We examine the effect of thermal runaway via electrons. Let the space be filled with a fully ionized plasma whose electron density and temperature vary along a single coordinate x. Assume that the plasma is not magnetized or that the magnetic field is directed along x. For simplicity we assume also that the electron temperature varies monotonically from a value T_{e0} as $x \to -\infty$ to $T_{e1} < T_{e0}$ as $x \to +\infty$. The electron temperature gradient will be assumed quite weak (1).

The kinetic equation for the electron distribution $f(x, v, \theta)$ is of the form

$$\frac{\partial f}{\partial t}+v\cos\theta\frac{\partial f}{\partial x}+\frac{e}{m}\nabla\Phi\frac{\partial f}{\partial \mathbf{v}}+S(f)=0. \tag{3}$$

Here θ is the angle between the velocity $\mathbf{v}$ and the x axis, Φ is the electric field potential, and S is the collision integral. We take into account the fact that under condition (1) the distribution function is strongly perturbed by the thermal runaway only in the region of high electron energies $\varepsilon \gtrsim \varepsilon_0 \gg T_e$(2). This simplifies Eq. (3) substantially.

In fact, assuming that the electric field Φ is not produced by external sources but is the intrinsic polarization field of the plasma, we can neglect its influence, since

$$|\Phi(x\to-\infty)-\Phi(x\to+\infty)|\sim T_e/e.$$

Next, recognizing that the plasma is quasineutral and is characterized by hydrodynamic motions with velocities of the order of thermal ion velocities, we can assume the electron distribution to be quasistationary, i.e., we can neglect in (3) the term $\partial f/\partial t$ (first-order approximation in the parameter m/M_i). Finally, since the number of fast electrons is small, we can neglect the collisions between them and express the collision integral in (3) in a linearized form[1,3]:

$$S(f)=-\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^3\nu(v)\left[\frac{T_e}{m}\frac{\partial f}{\partial v}+vf\right]\right\}-k\nu(v)\frac{\partial}{\partial\cos\theta}\left[\sin^2\theta\frac{\partial f}{\partial\cos\theta}\right]. \tag{4}$$

Here $\nu(v)$ is the electron collision frequency

$$\nu(v)=\frac{4\pi e^4 N\ln\Lambda}{m^2v^3}, \tag{5}$$

$\ln\Lambda$ is the Coulomb logarithm, and $k=(1+Z)/2$, where

$$Z=\sum_i\frac{Z_i^2N_i}{N}$$

is the effective charge of the ions.

We now introduce the dimensionless variables

$$n=\frac{N_e(x)}{N_0},\quad t=\frac{T_e(x)}{T_{e0}},\quad \mu=\cos\theta,$$
$$u^2=\frac{mv^2}{T_{e0}},\quad \lambda=\frac{1}{l_T}\int_0^x n(x)\,dx. \tag{6}$$

Here l_T is the mean free path of the thermal electrons

$$l_T=\left(\frac{T_{e0}}{m}\right)^{1/2}\Big/\nu\left[\left(\frac{T_{e0}}{m}\right)^{1/2}\right].$$

Equation (3) assumes in these variables the form[1)]

$$u\mu\frac{\partial f}{\partial\lambda}-\frac{1}{u^2}\frac{\partial}{\partial u}\left\{\frac{1}{u}\left(t\frac{\partial f}{\partial u}+uf\right)\right\}-\frac{k}{u^3}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial f}{\partial\mu}\right]=0. \tag{7}$$

We replace, as before,[1] f by a new function φ:

$$f=e^{-\varphi} \tag{8}$$

and in addition, separate in explicit form the small parameter of the problem

$$\gamma=\frac{l_T}{T_{e0}}\left|\frac{dT_e}{dx}\right|_0, \tag{9}$$

where $(dT_e/dx)_0$ is the maximum value of the temperature gradient. To this end we introduce in place of u^2 and λ the new variables

$$z=\gamma^{1/2}u^2,\qquad \tau=\gamma\lambda. \tag{10}$$

For the function $\varphi(\tau, z, \mu)$ we obtain the equation

$$z^2\mu\frac{\partial\varphi}{\partial\tau}-2z\frac{\partial\varphi}{\partial z}\left(1-2t\gamma^{1/2}\frac{\partial\varphi}{\partial z}\right)-4\gamma^{1/2}tz\frac{\partial^2\varphi}{\partial z^2}$$
$$+k\left\{2\mu\frac{\partial\varphi}{\partial\mu}+(1-\mu^2)\left(\frac{\partial\varphi}{\partial\mu}\right)^2-(1-\mu^2)\frac{\partial^2\varphi}{\partial\mu^2}\right\}=0. \tag{11}$$

