relative change that occurs in the flux density of the wave over a unit length along the propagation direction as a result of the conversion of the wave into a plasma wave in the region of the jump in the form

$$\frac{\delta S_p}{S_w}\sim\frac{v_T}{c}\frac{(E_\perp)_1^2}{E_g^2}\left(\frac{\varepsilon_+}{\varepsilon_2}\right)^{1/2}\left(1+\frac{\varepsilon_+}{|\varepsilon_-|}\right)\frac{1}{|\varepsilon_1|}.$$

This ratio is small under the conditions of the inequality (4.4).

The obtained solution can be used to construct a more complex solution—of the type of a plasma facula (layer) in an unbounded plasma with $\varepsilon_0>0$–having even two jumps. Treating the $z=0$ plane as a middle plane, and seeking the solution that describes the decreasing field for $|z|\to\infty$, we arrive on account of the symmetry of the problem to a solution of the surface-wave type. However, the dispersion properties of such a wave ($\infty>\eta\geqslant\varepsilon_0$) will now be determined by another factor: the law of conservation of the wave-energy flux along the axis of the facula:

$$S_0=\frac{c}{2\pi}E_g^2\frac{\eta^{1/2}}{k}\int_0^\infty\frac{b^2(\zeta;\eta)}{|\varepsilon|}d\zeta.$$

Introducing the effective skin depth

$$\delta=\frac{1}{kB_2}\int_0^\infty b(\zeta)d\zeta,$$

where B_2 is the amplitude of the field at $z=0$, we obtain for a sufficiently large wave amplitude[1]

$$S_0\approx\frac{c}{2\pi}E_g^2\eta^{1/2}\Delta,$$

where Δ does not depend on the wave amplitude.

[1]A. G. Boev and A. V. Prokopov, Zh. Eksp. Teor. Fiz. **69**, 1208 (1975) [Sov. Phys. JETP **42**, 617 (1975)].
[2]A. V. Gurevich and L. P. Pitaevskiĭ, Zh. Eksp. Teor. Fiz. **45**, 1243 (1963) [Sov. Phys. JETP **18**, 855 (1964)].
[3]V. B. Gil'denburg, Zh. Eksp. Teor. Fiz. **46**, 2156 (1964) [Sov. Phys. JETP **19**, 1456 (1964)].
[4]V. B. Gil'denburg and G. M. Fraĭman, Zh. Eksp. Teor. Fiz. **69**, 1601 (1975) [Sov. Phys. JETP **42**, 816 (1975)].
[5]A. G. Litvak and G. M. Fraĭman, Zh. Eksp. Teor. Fiz. **68**, 1288 (1975) [Sov. Phys. JETP **41**, 640 (1975)].
[6]V. E. Golant, Usp. Fiz. Nauk **65**, 39 (1958).

Translated by A. K. Agyei

Linear wave interaction in a plasma with an inhomogeneous magnetic field

V. V. Zheleznyakov, V. V. Kocharovskiĭ, and Vl. V. Kocharovskiĭ

Institute of Applied Physics, USSR Academy of Sciences

(Submitted 21 December 1978)
Zh. Eksp. Teor. Fiz. **77**, 101–113 (July 1979)

The linear interaction of waves in the region where the geometrical-optics approximation is violated is analyzed qualitatively on the basis of the Budden–Kravtsov equations that describe the propagation of the electromagnetic waves in a smoothly inhomogeneous magnetoactive plasma. It is shown that the interaction sets in when the polarization of the geometrical-optics waves is substantially altered over the spatial period of the beats between these waves. Conditions are obtained under which the efficiency of the interaction is characterized by a single parameter whose form can be established without solving the equations that describe this phenomenon. The plasma-parameter regions in which the interaction is the most effective at a specified scale of the inhomogeneity of the magnetic field are determined. The exact solution of the standard problem that describes the linear interaction in plasma layers of the transition type is analyzed.

PACS numbers: 52.35.Hr, 52.40.Db

The study of the sources of cosmic radio emission calls for an exact account of those changes that the emission undergoes in the plasma located on the path from the source to the observer. Because of the weak inhomogeneity of the electron density and of the magnetic fields in the cosmic plasma (over scales comparable with the wavelength), these changes can usually be described in the geometrical-optics approximation, so that it is rather a simple matter to take them into account. However, if the emission passes through regions where the geometrical-optics approximation does not hold, the situation becomes much more complicated: a linear interaction arises between the waves and causes the amplitude and phases of the emerging ordinary and extraordinary waves to differ substantially from those expected in the geometrical-optics approximation.

