Quantum theory of self-induced transparency

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A quantum-mechanical investigation of the processes of short-pulse propagation in a two-level resonance medium is carried out. The coherent-state representation for fields and matter is used in the calculations. A consistent system of Bloch and electromagnetic-field equations similar to the set of Lagrange-Poisson equations is obtained. It is shown that this system goes over in a particular case into the Korteweg–de Vries equations. Self-induced transparency and its distinctive features in the cases of low- and high-power pulses are also investigated. As was to be expected, the obtained expressions go over in a particular case into the McCall-Hahn solutions.

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The production of short and ultrashort high-power light pulses has stimulated intensive investigation of their propagation in resonantly absorbing media.[1–5] In all the papers without exception the theoretical description of the propagation process is based on a semiclassical approach in which the two-level medium is treated quantum mechanically, while the field is described by the Maxwell equations.

In the present paper we carry out a consistent quantum-mechanical analysis of the interaction of short light pulses with a two-level medium. For this purpose, we use the coherent-state representation for the field and the material.[6–8] A continuous representation of this sort enables us to reduce the evolution of the quantum-mechanical system to the evolution of a classical system during which the quantum-mechanical relations between the physical quantities are preserved. It is on this basis that we consider the phenomenon of self-induced transparency. The density matrix, the field characteristics, and the angular momentum are used in the Boson representation.

1. THE KORTEWEG-DE VRIES EQUATIONS

For the unified description of the field and the ensemble of two-level particles, let us choose the coherent-state representation[6–8]:

$$b|\beta\rangle=\beta|\beta\rangle,\quad a_\pm|\alpha_+,\alpha_-\rangle=\alpha_\pm|\alpha_+,\alpha_-\rangle, \tag{1}$$

where b is the photon-annihilation operator, the $a_\pm$ are the effective spin operators for the excited and ground states repsectively, $|\beta\rangle$ is a coherent state of the photon field, and $|\alpha_+,\alpha_-\rangle$ is a coherent state of the angular momentum.

The operators corresponding to the cooperative angular momentum R_α and its components are expressed in the second-quantization formalism in terms of the operators $a_\pm^+, a_\pm$ as follows:

$$R_\alpha=\sum_{\lambda,\mu}a_\lambda^+\langle\lambda|r_\alpha|\mu\rangle a_\mu,$$
$$R_\pm=a_\pm^+a_\mp,\quad R_3={}^1/_2(a_+^+a_+-a_-^+a_-), \tag{2}$$

where r_α is the effective-spin operator for samples of length smaller than the wavelength, and $\lambda(\mu)$ is the final (initial) state.

In the case of a single mode R_α can be written in the form

$$R_{\alpha k}=\sum_{j=1}^{N}r_{\alpha j}\exp(i\mathbf{kx}_j), \tag{3}$$

where $\mathbf{k}$ is the wave vector and $\mathbf{x}_j$ is the position vector of the particle. On introducing the single-particle phase states

$$|\pm\rangle_{jk}=\exp(-i\mathbf{kx}_j)|\pm\rangle_j,\quad r_{\alpha j}\exp(i\mathbf{kx}_j)|\pm\rangle_{jk}=r_{\alpha j}|\pm\rangle \tag{4}$$

and the dipole interaction between the single field mode and the medium, we can choose the R_α operators in the form (2) both for samples of small length and for extended media.

In the Heisenberg representation, the equation of motion for some physical operator A has the form

$$\frac{dA}{dt}=\frac{i}{\hbar}[A,\mathcal{H}], \tag{5}$$

where $\mathcal{H}$ is the total Hamiltonian of the field + medium system. In the rotary-wave approximation this Hamiltonian will have the form

$$\mathcal{H}=\hbar\omega a^+a-{}^1/_2\hbar\omega(a_+^+a_+-a_-^+a_-)+\hbar(gaR_++g^*aR_-). \tag{6}$$

Here $\hbar\omega$ is the energy of the excited level of the active center and g is the interaction constant.