It is natural to seek the solution of (11) in the form of a series in powers of the small parameter γ:

$$\varphi=\frac{1}{\gamma^{1/2}}\varphi_0+\frac{1}{\gamma^{1/4}}\varphi_1+\varphi_2+\dots. \tag{12}$$

Substituting the expansion (12) in (11) and equating terms at equal powers of γ, we obtain

$$(1-\mu^2)\left(\frac{\partial\varphi_0}{\partial\mu}\right)^2=0, \tag{13}$$

$$\mu z^2\frac{\partial\varphi_0}{\partial\tau}-2z\frac{\partial\varphi_0}{\partial z}\left(1-2t\frac{\partial\varphi_0}{\partial z}\right)+k(1-\mu^2)\left(\frac{\partial\varphi_1}{\partial\mu}\right)^2=0, \tag{14}$$

$$\mu z^2\frac{\partial\varphi_1}{\partial\tau}-2z\frac{\partial\varphi_1}{\partial z}\left(1-4t\frac{\partial\varphi_0}{\partial z}\right)+k\left[2\mu\frac{\partial\varphi_1}{\partial\mu}\right.$$
$$\left.-(1-\mu^2)\frac{\partial^2\varphi_1}{\partial\mu^2}+2(1-\mu^2)\frac{\partial\varphi_1}{\partial\mu}\frac{\partial\varphi_2}{\partial\mu}\right]=0. \tag{15}$$

The chain of equations (13)–(15) defines the sought distribution function.

3. SPHERICALLY SYMMETRICAL PART OF THE DISTRIBUTION FUNCTION. KINETIC DISCONTINUITIES

It follows from (13) that the main function is spherically symmetrical:

$$\varphi_0=\varphi_0(\tau, z). \tag{16}$$

The equation for the function $\varphi_0(\tau, z)$ follows from the condition for the solvability of Eq. (14) for φ_1. Indeed, recognizing, just as in Refs. 1, 2, and 4 that in the absence of an additional electron source the derivative $\partial\varphi_1/\partial\mu$ has in velocity space no singularities as $\mu\to1$, we get from (14)

$$z\frac{\partial\varphi_0}{\partial\tau}-2\frac{\partial\varphi_0}{\partial z}\left(1-2t\frac{\partial\varphi_0}{\partial z}\right)=0. \tag{17}$$

Introducing

$$y=2\frac{\partial\varphi_0}{\partial z}, \tag{18}$$

we rewrite (17) in the form

$$\frac{\partial y}{\partial \tau}-\frac{\partial}{\partial z}\left[\frac{2}{z}y(1-ty)\right]=0. \quad (19)$$

According (18), (12), (10), and (6) the quantity y has the physical meaning of the reciprocal local effective temperature:

$$y=\frac{T_{e0}}{T_e(v,x)}, \quad T_e(v,x)=-mv\Big/\frac{\partial \ln f}{\partial v}. \quad (20)$$

Recognizing that as $v\to 0$ the distribution function is close to a locally Maxwellian function with a specified temperature profile $T_e(x)$, we have the following boundary condition:

$$y(z,\tau)=1/t(\tau) \quad \text{as} \quad z\to 0. \quad (21)$$

We consider first the electron distribution function at

$$\tau\gg 1, \quad (22)$$

i.e., in the region of a cold plasma far from the temperature transition. In this region, the temperature is constant:

$$t(\tau)=t=\text{const}. \quad (23)$$

In addition, the temperature transition itself, whose width is $\Delta\tau\sim 1$, can be regarded under the conditions (22) as sufficiently abrupt, i.e., accurate to $\sim 1/\tau$ we can neglect the concrete structure of the transition and assume $t(\tau)$ to be a discontinuous function:

$$t(\tau)=\begin{cases}1 & \text{as} \quad \tau<0\\ t=\text{const} & \text{as} \quad \tau>0\end{cases}. \quad (24)$$