If the polarization characteristics of the radiation source are known, then the parameters of the cosmic plasma in the interaction region can be evaluated from the observed polarization that is produced as a result of the linear wave transformation. Diagnostics of this type uncovers new possibilities of studying the plasma near the earth and between the planets by reception of radiowaves from spacecraft. It is of interest also for the study of processes in a laboratory plasma.

In a tenuous plasma ($\omega_L\ll\omega$, where ω_L is the plasma

0038-5646/79/070051-08$02.40

frequency and ω is the radiation frequency), a sizable interaction is realized only when the magnetic field is sufficiently inhomogeneous (see Ref. 1 and Sec. 2 of the present article). Therefore an investigation of the interaction in a plasma situated in an inhomogeneous magnetic field becomes particularly significant. So far, however, only preliminary qualitative concepts have been developed,[1,2] based on Cohen's initial idea,[3] and two problems, of importance from the viewpoint of astrophysical applications,[4] were solved concerning the interaction occurring when waves pass through a transverse or zero magnetic field.[5-8] What is common to these problems is the reversal of the sign of the projection of the magnetic field on the direction of propagation in the region of the wave interaction. At the same time, considerable interest attaches to the problem of the interaction in plasma layers of the transition type—from a quasihomogeneous plasma with one set of parameters to a quasihomogeneous with another set of values of these parameters. This problem includes, as a particular case, the problem of radiation from a magnetoactive plasma into an isotropic plasma or into vacuum. Under these conditions the wave transformation leads to the appearance of the so called effective limiting polarization, when the polarization of the radiation that emerges from the magnetoactive plasma differs noticeably from the polarization calculated in the geometrical-optics approximation.

A qualitative analysis presented in this article of the problem of linear interaction applies to all the indicated problems dealing with the propagation of electromagnetic waves in a plasma with an inhomogeneous magnetic field B_0. We confine ourselves for simplicity to an investigation of only the outgoing waves (without the reflected ones) under conditions when there is no shear of the magnetic field (i.e., the component of the vector $\mathbf{B}_0$ in a plane perpendicular to the wave vector $\mathbf{k}$ retains its orientation along the beam). The shear-induced wave transformation in a plasma is considered in another article.[9] The choice of the standard problem investigated in Sec. 3 was dictated by the almost complete absence of published investigated exact solutions for transition layers with inhomogeneous magnetic fields (see Ref. 10 in this connection).

1. INITIAL EQUATIONS

It is known[11] that in a homogeneous plasma with a constant magnetic field $\mathbf{B}_0$ there propagate two normal waves $\exp(i\omega t - i\mathbf{k}\cdot\mathbf{r})$ with refractive indices

$$n_{1,2}^2=\varepsilon\pm C(1+q^2)^{1/2} \quad (1.1)$$

and with polarization coefficients

$$K_{1,2}=-iE_y^{(1),(2)}/E_x^{(1),(2)}=q-(\pm\sqrt{1+q^2}). \quad (1.2)$$

The index 1 and the plus sign pertain here to the wave of type 1, while the index 2 and the minus sign pertain to the wave of type 2. If $q>0$ [see (1.4)], then these are respectively the extraordinary and ordinary waves, and vice versa if $q<0$. The other quantities in (1.1) and (1.2) are

$$2\varepsilon=\frac{2(1-v)(u+v-1)+uv\sin^2\alpha}{u-(1-v)-uv\cos^2\alpha},\quad C=\frac{v\sqrt{u}(1-v)\cos\alpha}{u-(1-v)-uv\cos^2\alpha}, \quad (1.3)$$

$$q=\frac{u^{1/2}\sin^2\alpha}{2(1-v)\cos\alpha}, \quad (1.4)$$

E_x and E_y are the components of the electric field in a plane orthogonal to the wave vector $\mathbf{k}$ (the y axis lies in the plane of the vectors $\mathbf{B}_0$ and $\mathbf{k}$, while the x axis is perpendicular to the latter), α is the angle between $\mathbf{B}_0$ and $\mathbf{k}$, $u=\omega_B^2/\omega^2$, $v=\omega_L^2/\omega^2$, where ω is the wave frequency, and ω_B and ω_L are respectively the electron gyrofrequency and the plasma frequency. It is assumed throughout the article that there is no absorption or spatial dispersion in the plasma.

In the case of quasilongitudinal propagation ($q^2\ll 1$) the polarization of the waves of both types is close to circular: $K_{1,2}\approx\mp 1$; in quasitransverse propagation, when $q^2\gg 1$, the polarization is almost linear: $K_2=-K_1^{-1}\gg 1$ or $K_2=-K_1^{-1}\ll 1$.

