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A soliton system subject to perturbation. Oscillatory shock waves.

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We study a system of solitons of the perturbed Korteweg–de Vries equation with nearly equal amplitudes. We show that in that case there may exist quasistationary systems with a large number of solitons for well-defined relations between the amplitudes. Such systems become stationary when there is a piston which compensates for the damping of the solitons and their mutual repulsion. Using such an approach we give a detailed description of the soliton structure of oscillatory shock waves.

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1. In the present paper we study effects which occur when a permanent perturbation acts on a system of solitons, and we consider from that point of view the structure of shock waves in weakly dispersive media. To fix our ideas we consider here waves which are described by the perturbed Korteweg–de Vries (KdV) equation

$$u_t-6uu_x+u_{xxx}=\varepsilon R[u], \qquad (1)$$

where R is a (generally speaking non-linear) operator acting upon the function $u(x,t)$. The general approach discussed here and several of the results turn out to be valid also in a number of other cases.

The evolution of a single perturbed KdV soliton is described by the equations[2-4]

$$\begin{gathered} u(x,\ t)=u_s(z,\ \varkappa(t))+\delta u(x,\ t),\\ u_s(z,\ \varkappa)=-2\varkappa^2\operatorname{sech}^2 z,\qquad z=\varkappa[x-\xi(t)], \end{gathered} \qquad (2)$$

where

$$\frac{d\varkappa}{dt}=-\frac{\varepsilon}{4\varkappa}\int_{-\infty}^{\infty}R[u_s]\operatorname{sech}^2 z\,dz, \qquad (3)$$

$$\frac{d\xi}{dt}=4\varkappa^2-\frac{\varepsilon}{4\varkappa^3}\int_{-\infty}^{\infty}R[u_s](z\operatorname{sech}^2 z+\operatorname{th} z+\operatorname{th}^2 z)\,dz. \qquad (4)$$

As far as the deformation $\delta u(x,t)$ is concerned we shall discuss here only its "tail" part which is described by the expressions[2,3]

$$\delta u_-=\lim_{x\to-\infty}\delta u=\varkappa^2\varepsilon q,$$

$$q=\frac{1}{4\varkappa^5}\int_{-\infty}^{\infty}R[u_s]\operatorname{th}^2 z\,dz. \qquad (5)$$

Indeed, $\delta u(x,t)$ is transformed into a flat tail already at a few soliton lengths behind the soliton, and this can also be seen from numerical solutions[5] obtained for $R=\partial^2/\partial x^2$).

The characteristic time scale, defined by the perturbation, is[1] $t_p=t_s/\varepsilon q$, where $t_s=(2\varkappa)^{-3}$ is the characteristic time connected with the unperturbed soliton. If, therefore, there are two solitons with greatly different amplitudes ($\delta\varkappa=\varkappa_2-\varkappa_1\sim\varkappa_{1,2}$), the time it takes the larger soliton to pass through the smaller one is of the order of t_s. As $t_s\ll t_p$, the interaction of the solitons does not appreciably interfere with the effects of the perturbation.

However, this interference may turn out to be important if the solitons have almost the same amplitudes, i.e., $\varkappa_1\approx\varkappa_2\gg|\delta\varkappa|$. We shall therefore consider just this case.[1)] From the point of view of the general perturbation theory apparatus[1-4,6,7] the problem reduces to evaluating a few matrix elements using two-soliton wave functions. Such an approach is, however, very cumbersome since the expression for the two-soliton solution is particularly difficult to visualize just for the case when $\delta\varkappa$ is small.