Neither Eq. (19) itself with the function $t(\tau)$ (24) nor the boundary condition (21) contains parameters that specify the characteristic dimensions in terms of the variables τ and z. Consequently, the solution of (19) in the present case can depend only on the ratio z/τ^{α}, i.e., it should be self-similar. It follows from (19) that $\alpha=\frac{1}{2}$. Putting therefore

$$p=z/\tau^{1/2}, \quad (25)$$

we get from (19)

$$p\frac{dy}{dp}\left(1-2ty+\frac{p^2}{4}\right)=y(1-ty). \quad (26)$$

Equation (26) has singular solutions:

$$y=0, \quad y=1/t.$$

The general solution, obtained by the substitution $p=y(1-ty)p_1(y)$, is of the form

$$p=2y(1-ty)/(C-y^2+{}^2/_3ty^3)^{1/2}, \quad (27)$$

where C is an arbitrary constant.

The boundary conditions (21) and (24) in terms of the self-similar variables are the following:

$$y=1/t, \quad p=0; \quad y=1, \quad p=\infty. \quad (28)$$

Consequently, the continuous solution that satisfies the conditions (28) is

$$y=1/t, \quad 0\leqslant p\leqslant 2. \quad (29a)$$

$$p(y)=2\cdot 3^{1/2}ty/(1+2ty)^{1/2}, \quad 0\leqslant p\leqslant 2, \quad (29b)$$

$$p(y)=2y(1-ty)/[1-y^2-{}^2/_3t(1-y^3)]^{1/2}, \quad 0\leqslant p\leqslant\infty. \quad (29c)$$

The function $y(p)$ is shown in Fig. 1 (solid curves). In the region $0<p<2$ it is triple-valued. This cannot be, because actually (26) has no continuous solution that satisfies the boundary conditions (28). A strong discontinuity is produced and is shown dashed in Fig. 1.

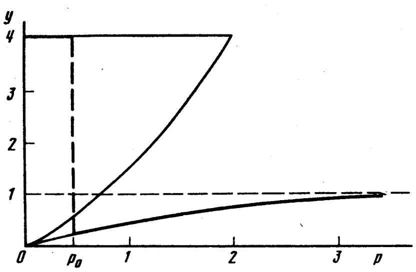

FIG. 1. Plots of the function $y(p)$ at $\tau\gg 1$ ($t=0.25$). The dashed line shows the position of the discontinuity p_0: $p=z/\tau^{1/2}$.

The conditions on the discontinuity follow directly from (19). We have

$$[y]_\tau+\left[\frac{2}{z}y(1-ty)\right]_z=0. \quad (30)$$

In the self-similar case relation (30) takes the form

$$\left[\frac{p}{4}y+\frac{y}{p}(1-ty)\right]_{p_0}=0. \quad (31)$$

Therefore, taking (29) into account, we obtain p_0, i.e., the discontinuity point:

$${}^1/_4p_0^2=1-(1-t)^{2/3}(2t+1)^{1/3}. \quad (32)$$

The value of y changes at the discontinuity point from $y=1/t$ to $y=y_0=p_0^2/4t$. The value of the jump $\Delta y=1/t-y_0$ has the following dependence on the cold-plasma temperature:

$$\Delta y=t^{-1}(1-t)^{2/3}(2t+1)^{1/3}.$$

The form of the distribution function of the electrons is shown in Fig. 2. The Maxwellian distribution with a local cold-plasma temperature t is valid only up to energies $z=z_0=p_0\tau^{1/2}$. At larger z it changes radically. It is interesting that the effective temperature at $z>z_0$ becomes at first very large, but then approaches rapidly

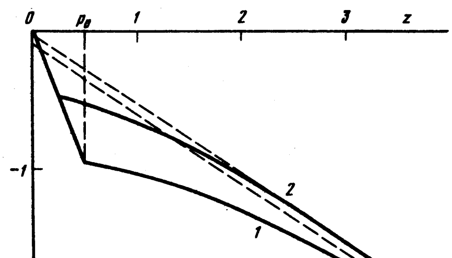

FIG. 2. Logarithm of the spherically symmetrical part of the distribution function of the electrons at $\tau\gg 1$ ($t=0.25$). The large initial slope corresponds to a Maxwellian distribution with local temperature. Next, at high energies, the distribution also tends to be Maxwellian with a temperature equal to the temperature of the hot plasma. Curve 1) $\tau=1$; 2) $\tau=1/4$.

the temperatures of the electrons in the hot region of the plasma. Thus, the distribution function turns out, in rough approximation, to be doubly Maxwellian: at low electron energies it is characterized by the local cold temperature, and at higher ones by the temperature of the hottest region of the plasma.