The form of the operator A is specified by the problem under consideration. On the basis of the equations of motion (5) and the Hamiltonian (6) for the quantities characterizing the field and the medium, we obtain the consistent system of differential equations:

$$\begin{aligned} dp/dt&=rq+mR_y, & dR_x/dt&=rR_y-qR_z;\\ dq/dt&=-rp-mR_x, & dR_y/dt&=-rR_x+pR_z;\\ dr/dt&=0, & dR_z/dt&=qR_x-pR_y,\end{aligned} \tag{7}$$

where

$$p=f^*\langle b^+\rangle+f\langle b\rangle,\quad R_x=\langle R_+\rangle+\langle R_-\rangle,$$
$$q=i\{f^*\langle b^*\rangle-f\langle b\rangle\},\quad R_y=i\{\langle R_+\rangle-\langle R_-\rangle\},$$
$$r=\omega,\quad R_z=2\langle R_3\rangle,\quad m=|f|^2,$$

f is the interaction constant. The system of equations (7) is a consistent set of Bloch equations (the right-hand side) and electromagnetic field equations (the left-hand side).

The averaging of the quantities entering into (7) was

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performed over the coherent states with the aid of the density operator ρ:

$$\rho=\rho_F\rho_A, \tag{8}$$

where $\rho_{F(A)}$ is the density operator of the field (medium). The factorized form of ρ is determined by the fact that, first, we choose a representation in which ρ does not vary in time; second, the field and the medium do not interact at the initial moment of time; third, the radiation is assumed to be totally coherent.

Using the Klauder-Sudarshan theorem,[9] we can express the solutions to the Eqs. (7) in terms of the integrals, I_i, of the motion, the mean values,

$$\langle I_i\rangle=\mathrm{Sp}\{\rho I_i\}=\int P(\alpha_\mu)\langle\alpha_\mu|I_i|\alpha_\mu\rangle d^2\alpha_\mu, \tag{9}$$

of which clearly do not vary in time, and are determined by the initial conditions. Here $|\alpha_\mu\rangle \equiv|\alpha_+,\alpha_-,\beta\rangle$, $d^2\alpha_\mu\equiv d^2\alpha_+d^2\alpha_-d^2\beta$, and $P(\alpha_\mu)=P_F(\beta)P_A(\alpha_+\alpha_-)$ is the weight function. Analyzing the system of equations (7), we see that the structure of the obtained equations is similar to the structure of the system of differential equations determining the motion of a solid about a fixed point (the Lagrange-Poisson equations),[10] which significantly facilitates the determination of the solution.

Introducing the local time $\tau=t-x/v$, and omitting the apparent, but tedious calculations, we can derive from the system (7) the well-known Koretweg-de Vries equations:

$$\frac{d^3n}{d\tau^3}=g^2\left\{6n\frac{dn}{d\tau}-4E\frac{dn}{d\tau}\right\},$$
$$\frac{d^3R_3}{d\tau^3}=g^2\left\{6R_3\frac{dR_3}{d\tau}+4E\frac{dR_3}{d\tau}\right\}, \tag{10}$$

where $n=\langle b^+b\rangle=n(x,t)$ is the photon density, $R_3=\langle\frac{1}{2}R_z\rangle$ is the inversion density of the medium, $E=n(0)+R_3(0)$; $n(0)$ and $R_3(0)$ are the initial values of the photon and inversion densities, respectively. The general solution to (10) has the form

$$X=S_1+(S_2-S_1)\mathrm{sn}^2(\theta,\ m), \tag{11}$$

where $X=n,R_3$; S_i,m are constants that depend on the initial conditions, and $\theta=\Omega(t-x/v)$. The quantities p, q, R_x, and R_y are connected with n and R_3 by the relations

$$(p^2+q^2)/4g^2=n,\quad R_x^2+R_y^2=d^2=R^2-R_z^2. \tag{12}$$

Since the solutions to Eq. (10) exist in the form of running periodic running waves, we can, using the standard method[10] for the Lagrange-Poisson equations (7), show that $\varphi=\omega t-kx$. Then

$$p=2g(n(x,t))^{1/2}\sin(\omega t-kx),\quad R_x=d(x,t)\cos(\omega t-kx),$$
$$q=2g(n(x,t))^{1/2}\cos(\omega t-kx),\quad R_y=d(x,t)\sin(\omega t-kx). \tag{13}$$