It turns out, however, that one can consider this limiting case very simply, if one uses the observation by Zabusky and Kruskal[8] and Lax's results,[9] from which it follows that the solution of the unperturbed KdV equation can, when $|\delta\varkappa| \ll \varkappa_{1,2}$, approximately be written as the superposition of two solitons with slowly changing amplitudes. These solitons approach each other up to a certain minimum distance, which is large compared to the dimensions of the solitons, and after that they diverge. It turns out in this way one can lucidly describe the process of the collision of solitons with a small $\delta\varkappa$ and obtain a phase shift which is to a high accuracy the same as the one obtained from the exact two-soliton solution of the unperturbed KdV equation.[9,10] This simple picture is valid also when $\varepsilon \neq 0$. It is the basis of the approach made in the present paper which is given in Sec. 2 for a system of two solitons and in Sec. 3 for an arbitrary number of almost identical solitons. It then turns out that for well-defined conditions there exists a state of a system of solitons such that the amplitudes of all solitons except the last one (the one furthest to the left) are time-independent while their velocities are the same. This is possible when the change in the soliton amplitudes due to their mutual repulsion is compensated by the action of the perturbation $\varepsilon R[u]$. As regards the last soliton, such a compensation is possible for it only when some "external force" (such as a piston) is acting. If such a force which produces a stationary state ceases to act, the equilibrium is violated, but the more slowly the more solitons there are in the system. One can thus speak of the possibility of a quasi-bound state for a system with a sufficiently large number of solitons. On the other hand, the above-mentioned action of a piston leads under certain conditions to the formation of a stationary shock wave, the front of which can be considered to be a system of solitons with smoothly varying parameters. Such shock waves have already been known for a long time in plasmas,[11,12] in non-linear electrical circuits,[13] and in dispersive media with viscoisty and thermal conductivity.[14] The theory given below, in Sec. 4, allows us to give a detailed description of their soliton structure for a very wide class of perturbations $\varepsilon R[u]$, including as particular cases the perturbations acting in the above-mentioned cases.

2. We shall look for a solution of Eq. (1) in the form of a superposition of two expressions of the form(2)

$$u = u_{1s} + u_{2s} + \delta u_1 + \delta u_2,$$

where we have substituted the quantities $\varkappa_i$ and $\xi_i (i = 1, 2)$ for $\varkappa$ and ξ which satisfy Eqs. (3) and (4) in which $\varepsilon R[u_s]$ is replaced by

$$\varepsilon R[u_{is}] + \varepsilon' R_{12}' + \varepsilon'' R_i''. \qquad (6)$$

Here $\varepsilon' R_{12}'$ is the perturbation caused by the overlapping of the two solitons, while $\varepsilon'' R_i''$ is connected with the

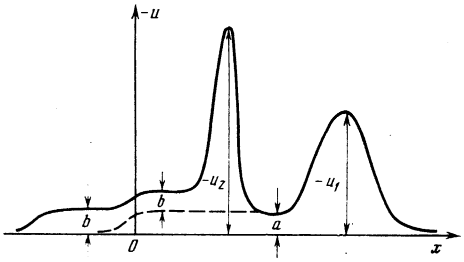

FIG. 1. Case when the second soliton "sits on the tail" of the first one; $a = -(\delta u_-)_1$, $b = -(\delta u_-)_2$. We assume that the perturbation was switched on when the center of the first soliton was at the point $x = 0$. We have not drawn here the oscillations at the end of the tails (they are unimportant for the effects considered in the present paper).

fact that one of the solitons is in the region of the tail of the other soliton. It will become clear from what follows that $\varepsilon' \sim (\delta\varkappa/\varkappa)^2 \ll 1$ and $\varepsilon'' \sim \varepsilon$. Neglecting small quantities of second order (and, in particular, $\varepsilon\varepsilon'$) we can write

$$\varepsilon' R_{12}' = 6\frac{\partial}{\partial x}(u_{1s}u_{2s}) = 24\varkappa_1^2\varkappa_2^2\frac{\partial}{\partial x}(v_1 v_2) \qquad (7)$$