The onset of the discontinuity can be easily understood if it is recognized that Eq. (19) is similar to the equation of a simple Riemann wave.[5] It is known that in the course of time the profile of a Riemann wave breaks (topples over), and this leads in hydrodynamics to the onset of a shock wave. Naturally, in our case the profile $y(z)$, specified as monotonic at a certain initial value τ_0, can also break with increasing τ. This leads to formation of strong discontinuities in the distribution of $y(z)$. We shall call them kinetic discontinuities.

Similar singularities of the distribution function arise also in the region of the temperature transition, where the $t(\tau)$ dependence becomes significant. The equations of the characteristics for (19) are of the form

$$\frac{dz}{d\tau}=-\frac{2(1-2ty)}{z},\quad \frac{dy}{d\tau}=-\frac{2y(1-ty)}{z^2}. \tag{33}$$

The numerically obtained dependence of y on $z\varkappa^{1/2}$ at various τ is shown in Fig. 3 (the temperature profile $t(\tau)=\frac{5}{8}-\frac{3}{8}\tanh\varkappa\tau$). It is seen that with increasing τ, i.e., on going from the heated to the cold plasma, the monotonic distribution gradually goes over into a solution with a kinetic discontinuity, shown dashed in the figure.

The obtained numerical solution $y(z,\tau)$ in the region of the maximum change of the temperature is well described analytically if we put

$$dt/d\tau=(dt/d\tau)_0=\text{const}.$$

In this case, just as in the region far beyond the temperature transition (22), the solution $y(z,\tau)$ is a triply valued function of the energy at $z\lesssim 1$. The function $y(z)$ satisfying the boundary condition $y=1$ at $z=\infty$ is given by the formula

$$z=3^{1/2}y(1-ty)/(1-y^3)^{1/2}. \tag{34}$$

The other solution, satisfying the condition $y=1/t$ at $z=0$, is obtained from (19) by introducing the self-similar variable $\eta=z^2/t(\tau)$ and the function $g=yt(\tau)$. As a result we obtain an ordinary differential equation for g:

$$\eta\frac{dg}{d\eta}[\eta-4(1-2g)]=-g[\eta+2(1-g)]. \tag{35}$$

The solution of (35), starting out from the point $g=1$ and $\eta=0$, turns back at $\eta=0.781$, and at $\eta=0$ the value of g again becomes equal to zero. Both solutions cross at $t\geqslant 0.6$. In this case y goes over continuously from one branch to the other and undergoes only a weak discontinuity—a discontinuity of the derivative. At $t<0.6$ there is no crossing and a jumplike transition from one branch to the other takes place. In this case we have a kinetic discontinuity. The equation for the coordinate z_0 of the discontinuity point has as before the form (30). Solution of this equation yields

$$z_0^2\approx 1.5t^2,\quad t<0.6.$$

Thus, quantitatively the solutions $y(z,\tau)$ in the region of the temperature jump and beyond it are similar. In the former case, however, the position of the discontinuity point is almost immobile ($z_0\sim 1$), while in the latter it moves with increasing distance from the transition region ($z_0\sim\tau^{1/2}$).

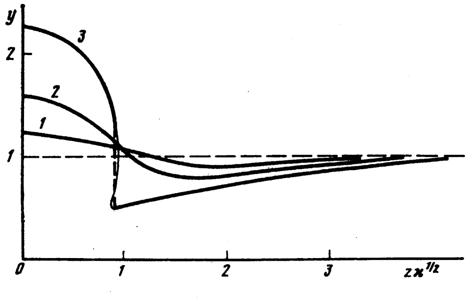

FIG. 3. The function $y(z)$ for three different values of τ at a temperature profile $t(\tau)=\frac{5}{8}-\frac{3}{8}\tanh\varkappa\tau$. These values of τ correspond to the following temperatures: 1) $t=\frac{13}{16}$, 2) $t=\frac{5}{8}$; 3) $t=\frac{7}{16}$.