In a smoothly inhomogeneous medium, where

$$\frac{\omega}{c}n_{1,2}\Lambda\gg 1 \quad (1.5)$$

(Λ is the characteristic scale of variation of the material properties that determine the wave propagation), the geometrical-optics approximation is usually valid. It is described by the first terms of the asymptotic expansion of the exact solution of the equations of the electrmoagnetic field in the parameter $1/k_0\equiv c/\omega$. This approximation, however, may be violated in those regions (1.5) where the condition

$$\frac{\omega}{c}|n_1-n_2|\Lambda\gg 1 \quad (1.6)$$

is violated. In these regions, obviously, the dispersion branches $n_1(\mathbf{r})$ and $n_2(\mathbf{r})$ come sufficiently close to each other, and the properties of the medium change substantially over the period of the spatial beats between the waves of the two types.

If the waves pass in the course of their propagation through a region where the geometrical-optics approximation is not valid, then the asymptotic expansion of the exact solution of the linear field equations takes different forms on different sides of the indicated region (the ratio of the complex amplitudes of the waves of the two types changes). In this case one speaks of linear interaction of the waves in the inhomogeneous medium and of the conversion of waves of one type into waves of another type.

An investigation of the problem of interaction in a three-dimensionally inhomogeneous magnetoactive plasma becomes simpler if in addition to the criterion of smooth inhomogeneity (1.5) we impose the condition of weak anisotropy of the medium:

$$|\varepsilon_{ij}-\varepsilon\delta_{ij}|\ll\varepsilon\approx{}^1/_3\mathrm{Sp}\,\varepsilon_{ij} \quad (1.7)$$

(ε_{ij} is the Hermitian dielectric tensor of the magnetoactive plasma, $Sp\,\varepsilon_{ij}$ is the trace of the matrix ε_{ij}, and δ_{ij} is the Kronecker symbol).

In analogy with the method of the "quasi-isotropic" approximation of geometrical optics,[12] we seek the induction of the electric field $\mathbf{D}$ and of the magnetic field $\mathbf{B}$ of the radiation in the form of an expansion in powers of $1/k_0$, just as is done in geometrical optics of an isotropic medium. Then, using the smooth inhomogeneity (1.5) and the weak anisotropy (1.7) as small parameters, we arrive at a system of equations for the complex wave amplitudes $f_{1,2}$:

$$f_1'+in_1f_1=-\Psi f_2,\qquad f_2'+in_2f_2=\Psi f_1, \tag{1.8}$$

in which

$$\Psi=-q'/2(1+q^2). \tag{1.9}$$

In (1.8) and (1.9) the differentiation is with respect to the variable $\zeta=k_0l$, where l is the length of the quasi-isotropic beam corresponding to a medium with refractive index[1)] $n=(n_1+n_2)/2\approx\varepsilon^{1/2}$. The form of this beam is determined by the usual eikonal equation with refractive index n. The high-frequency electric field is connected with the functions $f_{1,2}$ by the relations

$$E_x=\Phi_0\{f_1/(1+K_1^2)^{1/2}+f_2/(1+K_2^2)^{1/2}\},\qquad E_y=\Phi_0\{iK_1f_1/(1+K_1^2)^{1/2}+iK_2f_2/(1+K_2^2)^{1/2}\}. \tag{1.10}$$

Here the function $\Phi_0(l)$ satisfies the equation $\mathrm{div}(n\Phi_0^2\mathbf{l})=0$ ($\mathbf{l}$ is a unit vector along the quasi-isotropic beam). Since, as follows from the system (1.8)

$$|f_1|^2+|f_2|^2=\mathrm{const} \tag{1.11}$$

along the beam, the equation for Φ_0 is in fact the consequence of the energy conservation law $\mathrm{div}\,\mathbf{S}=0$ for a monochromatic wave ($\mathbf{S}$ is the Poynting vector).