[$v_i = \mathrm{sech}^2 z_i$ and $z_i = \varkappa_i(x - \xi_i)$]. Assuming that $\xi_1 > \xi_2$ we have for the case when the second soliton enters the region of the tail of the first soliton (see Fig. 1)

$$\varepsilon'' R_2'' = 6\varepsilon\varkappa^2 q_1 \partial u_{2s}/\partial x, \qquad \varepsilon'' R_1'' = 0. \qquad (8)$$

Using (6) to (8) and the smallness of $\delta\varkappa$ we find after simple calculations

$$\frac{d\varkappa_1}{dt} = 64\varkappa^4 e^{-2\varkappa r} - \frac{\varepsilon}{4\varkappa}\int_{-\infty}^{\infty} R[u_s]\,\mathrm{sech}^2 z\,dz, \qquad (9)$$

$$\frac{d\varkappa_2}{dt} = -64\varkappa^4 e^{-2\varkappa r} - \frac{\varepsilon}{4\varkappa}\int_{-\infty}^{\infty} R[u_s]\,\mathrm{sech}^2 z\,dz, \qquad (10)$$

$$\frac{d\xi_1}{dt} = 4\varkappa_1^2 - 16\varkappa^2 e^{-2\varkappa r} - \frac{\varepsilon}{4\varkappa^3}\int_{-\infty}^{\infty} R[u_s](z\,\mathrm{sech}^2 z + \mathrm{th}\, z + \mathrm{th}^2 z)\,dz, \qquad (11)$$

$$\frac{d\xi_2}{dt} = 4\varkappa_2^2 + 112\varkappa^2 e^{-2\varkappa r} - 6\varkappa^2\varepsilon q\theta(\xi_2)$$

$$-\frac{\varepsilon}{4\varkappa^3}\int_{-\infty}^{\infty} R[u_s](z\,\mathrm{sech}^2 z + \mathrm{th}\, z + \mathrm{th}^2 z)\,dz. \qquad (12)$$

Here $r = \xi_1 - \xi_2 > 0$, $\varkappa = 1/2(\varkappa_1 + \varkappa_2)$, $\theta(\xi) = 1 (\xi > 0)$, $\theta(\xi) = 0$ $= 0 (\varkappa < 0)$, and we assumed that the perturbation εR was switched on at the moment when the first soliton was at the point $x = 0$ so that the left-hand limit of its tail lies in the vicinity of the point $x = 0$.[2,3] It will become clear in what follows that always $|(\varkappa_1 - \varkappa_2)r| \ll 1$ and $\exp(-2\ r) \lesssim [(\varkappa_{10} - \varkappa_{20})/\varkappa]^2$. In first-order perturbation theory we neglected in the right-hand sides of (9) to (12) the difference between $\varkappa_1$ and $\varkappa_2$ in the terms containing the exponential (and also ε).

We consider first the results of these equations at $\varepsilon = 0$, i.e., when we have the usual two-soliton solution. It then follows from (9) and (10) that $\varkappa_1 + \varkappa_2 = \mathrm{const.}$, i.e., $\varkappa$ is an integral of the motion. Introducing the notation

$$p(t) = \varkappa_1(t) - \varkappa_2(t), \qquad \Delta\varkappa = -p(-\infty) > 0, \qquad (13)$$

we get furthermore[2)]

$$dp/dt=128\varkappa^4 \exp(-2\varkappa r), \tag{14}$$

$$dr/dt=8\varkappa p-128\varkappa^2 \exp(-2\varkappa r). \tag{15}$$

These equations without the second term in (15) were obtained in Ref. 10 by another method. One can neglect this term when $p \gg 16\varkappa\exp(-2\varkappa r)$, i.e., at sufficiently large r. We consider that region first. We then get from (14) and (15)

$$p=\mp 4\varkappa[\exp(-2\varkappa r_0)-\exp(-2\varkappa r)]^{1/2}, \tag{16}$$

$$dr/dt=\mp 32\varkappa^2[\exp(-2\varkappa r_0)-\exp(-2\varkappa r)]^{1/2}. \tag{17}$$