On the other hand, energy is conserved during the propration of electromagnetic waves in a medium (the Poynting theorem):

$$\frac{\partial U(x,t)}{\partial t}+\nabla\mathbf{S}(x,t)=-\mathbf{E}(x,t)\frac{\partial\vec{\mathscr{P}}}{\partial t}(x,t)=-\frac{\partial W}{\partial t}. \tag{14}$$

$\mathbf{E}(x,t)$ and $\vec{\mathscr{P}}$ can be found by solving (10). The electric-field intensity, $\mathbf{E}(x,t)$, is then described by the formula

$$\mathbf{E}(x,t)=[2\pi\hbar\omega n(x,t)]^{1/2}\{\exp[i(\omega t-kx)]+\exp[-i(\omega t-kx)]\}\mathbf{e}_3$$
$$=\vec{\mathscr{E}}(x,t)\cos(\omega t-kx). \tag{15}$$

The external-electromagnetic-field-induced polarization, $\vec{\mathscr{P}}$, of the medium will have the form

$$\vec{\mathscr{P}}=\mu\{R_x\mathbf{e}_1\cos(\omega t-kx)-R_y\mathbf{e}_2\sin(\omega t-kx)\}$$
$$=\mathbf{X}\cos(\omega t-kx)-\mathbf{Y}\sin(\omega t-kx). \tag{16}$$

In the formulas (14)–(16) we have used the following notation: $U(x,t)=n^2(4\pi)^{-1}E^2(x,t)$ is the energy density of the electromagnetic field; $\mathbf{S}(x,t)=\mathbf{e}_xc\eta^{-1}U(x,t)$ is the Poynting vector, $W=\hbar\omega R_3(x,t)$ is the energy density stored by the medium, η is the refractive index, $\mathbf{e}_x(x=1,2,3)$ is the unit polarization vector, and μ is the transition dipole moment.

For the steady-state process, and for $\tau=x-x/v$, (14) goes over into

$$\frac{\eta^2}{4\pi}\left(1-\frac{c}{v}\right)\frac{dU(\tau)}{d\tau}=-\frac{dW(\tau)}{d\tau}. \tag{17}$$

The integration of (17) yields the first integral:

$$\frac{\eta^2}{4\pi}\left(1-\frac{c}{v}\right)E^2(\tau)+W(\tau)=I. \tag{18}$$

Since Eqs. (7) and (14) describe one and the same process, (18) will be the integral of motion for (7) as well. Substituting the formulas (15) and (16) into (14), we can write (18) in terms of the envelopes:

$$I=(c/v-1)n(x,t)+R_3(x,t)=n(0)+R_3(0). \tag{19}$$

The expressions (7), (14), and (19) describe the most general case (in the rotary-wave approximation) of the resonance interaction of a photon field with a two-level medium. Each particular effect (boson avalanche, self-induced transparency, nutation) is determined by specifying the first integrals I_i and the constants S_i and k.

2. SELF-INDUCED TRANSPARENCY

We shall assume that the two-level active centers of the material interact only via the radiation field. We shall characterize the high-frequency polarization induced in such a medium by the pseudo-spin vector (the Bloch vector):

$$\mathbf{P}=\mathbf{e}_1\langle R_x\rangle+\mathbf{e}_2\langle R_y\rangle+\mathbf{e}_3\langle R_z\rangle. \tag{20}$$

For the case of the electromagnetic interaction with the medium of pulses of short duration (right down to $\tau_p\sim10^{-11}$ sec, where τ_p is the pulse length in the medium), the condition $\tau_p^{-1}\gg\omega$ is naturally fulfilled. We assume, as usual, that τ_p is much less than all the relaxation times.