4. STRUCTURE OF KINETIC DISCONTINUITY

To investigate the structure of the kinetic discontinuity we turn to Eq. (11). Recognizing that in our approximation the discontinuity is spherically symmetrical, we can neglect the dependence of φ on μ. Assuming furthermore, in accordance with (17), that $\mu=1$ in the first term, we rewrite (11) in the region near the discontinuity in the form

$$z\frac{\partial\varphi}{\partial\tau}-2\frac{\partial\varphi}{\partial z}\left[1-2t\gamma^{1/2}\frac{\partial\varphi}{\partial z}\right]-4t\gamma^{1/2}\frac{\partial^2\varphi}{\partial z^2}=0. \tag{36}$$

Equation (36) differs from (17) in the last term that describes the diffusion of the electrons in energy as a result of the collision. It is this term which determines the smearing of the discontinuity.

Taking the expansion (12) into account, introducing the function y in accord with (18), and changing over to the new variable

$$\xi=(z-z_0)/\gamma^{1/2}, \tag{37}$$

we obtain

$$z_0\frac{dz_0}{d\tau}\frac{dy}{d\xi}+2(1-2t_0y)\frac{dy}{d\xi}+4t_0\frac{d^2y}{d\xi^2}=0. \tag{38}$$

Here $z_0=z(\tau_0)$ is the location of the discontinuity at the given instant τ_0, and $t_0=t(\tau_0)$. The boundary conditions for (38) are

$$y\to y^+\ \text{as}\ \xi\to+\infty,\quad y\to y^-\ \text{as}\ \xi\to-\infty, \tag{39}$$

with $y^+=y_1(\tau_0,z_0)$ and $y^-=y_2(\tau_0,z_0)$ the solutions of Eq. (17); they are connected, in addition, by the relation (30).

The solution of (38) with the condition (39) is

$$y={}^1/_2(y^++y^-)+{}^1/_2(y^+-y^-)\ \text{th}\,[{}^1/_4\xi(y^--y^+)]. \tag{40}$$

We see therefore that the width of the discontinuity region is small in terms of the variables z —of the order of $\gamma^{1/2}$. It increases with increasing jump $y^- - y^+$.

5. DIRECTIONAL PART OF THE DISTRIBUTION FUNCTION

The function φ_1 is defined by Eq. (14). It follows from (14) and (17) that

$$\varphi_1(z,\mu,\tau)=(1+\mu)^{1/2}\varphi_{11}(z,\tau)+\varphi_{10}(z,\tau), \qquad (41)$$
$$\varphi_{11}=-2k^{1/2}z(\partial\varphi_0/\partial\tau)^{1/2}.$$

The sign in front of the square root is chosen such that the distribution function (8) has a directional character, i.e., it has a maximum at $\mu=1$. Equation (41) defines the angular dependence of the distribution function.

The equation for the function $\varphi_{10}(z,\tau)$ follows, as before, from the condition of the solvability of Eq. (15) for φ_2. Recognizing that the function φ_2 must not have any singularities as $\mu\to 1$, we obtain from (15)

$$z\frac{\partial\varphi_1^*}{\partial\tau}-2\frac{\partial\varphi_1^*}{\partial z}\left[1-4t\frac{\partial\varphi_0}{\partial z}\right]=-\frac{k}{z2^{1/2}}\varphi_{11}, \qquad (42)$$
$$\varphi_1^*=\varphi_{10}(z,\tau)+2^{1/2}\varphi_{11}(z,\tau).$$

We consider first the cold-plasma region far from the temperature transition (22), where the self-similar solution (29) is valid. Up to the discontinuity point p_0 we have

$$y=1/t,\quad \varphi_0=z/2t,\quad p<p_0. \qquad (43)$$

We see therefore that

$$\partial\varphi_0/\partial\tau=0,\quad \varphi_{11}=0,\quad \varphi_1^*=0,\quad \varphi_{10}=0,$$

i.e., the solution (29a) at $p<p_0$ is not perturbed and remains spherically symmetrical, Maxwellian, with a local temperature T. In the region beyond the discontinuity $p>p_0$, i.e., at

$$z>z_0=p_0\tau^{1/2}, \qquad (44)$$

it follows from (17) that

$$\frac{\partial\varphi_0}{\partial\tau}=\frac{1}{z}y(1-ty),$$

and the distribution function becomes sharply directional, with a directivity that increases with increasing energy like $\sim z^{1/2}$. Thus, in the region of the kinetic discontinuity we have not only a jump of the effective electron temperature $T_e(v)$, but also a transformation of the distribution function from spherically symmetrical to sharply directional.