Here r_0 is an integration constant, the magnitude of which can be found by putting $r=\infty$ in (16) and using (13):

$$r_0=(1/\varkappa)\ln(4\varkappa/\Delta\varkappa). \tag{18}$$

It is clear from (16) that in the vicinity of the point $r=r_0$ one cannot neglect the second term in (15) and one must thus reconsider the solution (16), (17). To obtain a more exact result we note that when $\varkappa(r-r_0) \ll 1$ we can put $r=r_0$ in the right-hand side of (15). One can easily verify that then one obtains the solution of the set (14), (15) by adding to (16) the constant term $16\varkappa\exp(-2\varkappa r_0)$, i.e., by writing instead of (16)

$$\begin{aligned} p&\approx\mp 4\varkappa[\exp(-2\varkappa r_0)-\exp(-2\varkappa r)]^{1/2}+16\varkappa\exp(-2\varkappa r_0)\\ &\approx\mp 4\varkappa[2\varkappa(r-r_0)]^{1/2}\exp(-\varkappa r_0)+16\varkappa\exp(-2\varkappa r_0) \end{aligned} \tag{19}$$

and retaining (17) without change.

It is clear that (19) is valid in the whole range $(r-r_0) \ll 1$. Remaining in that region, but assuming that

$$[2\varkappa(r-r_0)]^{1/2} \gg 4\exp(-\varkappa r_0)=\Delta\varkappa/\varkappa, \tag{20}$$

we can neglect the second term in (19). The second terms on the right-hand sides of (15) and (19) are thus important only when

$$2\varkappa(r-r_0) \leqslant (\Delta\varkappa/\varkappa)^2. \tag{21}$$

It follows from (17) that the quantity r_0 is the minimum distance between the solitons. It is convenient to assume that $r_0=r(0)$. In that case $r(t)$ decreases at $t<0$ and increases at $t>0$, i.e., the negative (positive) signs in (16), (17), and (19) must be taken at $t<0$ ($t>0$). At $t<0$ the amplitude of the first soliton (the one on the right) increases and that of the second one decreases. The magnitudes of $\varkappa_i(t)$ become equalized at the point r' which we find from (19):

$$\varkappa(r'-r_0)={}^1/_2(\Delta\varkappa/\varkappa)^2. \tag{22}$$

In what follows we have $\varkappa_1(t)>\varkappa_2(t)$ at $t>t'$. As $t\to\infty$, where (16) is again valid, we get $p(\infty)=\Delta\varkappa$, i.e.,

$$\varkappa_1(\infty)-\varkappa_2(\infty)=\varkappa_2(-\infty)-\varkappa_1(-\infty).$$

It follows from this and from the fact that $\varkappa$ is constant that the amplitudes are exchanged as a result of the collision.[8,9]

We turn to the case $\varepsilon\neq 0$. In that case Eq. (14) remains unchanged and instead of (15) we get

$$dr/dt=8\varkappa p-128\varkappa^2\exp(-2\varkappa r)+6\varkappa^2\varepsilon q\theta(\xi_2). \tag{23}$$

The quantity $\varkappa(t)$ now changes according to the equation

$$\frac{d\varkappa}{dt}=-\frac{\varepsilon}{4\varkappa}\int_{-\infty}^{\infty}R[u_s]\operatorname{sech}^2 z\,dz, \tag{24}$$

which follows from (9) and (10).