In the preceding section we showed that the self-consistent system of equations, (7), for the coherent interaction of the field with the pseudospin reduces to the Korteweg-de Vries equation, (10), for $\langle n\rangle$ and $\langle R_3\rangle$. We shall seek the particular form of the solution to (10) for an absorbing medium ($\langle R_3(0)\rangle=-R$). In this case the quantity $I=n(0)-R$ will serve as the integral of the motion; $h=an(0)(|R_3(0)|)^{1/2}$ is the interaction energy; R is the cooperative angular momentum for the electric-dipole transitions. The successive integration of the Korteweg-de Vries equations leads to the expressions

$$(dn/d\theta)^2=g^2\{n(-n^2+2In+R^2-I^2)-h^2\},$$
$$(dR_3/d\theta)^2=g^2\{(I-R_3)(R^2-R_3^2)-h^2\}, \tag{21}$$

which are also the equations of motion for the nonlinear oscillator, i.e., the Korteweg-de Vries equations and the equations of motion of the spherical pendulum are equivalent. Consequently, the processes of nonlinear resonance interaction between a field and a two-level medium are described by the theory of nonlinear oscillations, which is represented in our paper by Eqs. (21).

The general solutions for n and R_3 will be

$$n=l_1-(l_1-l_2)\,\mathrm{sn}^2(\theta,m),\quad R_3=U_1+(u_2-u_1)\,\mathrm{sn}^2(\theta,m), \tag{22}$$

where the l_i and u_i are the roots, expressed in terms of the integrals of the motion, of the right-hand sides of the equalities (21), $\theta=\Omega(t-x/v)$, m is the modulus of the elliptic function sn, and Ω is a characteristic oscillation frequency.[11,12]

Let us assume that a weak pulse $[n(0)<R]$ is incident on the medium. Then the first integrals and the constants will have the form

$$I=n(0)-R<0,\quad h^2=2n(0)R<R^3,\quad \langle R\rangle=R,$$
$$l_1=\frac{(I+R)-\Delta}{\alpha^2}=\frac{n(0)-\Delta}{\alpha^2},\quad l_2=\frac{2\Delta}{\alpha^2},$$
$$l_3=\frac{(I-R)+\Delta}{\alpha^2}=\frac{n(0)-2R+\Delta}{\alpha^2},$$
$$u_1=-R+\Delta,\quad u_2=I-2\Delta,\quad u_3=R+\Delta, \tag{23}$$
$$\Omega=g\alpha^{-1}(l_3-l_1)^{1/2}=g\alpha^{-1}(u_3-u_1)^{1/2}=g\alpha^{-1}(2R)^{1/2}=g\alpha^{-1}N^{1/2},$$
$$\alpha^2=\left(\frac{c}{v}-1\right),\quad m^2=\frac{u_2-u_1}{u_3-u_1}=\frac{l_1-l_2}{l_1-l_3}$$
$$=\frac{n(0)+\Delta}{2R}\approx\frac{n(0)}{2R},\quad \Delta\cong\frac{h^2R}{R^3}<1;$$

if $n(0)\ll R$, then $m^2\to 0$ and $sn\to\sin$.