The condition of weak anisotropy of the plasma, under which the system (1.8) is valid, is satisfied either in a weak magnetic field

$$u^{1/2}\ll 1-v,\quad v<1, \tag{1.12}$$

or in a tenuous plasma

$$v\ll 1,\ |1-u^{1/2}|\gg v. \tag{1.13}$$

In either case at $u\ll 1$ we have approximately $\varepsilon\approx 1-v$, and in the case (1.13) we have $\varepsilon\approx 1$ at $u\gg 1$. The condition (1.12) corresponds to an almost isotropic plasma, while the inequality $v<1$ ensures transparency of the medium. The condition (1.13) indicates closeness to vacuum, while the inequality $|1-u^{1/2}|\gg v$ excludes in this case the region of the electron gyroresonance $u\approx 1$ where the strong anisotropy is preserved (without allowance for spatial dispersion) even in a tenuous plasma. In the case $u\ll 1$ and $v\ll 1$ the system (1.8) goes over into a system of equations equivalent to the system given in the article of Kravtsov and Naida,[6] for the case $v\sim 1$ see also the paper by Naida.[7]

We note that in the one-dimensional case the system (1.8) was obtained earlier by Budden[13] under weaker assumptions, namely that the reflected waves are neglected and that the refractive indices are relatively close

$$|n_1-n_2|\ll n_1+n_2. \tag{1.14}$$

This relation is equivalent to the following two inequalities:

$$vu\sin^2\alpha\ll|1-v||1-v-u|,\quad vu^{1/2}|\cos\alpha|\ll|1-v-u| \tag{1.15}$$

and can be realized even in a strongly anisotropic plasma, where the criterion (1.7) is violated. According to (1.15) in a strong magnetic field $u\gg 1$ the refractive indices come close together if the propagation is close to longitudinal: $\sin^2\alpha\ll 1$, even in a dense plasma ($1<v\ll u^{1/2}$). In similar cases of sufficiently strong anisotropy, the system (1.8) describes normal incidence of waves in a smoothly inhomogeneous plane-stratified medium (see also Ref. 14).

The system (1.8) makes it possible to obtain the values of the functions f_1^{out} and f_2^{out} and accordingly the radiation field at the exit from the plasma layer from the given values f_1^{in} and f_2^{in} at the entrance. Because of the linearity of the system, f_1^{out} and f_2^{out} can be represented as linear combinations of f_1^{in} and f_2^{in}:

$$f_1^{\mathrm{out}}=F_{11}f_1^{\mathrm{in}}+F_{12}f_2^{\mathrm{in}},\quad f_2^{\mathrm{out}}=F_{21}f_1^{\mathrm{in}}+F_{22}f_2^{\mathrm{in}}.$$

It is easy to verify that the four quantities F_{ij} which make up the transformation matrix $\hat{F}$ take the following form

$$\hat{F}=\begin{pmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{pmatrix}=\begin{pmatrix}(1-Q)^{1/2}e^{-i\gamma} & Q^{1/2}e^{i(\varphi+\gamma)}\\ -Q^{1/2}e^{-i(\varphi+\gamma)} & (1-Q)^{1/2}e^{i\gamma}\end{pmatrix}\exp\left(-i\int n\,d\zeta\right). \tag{1.16}$$

In fact, the substitution $f_{1,2}=\delta_{1,2}\exp(-i\int n\,d\zeta)$ leads to a symmetrical system of equations

$$\delta_1'+{}^1/_2i(n_1-n_2)\delta_1=-\Psi\delta_2,\qquad \delta_2'-{}^1/_2i(n_1-n_2)\delta_2=\Psi\delta_1.$$

The first column of the transformation matrix is obtained if the solution (δ_1,δ_2) is written for the boundary condition $\delta_1^{\mathrm{in}}=1$ and $\delta_2^{\mathrm{in}}=0$. Since the energy conservation law (1.11) already specifies the relation between the moduli of δ_1^{out} and δ_2^{out}, we can introduce the notation

$$\delta_1^{\mathrm{out}}=(1-Q)^{1/2}e^{-i\gamma},\quad \delta_2^{\mathrm{out}}=-Q^{1/2}e^{-i\gamma-i\varphi},$$

where the possible phase differences are taken into account by the factors $\exp(-i\gamma)$ and $-\exp(-i\gamma-i\varphi)$. The second column of the transformation matrix is obtained under the boundary condition $\delta_1^{\mathrm{in}}=0$ and $\delta_2^{\mathrm{in}}=1$. Simple substitution in the system presented above shows that the solution can be assumed here in the form $(-\delta_2^*,\delta_1^*)$, where δ_1 and δ_2 are the previous solutions used to determine the first column. This enables us to write down the transformation matrix in the form (1.16) if account is taken also of the connection between $\delta_{1,2}$ and $f_{1,2}$.