We consider again the collision of two solitons, assuming that "prior to the collision" the second soliton is outside the region of the tail of the first soliton and after the collision it is in the tail region. In that case the collision process is irreversible. Assuming that

$$\frac{\varepsilon}{4\varkappa^5}\int_{-\infty}^{\infty}R[u_s]\operatorname{sech}^2 z\,dz \leqslant \left(\frac{\Delta\varkappa}{\varkappa}\right)^2, \tag{25}$$

we can neglect the change of $\varkappa$ in Eqs. (14) and (15). Proceeding as in the case $\varepsilon=0$, we are led again to Eq. (17) but instead of (19) we shall have

$$\begin{aligned} p&\approx\mp 4\varkappa[\exp(-2\varkappa r_0)-\exp(-2\varkappa r)]^{1/2}-6\varkappa^2\varepsilon q\theta(\xi_2), \quad [2\varkappa(r-r_0)]^{1/2}\gg\Delta\varkappa/\varkappa,\\ p&\approx\mp 4\varkappa[2\varkappa(r-r_0)]^{1/2}+16\varkappa\exp(-2\varkappa r_0)-6\varkappa^2\varepsilon q, \quad \varkappa(r-r_0)\ll 1. \end{aligned} \tag{26}$$

In the point $r=r_0$ where the soliton velocities become equal ($dr/dt=0$)

$$p(r_0)=16\varkappa\exp(-2\varkappa r_0)-{}^3/_4\varkappa^2\varepsilon q. \tag{27}$$

3. We turn now to a study of the interaction of an arbitrary number of solitons, assuming that the conditions

$$\varepsilon\int_{-\infty}^{\infty}R[u_s]\operatorname{sech}^2 z\,dz>0, \quad \varepsilon q<0 \tag{28}$$

are satisfied. The first of them means that the perturbation $\varepsilon R[u]$ leads to a decrease in the soliton amplitudes when they interact with one another, while the second one gives $(\delta u_-)_i<0$.

Introducing the notation

$$p_i=\varkappa_i-\varkappa_{i+1}, \quad r_i=\xi_i-\xi_{i+1}, \quad p_i/\varkappa_i\ll 1 \tag{29}$$

and using the same expressions as before for the perturbations which describe the interaction of a soliton with its nearest neighbors we get the following equations

$$d\varkappa_1/dt=64\varkappa_1^4\exp(-2\varkappa_1 r_1)-\varepsilon A_1, \tag{30}$$

$$d\varkappa_m/dt=-64\varkappa_{m-1}^4\exp(-2\varkappa_{m-1}r_{m-1})+64\varkappa_m^4\exp(-2\varkappa_m r_m)-\varepsilon A_m, \tag{31}$$

$$d\varkappa_N/dt=-64\varkappa_{N-1}^4\exp(-2\varkappa_{N-1}r_{N-1})-\varepsilon A_N, \tag{32}$$

$$d\xi_1/dt=4\varkappa_1^2-16\varkappa_1^2\exp(-2\varkappa_1 r_1)-\varepsilon B_1, \tag{33}$$

$$\begin{aligned} d\xi_m/dt&=4\varkappa_m^2+112\varkappa_{m-1}^2\exp(-2\varkappa_{m-1}r_{m-1})\\ &-16\varkappa_m^2\exp(-2\varkappa_m r_m)-\sum_{j=1}^{m-1}6\varepsilon q_i\varkappa_j^2-\varepsilon B_m, \end{aligned} \tag{34}$$

$$d\xi_N/dt=4\varkappa_N^2+112\varkappa_{N-1}^2\exp(-2\varkappa_{N-1}r_{N-1})-\sum_{j=1}^{N-1}6\varepsilon q_j\varkappa_j^2-\varepsilon B_N,$$

where $m=2,3,\ldots,N-1$ and

$$A_m=\frac{1}{4\varkappa_m}\int_{-\infty}^{\infty}R[u_{sm}]\operatorname{sech}^2 z_m\,dz_m,$$

$$B_m=\frac{1}{4\varkappa_m^3}\int_{-\infty}^{\infty}R[u_{sm}](z_m\operatorname{sech}^2 z_m+\operatorname{th} z_m+\operatorname{th}^2 z_m)\,dz_m.$$