For the above-given values of the constants, the solutions, (22), of the equations will be

$$n=n(0)(c/v-1)^{-1}\cos^2\{g[N(c/v-1)^{-1}]^{1/2}(t-x/v)\},$$
$$R_3=-R+n(0)\sin^2\{g[N(c/v-1)^{-1}]^{1/2}(t-x/v)\}. \tag{24}$$

As can be seen from here, n and R_3 oscillate with the characteristic oscillation frequency $\Omega=g\alpha N^{1/2}=\tau_p^{-1}$, determined by only the density, N, of the active centers, it being independent of the density of the incident pulse. For the oscillations to be observed, it is necessary that $\Omega>T_2^{-1}$ (T_2 is the characteristic time of the transverse irreversible phase relaxation). But $\Omega^{-1}=\Omega_0^{-1}\alpha$, where Ω_0 is the rate of collective spontaneous decay,[12] $\tau_0\equiv\Omega_0^{-1}$. Consequently, the oscillations are possible if $\alpha\tau_0\sim T_2$. However, as a rule, τ is little less than T_2 (Ref. 13), and in the case of weak pulses α is large, i.e., $\alpha^2\to\infty$, $v\ll c$, $\alpha\tau_0\gg T_2$, since the magnitude of the energy density of the exciting pulse of significantly less than the concentration of the active centers. Consequently, after the period of time T_2 there occurs a nonoscillatory, noncoherent damping of the weak oulse as a result of the dephasing action of the relaxation characterized by T_2.

Let us consider the transmission of a high-power pulse $[n(0)\approx 2R]$. In this case the integrals of the motion and the constants assume the form

$$I=n(0)-R=R,\quad h^2=4R^2<R^3,\quad \langle R\rangle=R,$$
$$l_1=(2R-\Delta)/\alpha^2=(n(0)-\Delta)/\alpha^2,\quad l_2=l_3=2\Delta\alpha^2,$$
$$u_1=-R+\Delta,\quad u_2=R-2\Delta,\quad u_3=I+\Delta,$$
$$\Delta\cong 4\ll R,\quad \Omega=g\alpha^{-1}N^{1/2}=\alpha^{-1}g(n(0))^{1/2}, \tag{25}$$
$$m^2=\frac{u_2-u_1}{u_3-u_1}=\frac{l_1-l_2}{l_1-l_3}=\frac{n(0)}{2R-\Delta}\approx\frac{n(0)}{2R}\approx 1.$$

Then the solutions (22) assume the form

$$n=n(0)\alpha^{-2}\,\mathrm{sech}^2\,\Omega\tau,\qquad R_3=-R+n(0)\,\mathrm{th}^2\,\Omega\tau. \tag{26}$$

Thus, it can be seen from (25) and (26) that a soliton is formed provided $n(0)=2R$, i.e., the incident-energy density of the electromagnetic field is sufficient for it to make the medium go over completely into the superemissive state. Indeed, if we define the intensity as energy referred to the lifetime of a soliton in a medium of length L, i.e., if

$$J(t)=\frac{N}{\alpha^2\tau_3}\,\mathrm{sech}^2\,\Omega\tau, \tag{27}$$

then we obtain

$$J(t)=\frac{g^2N^2}{(c/L)A^2}\,\mathrm{sech}^2\,\Omega\tau. \tag{28}$$

Here $A=kL\alpha$, k being the absorption coefficient. The formula (28) coincides with the intensity obtained in the semiclassical approach[13] up to the correction factor A, which takes account of the delay of the pulse (or boson avalanche).

Further, the inverted state decays. The medium goes over into the ground state, forming a soliton. Thus, the return of the medium into the initial state is similar to a boson avalanche,[14,15] and can be described by the corresponding equations.[11,12] A pulse with $n(0)=2R$ is a 2π pulse, since its height does not vary (the amplitude $\varepsilon^2=n(0)\alpha^{-2}\approx n(0)$, since $\alpha^2\to 1$), returns the medium to the ground state, has the shape of a secant, and its length $\tau_p\approx\tau_0<T_2$.