We examine now the manner in which the distribution function becomes directional in the discontinuity region. To this end we represent the function $\varphi(z,\tau,\mu)$ in the form of a series

$$\varphi(z,\mu,\tau)=\bar\varphi_0(\mu,z,\tau)+\bar\varphi_1(\mu,z,\tau)+\bar\varphi_2(\mu,z,\tau)+\ldots, \qquad (45)$$

assuming in accordance with (12) that

$$|\bar\varphi_0|\gg|\bar\varphi_1|\gg|\bar\varphi_2|,\quad |\bar\varphi_2|\approx 1. \qquad (46)$$

The correctness of this assumption will be proved directly by constructing the solution. Substituting the expansion (45) in (11) and recognizing that in the zeroth approximation, by virtue of (46), the quadratic term $\sim(\partial\bar\varphi_0/\partial\mu)^2$, is the principal one, we find that the function $\bar\varphi_0$ is spherically symmetrical:

$$\bar\varphi_0=\bar\varphi_0(z,\tau) \qquad (47)$$

[the term $\sim(\partial\bar\varphi_0/\partial z)^2$ contains the small parameter $\gamma^{1/2}$].

In the next approximation, separating the principal terms (46), we have the equation

$$z^2\mu\frac{\partial\bar\varphi_0}{\partial\tau}-2z\frac{\partial\bar\varphi_0}{\partial z}\left[1-2\gamma^{1/2}t\frac{\partial\bar\varphi_0}{\partial z}\right]-4\gamma^{1/2}tz\frac{\partial^2\bar\varphi_0}{\partial z^2}+k(1-\mu^2)\left(\frac{\partial\bar\varphi_1}{\partial\mu}\right)^2=0. \qquad (48)$$

From the requirement that the derivative $\partial\bar\varphi_1/\partial\mu$ have no singularities as $\mu\to 1$, just as in Sec. 3, we obtain an equation that defines the function $\bar\varphi_0(z,\tau)$. We have

$$z^2\frac{\partial\bar\varphi_0}{\partial\tau}-2z\frac{\partial\bar\varphi_0}{\partial z}\left[1-2\gamma^{1/2}t\frac{\partial\bar\varphi_0}{\partial z}\right]-4\gamma^{1/2}tz\frac{\partial^2\bar\varphi_0}{\partial z^2}=0. \qquad (49)$$

This equation, naturally, coincides with (36). Its solution (40) describes, in the transition region, the structure of the spherically symmetrical part of the distribution function. From (48), taking (49) into account, we get for $\bar\varphi_1$:

$$\bar\varphi_1=(1+\mu)^{1/2}\bar\varphi_{11}(z,\tau)+\bar\varphi_{10}(z,\tau),\quad \bar\varphi_{11}=-2k^{1/2}z(\partial\bar\varphi_0/\partial\tau)^{1/2}. \qquad (50)$$

Thus, the angular dependence of the distribution function is determined by Eq. (41) everywhere with the exception of the discontinuity region. We have then in accordance with (18)

$$\bar\varphi_0=\frac{1}{2}\int_0^z\tilde y\,dz,\quad \frac{\partial\bar\varphi_0}{\partial\tau}=\frac{1}{2}\int_0^z\frac{\partial\tilde y}{\partial\tau}dz$$
$$=-\frac{1}{2}\frac{dz_0}{d\tau}\int_0^\xi\frac{d\tilde y}{d\xi}d\xi=\frac{1}{2}[\tilde y(0)-\tilde y(\xi)]\frac{dz_0}{d\tau}.$$

We have taken into account here the fact that in the transition region

$$\tilde y=2\frac{\partial\bar\varphi_0}{\partial z}=\tilde y(\xi),\quad \xi=(z-z_0)/\gamma^{1/2}.$$

The function $\tilde y(\xi)$ is defined by (40). In particular, in the self-similar case, recognizing that $z_0=p_0\tau^{1/2}$, $y(0)=1/t$, we have

$$\frac{\partial\bar\varphi_0}{\partial\tau}=\frac{1}{z_0}\tilde y_0(1-t\gamma^{1/2}\tilde y). \qquad (51)$$

Formulas (51) and (50), with (40) taken into account, describe the continuous transition from the spherically symmetrical distribution function at $z<z_0(\partial\bar\varphi_0/\partial\tau=0$ to the sharply directional (41) at $z>z_0$.