We elucidate now under what kind of conditions a stationary (or quasi-stationary) state of our system is possible. It follows from (30) and (31) that the condition for stationarity of the amplitudes of the first $N-1$ solitons has the form

$$r_m = -\frac{1}{2\varkappa_m}\ln\left[\varepsilon\sum_{j=1} A_j \Big/ 64\varkappa_m^4\right], \tag{36}$$

and the condition that their velocities (33) and (34) are equal is

$$p_m \approx \frac{3\varepsilon A_m}{16\varkappa_m^3} - \frac{3\varepsilon q_m \varkappa_m}{4} \tag{37}$$

$(m = 1, 2, \ldots, N-1)$. As regards the last, N–th soliton under these conditions $d\varkappa_N/dt \neq 0$ and $d\xi_N/dt \neq d\xi_{N-1}/dt$. Thus, if for some reason or other at some time conditions (36) and (37) are established, i.e., $d\varkappa_i/dt = 0$, $d\xi_i/dt = d\xi_1/dt$ for $i = 1, 2, \ldots, N-1$, at later times they are violated and the system will "collapse." However, the decay will proceed more slowly, the larger N. For large N the conditions (36) and (37) thus determine a quasi-bound state of a system of solitons[3)] with a lifetime that increases with increasing N.

4. The system considered might remain stationary, if there acted upon it yet another external force, apart from the perturbation $\varepsilon R[u]$, and compensated the tendency to decay. Such a situation is realized in a shock wave which for well-defined conditions is found as the result of the action of a piston moving with a constant velocity V and of an external perturbation $\varepsilon R[u]$. The leading part (front) of such a shock wave consists of a system of solitons with slowly changing amplitudes moving with the same velocity V. As the amplitudes of the solitons decrease, the wave profile tends to a constant value v (see Fig. 2).

The shock waves discussed here exist if Eq. (1) has stationary solutions of the form $u(x-Vt)$ satisfying the boundary conditions

$$u(x,t)|_{x\to-\infty} \to v, \quad u(x,t)|_{x\to\infty} \to 0. \tag{38}$$

Substituting $u(x-Vt)$ into (1) and integrating once we get

$$u_{xx} - 3u^2 - Vu = -\varepsilon \int_x^\infty R[u(x',t)]dx'. \tag{39}$$

Substituting $x \to -\infty$ and using (38) we get

$$3v^2 + Vv - \varepsilon T = 0, \quad T = \int_{-\infty}^{\infty} R[u(x,t)]dx. \tag{40}$$

From (40) we get

$$v \approx -V/3 - \varepsilon T/V. \tag{41}$$

When $\varepsilon = 0$ and using the second of conditions (38) we see that the solution of Eq. (39) has the form of a soliton moving with a velocity V. For sufficiently small ε we get the picture shown in Fig. 2, i.e., a shock wave with a leading part that can be considered to be a stationary sequence of slightly overlapping solitons having the same velocity V. The structure of such a system is described by the relations obtained in Sec. 3. Putting the velocity (33) of the first soliton equal to the quantity V we get the parameter $\varkappa_1$ determining its amplitude and size:

$$\varkappa_1 = \frac{1}{2}V^{1/2}\left(1 + \frac{\varepsilon A_1}{2V^2} + \frac{\varepsilon B_1}{2V}\right). \tag{42}$$

The amplitudes of all subsequent solitons and the distances between them are determined by Eqs. (36) and (37). The distances r_m between the solitons are then of the order of $\ln \varepsilon q_m$ and decrease slowly with increasing m; all $p_m > 0$ thanks to conditions (28) (i.e., $\varkappa_m - \varkappa_{m+1} > 0$), and the maxima of the profile $-u(x,t)$, which are situated in the peaks of the solitons decrease gradually. The difference between two consecutive maxima of the quantity $-u$ is equal to

$$u_m - u_{m+1} = -4\varkappa_m p_m - \varepsilon q_m \varkappa_m^2 \approx -\frac{3\varepsilon A_m}{4\varkappa_m^2} + 2\varepsilon q_m \varkappa_m^2 < 0. \tag{43}$$