For pulses lying in the intermediate region $[R\leqslant n(0)\leqslant 2R]$, the solutions (8) will have the form

$$n=n(0)\alpha^{-2}\,\mathrm{cn}^2(\Omega\tau,m),\quad R_3=-R+n(0)\,\mathrm{sn}^2(\Omega\tau,m). \tag{29}$$

Such a pulse also makes the medium go over into the superemissive state $R_3\leqslant R$, whose coherence is dependent upon the duration $\tau_p=\alpha\tau_0\lesssim T_2$ (in this case α is little less than unity). The deexcitation of states of this type also yields coherent pulses whose shape is close to that of a 2π pulse, since $cn(\Omega\tau,m)$ assumes as $m\to 1$ a shape close to that of the secant, and oscillations do not arise because of the closeness of the values of τ_p and T_2. It is clear that the initial-density value $n(0)=R$ specifies a π pulse.

For $n(0)>2R$ the parameters characterizing the evolution of the medium and the field during the propagation of the electromagnetic-field pulse are determined by the value of $n(0)$. In this case $\Omega<1$ ($\Omega=g(n(0))^{1/2}\alpha^{-1}$, $\alpha^2=N\varepsilon_0^{-2}$), which leads to the increase of the velocity of propagation of the pulse. If, for example, in the case of a 2π pulse the duration $\tau_p=g^{-1}(2/N)^{1/2}$, then the velocity of the pulse is twice the velocity of the 2π soliton. The densities $n(0)$ lying within the range from $2Rk$ to $2R(k+1)$ develop in the medium into $[2\pi k, 2\pi(k+1)]$ pulses. The dynamics and interaction of the 2π solitons can be more fully described with the aid of the conventional technique used in the investigation of nonlinear waves described by the Korteweg-de Vries equations (10).

The following conclusions are among the most interesting results that follow from the above-expounded theory.

1. The condition for self-induced transparency to be observed is not determined by the area of the pulse, but by the density of the incident photons $[n(0)]$.

2. Self-induced transparency is a development of a boson avalanche in an extended medium.

[1]S. L. McCall and E. L. Hahn, Phys. Rev. **183**, 457 (1969).
[2]P. G. Kryukov and V. S. Letokhov, Usp. Fiz. Nauk **99**, 169 (1969) [Sov. Phys. Usp. **12**, 641 (1970)].
[3]I. A. Poluéktov, Yu. M. Popov, and V. S. Roĭtberg, Usp. Fiz. Nauk **114**, 97 (1974) [Sov. Phys. Usp. **17**, 673 (1975)].
[4]C. L. Lamb, Rev. Mod. Phys. **43**, 99 (1971).
[5]K. J. Kaup, Phys. Rev. **A16**, 705 (1977).
[6]R. Glauber, in: Kogerentnye sostoyaniya v kvantovoĭ teorii (Coherent States in Quantum Theory), Mir, 1965, p. 91.
[7]J. Schwinger, in: Quantum Theory of Angular Momentum, Academic Press, New York, 1965, p. 229.
[8]V. V. Mikhaĭlov, Teor. Mat. Fiz. **15**, 367 (1973) [Theor. Math. Phys. **15**, 584 (1973)].
[9]J. Klauder and E. Sudarshan, Fundamentals of Quantum Optics, W. A. Benjamin, New York, 1968 (Russ. Transl., Mir, 1970).
[10]V. V. Golubev, Lektsii po integrirovaniyu uravneniĭ dvizheniya tverdogo tela okolo nepodvizhnoi tochki (Lectures on the Integration of the Equations of Motion of a Solid near a Fixed Point), Gostekhizday, 1958.
[11]E. I. Bogdanov and I. A. Nagibarova, Dokl. Akad. Nauk B. SSR **23**, 428 (1979).
[12]E. I. Bogdanov, in: Svetovoe i zvukovoe sverkhizluchenie (Light and Sound Super-radiation), Kazan', 1976, p. 116.
[13]A. Allen and J. H. Eberly, Optical Resonance and Two-Level Atoms, Wiley, New York, 1975 (Russ. Transl., Mir, 1978).
[14]R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[15]V. R. Nagibarov and U. Kh. Kopvillem, Zh. Eksp. Teor. Fiz. **54**, 313 (1968) [Sov. Phys. JETP **27**, 167 (1968)].

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