The equation for $\bar\varphi_{10}(z,\tau)$ is obtained by starting from the requirement that the derivative $\partial\bar\varphi_2/\partial\mu$ contain no singularities as $\mu\to 1$. We have

$$z\frac{\partial\bar\varphi_1^*}{\partial\tau}-2\frac{\partial\bar\varphi_1^*}{\partial z}\left[1-4t\gamma^{1/2}\frac{\partial\bar\varphi_0}{\partial z}\right]-4t\gamma^{1/2}\frac{\partial^2\bar\varphi_1^*}{\partial z^2}+\frac{k\bar\varphi_{11}}{z2^{1/2}}=0, \qquad (52)$$
$$\bar\varphi_1^*=\bar\varphi_{10}(z,\tau)+2^{1/2}\bar\varphi_{11}(z,\tau).$$

The solution of (52) shows that the function $\bar\varphi_1^*$ increases in the transition region from zero to a value

$$\bar\varphi_1^*=\left[\frac{2k(1-ty_0)}{y_0\gamma^{1/2}}\right]^{1/2},$$

and then increases slowly (like $z^{1/2}$) with increasing energy.

It is seen thus that $\bar\varphi_1^*\sim\gamma^{-1/4}\ll\bar\varphi_0$. It is easy to verify in similar fashion that $|\tilde{\varphi}_2|\approx 1$. Consequently, the conditions (46), which determine the validity of the expansion (45), are always satisfied.

6. HEAT FLUX. KINETIC CONVECTION

The heat flux along the x axis is given by

$$q_{\parallel}=\frac{m}{2}\int \mu v^3 f\,d\mathbf{v}=\frac{\pi m}{2}\gamma^{-3/2}\left(\frac{T_{e0}}{m}\right)^{3/2}\int_{-1}^{1}\mu\,d\mu\int_0^{\infty}z^2 f(z,\mu)\,dz, \tag{53}$$

$$f(\mu,z)=N(m/2\pi T_{e0}t)^{3/2}e^{-\varphi}.$$

We calculate first the heat flux in that space region (22) where t is constant. The heat is transported here only by convection of the fast particles with energy $z \gtrsim z_0$. Since the distribution function at lower energies is isotropic, the usual thermal-conductivity heat flux is absent—it is proportional to the temperature gradient. Since, as we have seen, the distribution function is strongly directional at $z>z_0$, we can confine ourselves for φ only to the first two terms of the expansion (12):

$$\varphi_0=\frac{z_0}{2t}+\frac{zy-z_0y_0}{2}-\tau^{1/2}\int_{y_0}^{y}y'(1-ty')r(y')\,dy',$$

$$\varphi_1=-2\left[\frac{zy(1-ty)}{k}\right]^{1/2}(1+\mu)^{1/2},\qquad z>z_0, \tag{54}$$

where

$$r(y)=[(1-y^2)-{}^2/_3t(1-y^3)]^{-1/2}.$$

Substituting (54) in (53) we get

$$q_{\parallel}=ANT_{e0}\left(\frac{T_{e0}}{m}\right)^{1/2}\frac{k^{1/2}}{(2t)^{1/2}}\gamma^{-3/4}z_0^{3/2}\exp\left\{-\frac{\gamma^{-1/2}z_0}{2t}\right\}$$

$$=A_1NT_{e0}\left(\frac{T_{e0}}{m}\right)^{1/2}k^{1/2}\lambda^{3/4}\exp(-\lambda^{1/2}). \tag{55}$$