The relations (42), (36), (37), and (43) completely describe the structure of the leading (soliton) part of the shock wave. Their correctness is violated when the perturbation theory which is the basis of the calculation ceases to be applicable. The conditions for the applicability of the perturbation theory are

$$2\varkappa_j r_j \gg 1, \quad |\varepsilon q_j| \ll 1. \tag{44}$$

As a simple example we consider the KdV–Burgers equation where $R[u] = \partial^2 u/\partial x^2$. In that case[3]

$$A_j = 8\varkappa_j^3/15, \quad \varepsilon q_j = -8\varepsilon/15\varkappa_j, \quad j = 1, 2, \ldots, \tag{45}$$

$$r_1 = (2\varkappa_1)^{-1}\ln(120\varkappa_1/\varepsilon). \tag{46}$$

Substituting (45) into (37) we get

$$p_j \approx \varepsilon/2, \quad \varkappa_j \approx \varkappa_1 - j\varepsilon/2. \tag{47}$$

For sufficiently small ε the number of solitons in the shock wave is large. For solitons with large numbers m we get

$$\varepsilon\sum_{j=1}^{m} A_j \approx \frac{16}{15}\int_{\varkappa_m}^{\varkappa_1} \varkappa^3 d\varkappa = \frac{4}{15}(\varkappa_1^4 - \varkappa_m^4), \tag{48}$$

$$r_m \approx -(2\varkappa_m)^{-1}\ln[(\varkappa_1^4/\varkappa_{m+1}^4 - 1)/240],$$
$$u_{m+1} \approx u_1 + (22\varepsilon m/15)(\varkappa_1 - \varepsilon m/4). \tag{49}$$

One checks easily that to first order in ε Eqs. (48) and (49) turn out to be valid also for small m $(m = 1, 2, \ldots)$. It is clear from (47) and (48) that the number of solitons in a shock wave is of the order $\varkappa_1/\varepsilon$ (in the limits considered ε has the dimension of $\varkappa$). In particular, for $m = 0.5\,\varkappa_1/\varepsilon$ we have

$$\varkappa_m/\varkappa_1 \approx 0.75, \quad r_m \approx 2.36\varkappa_m^{-1}, \quad u_m \approx -1.36\varkappa_1^2. \tag{50}$$

For $m = 0.75\,\varkappa_1/\varepsilon$ we have $\varkappa_m r_m \approx 1.88$, i.e., the first of conditions (44) is already not satisfied. We can thus assume that to a fair approximation $m = 0.5\,\varkappa_1/\varepsilon$ gives the number of solitons in the shock wave. It is interesting to note then that the quantity u_m in (50) is very close to the limiting value v in (38). Indeed, according to (41), (42) $v \approx -1.33\,\varkappa_1^2$ which differs only by $0.03\varkappa_1^2$ from the peak of the soliton with $m = 0.5\varkappa_1/\varepsilon$.

Using (50) we can estimate the length of the "soliton part" in the transitional region:

$$l_{(s)} = \sum_{i=1}^{2/\varepsilon} r_i \approx \frac{1}{\varepsilon}\int_{\varkappa_1}^{3\varkappa_1/4} \frac{d\varkappa}{\varkappa}\ln\left[\frac{1}{240}\left(\frac{\varkappa_1^4}{\varkappa^4} - 1\right)\right] \approx \frac{2}{\varepsilon}. \tag{51}$$

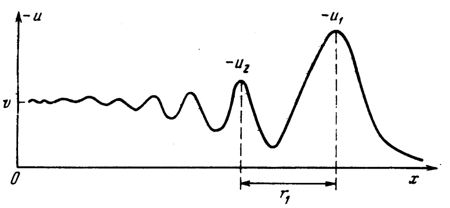

FIG. 2. Profile of an oscillatory shock wave.

The relations obtained agree with estimates following from the averaging method.[11,13]

Returning to the general case we must note that in order that to enable the perturbation $\varepsilon R[u]$ in (1) to lead to a shock wave it is necessary that certain conditions be satisfied and guarantee the continuous transition of the described soliton structure of its front to small profile oscillations, which are damped as $x \to -\infty$, against the background $u(-\infty) = v$. One condition is the convergence of the integral in (40). As already noted in Ref. 16, this condition is not fulfilled for a number of perturbations, for instance, for

$$R[u]=\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{u(x',t)}{x'-x}dx'.$$

This expression describes according to Ref. 17 the effect of Landau damping on non-linear ion-sound waves. Another perturbation for which T diverges is[16] $R[u]=u$ [we note that in that case it is impossible to satisfy also simultaneously the conditions (28) at any sign of ε].

We consider yet another condition which is necessary for a continuous transition to the region where $u \to v$. Putting in (39) $u = v + \tilde{u}(x - Vt)$, linearizing in $\tilde{u}$, and using (41) we get

$$\frac{d^2\tilde{u}}{dx^2}+V\tilde{u}=\varepsilon\int_{-\infty}^{x}R[v+\tilde{u}(x'-Vt)]dx'. \quad (52)$$

Equation (52) must have a solution satisfying the condition $\tilde{u}(x) \to 0$ $(x \to -\infty)$. For small ε this solution must be found from perturbation theory. We put

$$\tilde{u}(x)=D(\varepsilon x)\cos[V^{1/2}(x-x_0)]+\varepsilon\tilde{u}_1(x), \quad (53)$$

i.e., dD/dx and $\varepsilon\tilde{u}_1$ are assumed to be quantities of first order in ε and d^2D/dx^2 of second order. Substituting (53) into (52) we get after simple transformations the following equations:

$$\int_{-\pi/\alpha}^{\pi/\alpha}R[v+D(y)\cos\alpha x]dx=0,$$

$$\int_{-\pi/\alpha}^{\pi/\alpha}R[v+D(y)\cos\alpha x]\sin\alpha x\,dx=0, \quad (54)$$

$$2\pi\alpha\frac{dD(y)}{dy}+\int_{-\pi/\alpha}^{\pi/\alpha}R[v+D(y)\cos\alpha x]\cos\alpha x\,dx=0, \quad (55)$$

where $\alpha = V^{1/2}$ and $y = \varepsilon x$. Equations (54) are some conditions which the operator R must satisfy, while (55) is an equation for $D(y)$ which one must solve with the boundary condition $D(y) \to 0$ $(y \to -\infty)$. We illustrate this again with $R = d^2/dx^2$ as the example. One checks easily that (54) is satisfied automatically while (55) give $dD/dy = D/2$ which agrees with the asymptotic solution of Eq. (39) for this case.

[1)] We shall consider elsewhere another interesting case when $\varkappa_2 \gg \varkappa_1$, $t_{s_1} \sim t_{p_2}$.

[2)] We emphasize that in the exact two-soliton solution one can speak of solitons only at $t = \mp\infty$. We shall denote the parameters of the latter solitons by $\varkappa_{io}$ ($-\varkappa_{io}^2$ are the time-independent eigenvalues of the Schrödinger equation for the two-soliton potential). They are connected with our $\varkappa_i(t)$ (at $\varepsilon = 0$) as follows: $\varkappa_{10}=\varkappa_1(-\infty)$, $\varkappa_{20}=\varkappa_2(-\infty)$. According to (13) we retained the symbol $\Delta\varkappa$ for $\varkappa_{20}-\varkappa_{10}$.

[3)] Such states were apparently observed experimentally in Ref. 15. I use this opportunity to thank L. A. Ostrovskiĭ for drawing my attention to that paper.

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