In the last expression we have returned to the variables (6), T_{e0} is the temperature of the hot region of the plasma, and A_1 is a constant of the order of unity,[2)] namely

$$A_1=\left(\frac{2}{\pi}\right)^{1/2}[y_0(1-ty_0)r(y_0)]^{-3/2}c_1\lambda^{1/2}\int_{y_0}^{1}y(1-ty)r^{5/2}$$

$$\times[(1-2ty)r^{-2}(y)+y^2(1-ty)^2]\exp\left\{-\lambda^{1/2}\left[y^2(1-ty)r(y)\right.\right.$$

$$\left.\left.-y_0^2(1-ty_0)r(y_0)-\int_{y_0}^{y}y'(1-ty')r(y')\,dy'\right]\right\}dy.$$

In this case the energy is thus transported only convectively—by particles moving freely at high velocity relative to the plasma; the number of such particles decreases exponentially with the dimensionless distance λ(6).

We consider now the temperature transition region $t=t(\lambda)$. Here we have according to (34) and (35)

$$\varphi_0=\frac{1}{4t^{1/2}}\int_0^{z^2/t}g(\eta)\eta^{-1/2}\,d\eta,\qquad z\leqslant z_0.$$

$$\varphi_0=\frac{1}{4t^{1/2}}\int_0^{z_0^2/t}g(\eta)\eta^{-1/2}\,d\eta+\frac{1}{2}(zy-z_0y_0)-\frac{3^{1/2}}{2}\int_{y_0}^{y}\frac{y'(1-ty')}{(1-y'^3)^{1/2}}dy',\qquad z>z_0,$$

$$\varphi_1=-2\left[\frac{zy(1-ty)}{k}\right]^{1/2}(1+\mu)^{1/2}. \tag{56}$$

The contribution to the heat flux made by the thermal particles and by the low-energy particles ($z<z_0$) is proportional to the temperature gradient:

$$\delta q_{\parallel}=-\varkappa\nabla_{\parallel}T_e,$$

where $\varkappa$ is the thermal-conductivity coefficient[6]:

$$\varkappa=\alpha(k)N\left(\frac{T_{e0}}{m}\right)^{1/2}t^{5/2}l_T,$$

$\alpha(k)$ is a coefficient on the order of unity, whose value depends on the ion charge.

The particles with high energies ($z>z_0$) determine the convective heat flux, which is calculated just as (55). Thus, the total flux is

$$q_{\parallel}=NT_{e0}\left(\frac{T_{e0}}{m}\right)^{1/2}[\alpha(k)t^{5/2}\gamma+A_2k^{1/2}\gamma^{-3/4}\exp(-\gamma^{-1/2})]. \tag{57}$$

The coefficient A_2, just as A_1, is of the order of unity, and its value is

$$A_2=\frac{3^{5/4}}{2\pi^{1/2}}\gamma^{-1/2}\left(\frac{z_0}{2t}\right)^{-1/2}c_2\int_{y_0}^{1}\frac{y(1-ty)}{(1-y^3)^{3/4}}\left[(1-2ty)(1-y^3)\right.$$

$$\left.+\frac{3}{2}y^3(1-ty)\right]\exp\left\{-\frac{\sqrt{3}}{2}\gamma^{-1/2}\left[\frac{y^2(1-ty)}{(1-y^3)^{1/2}}-\frac{y_0^2(1-ty_0)}{(1-y_0^3)^{1/2}}\right.\right.$$

$$\left.\left.-\int_{y_0}^{y}\frac{y'(1-ty')}{(1-y'^3)^{1/2}}dy'\right]\right\}.$$

In the derivation of (57) we took into account the fact that $z_0\approx t$. It is seen from (57) that the convective term is exponentially small at small gradients. At $\gamma\gtrsim 10^{-2}$, however, it becomes comparable with the usual thermal-conduction term, which it can exceed at large gradients.

Thus, heat transport in an inhomogeneously heated plasma is effected by thermal-conductivity only at very small electron-temperature gradients, $\gamma\gtrsim 10^{-2}$. At large gradients, an important role can be played by the kinetic convection by fast electrons.

1) We note that Eqs. (3) and (7) are valid in fact also in the presence of a constant magnetic field in the plasma. If the temperature gradient of the magnetized electrons is directed at an angle α to the magnetic field, it is necessary to make in (7) the substitution $\lambda\to\lambda/\cos\alpha$.

2) The quantity $c_1\sim 1$ is determined by the third term of the expansion of the distribution function (12).

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