The quantity Q in (1.16) is the so called transformation coefficient and determines the relative intensity of the wave of one type (at the exit) if the wave of the other type is incident on the layer. We have

$$Q=\frac{|f_1^{\mathrm{out}}|^2}{|f_1|^2+|f_2|^2}=\frac{|f_1^{\mathrm{out}}|^2}{|f_2^{\mathrm{in}}|^2} \tag{1.17}$$

at $f_1^{\mathrm{in}}=0$ and

$$Q=\frac{|f_2^{\mathrm{out}}|^2}{|f_1|^2+|f_2|^2}=\frac{|f_2^{\mathrm{out}}|^2}{|f_1^{\mathrm{in}}|^2} \tag{1.18}$$

at $f_2^{\mathrm{in}}=0$. (It is obvious that $0\leqslant Q\leqslant 1$.) The coefficient of transformation into this same wave is $1-Q$. This is clear from the conservation law (1.11), which is taken into account also when the change is made to the last equations in (1.17) and (1.18). It follows from the symmetry of the system (1.8) that the quantities Q (1.17) and (1.18) are equal. These definitions of the transformation coefficient justify the representation of the amplitude factors of the matrix $\hat{F}$ in the form used in (1.16).

2. QUALITATIVE ANALYSIS. INTERACTION PARAMETERS

The qualitative analysis of the linear interaction can be conveniently carried out by using in addition to the system (1.8) also the Ricatti equation that follows from it for the ratio $P=-if_1/f_2$:

$$dP/d\eta=i(P^2-1)-2iG(\eta)P, \tag{2.1}$$

where

$$G(\eta)=\frac{n_2-n_1}{2\Psi}=\frac{1}{q'}(n_1-n_2)(1+q^2)=\frac{n_1-n_2}{2\eta'} \tag{2.2}$$

(the prime denotes differentiation with respect to the dimensionless coordinate $\zeta=k_0l$). On going from (1.8) to (2.1) we introduced a new independent variable $\eta=-\int\Psi d\zeta$. According to (1.2) and (1.9),

$$\eta={}^1/_4\pi+{}^1/_2\operatorname{arctg} q, \quad K_2=-K_1^{-1}=\operatorname{tg}\eta. \tag{2.3}$$

The ratio P determines directly the polarization coefficient of the radiation at any point along the beam [see (1.10)]:

$$K=-iE_y/E_x=K_1(K_2-iP)/(K_1-iP). \tag{2.4}$$

The elements of the transformation matrix (1.16) are expressed in terms of P in the following manner. The transformation coefficient Q and the phase φ are obtained from the relation

$$P_0(\eta_{\rm out})=-i\left(\frac{Q}{1-Q}\right)^{1/2}e^{i\varphi}, \tag{2.5}$$

in which $P_0(\eta_{\rm out})$ is the result of the solution of Eq. (2.1) under the condition $P_0(\eta_{\rm in})=0$ (only a wave of type 2 is incident on the layer). The validity of (2.5) becomes clear if we turn to formulas (1.16) and (1.17). The phase shift γ, in turn, is defined by the relation

$$\gamma-\int_{\eta_{\rm in}}^{\eta_{\rm out}}G(\eta)d\eta=-\operatorname{Re}\int_{\eta_{\rm in}}^{\eta_{\rm out}}P_0(\eta)d\eta, \tag{2.6}$$

in which $P_0(\eta)$ is the solution of (2.1) under the same boundary condition.

Formula (2.6) was obtained by integrating both parts of the substitution

$$P(\eta)=G(\eta)+i\left[f_2\exp\left(i\int n d\zeta\right)\right]^{-1}\frac{d}{d\eta}\left[f_2\exp\left(i\int n d\zeta\right)\right], \tag{2.7}$$

which makes it possible to change back from the Ricatti equation to the system (1.8).

It follows from (2.1) that the entire process of the change in the radiation polarization along the beam (incldiung the interaction region) is determined by the behavior of the characteristic function $G(\eta)$ over the interval of the change of the variable $\eta(l)$ in the considered plasma layer.

From the form of the function $\eta(q)$ (2.3) it is clear that when q changes from $-\infty$ to $+\infty$ the change of η takes place mainly where $q^2\sim1$, and consequently the values $K_{1,2}$ (1.2) correspond to elliptic polarization of the waves of both types. On the contrary, in the region of the quasitransverse propagation ($q^2\gg1$) the variable η remains practically unchanged and stays close to zero or $\pi/2$, while in the region of quasilongitudinal propagation ($q^2\ll1$) the values of η are close to $\pi/4$.

In the limiting case of infinitely large values of the characteristic function $G(\eta)$ over the thickness of the entire plasma layer (i.e., when $n_1\neq n_2$ and η varies arbitrarily slowly along the beam) we can neglect in (2.1) the term $i(P^2-1)$. Then the solution of the equation takes the form

$$P(\eta_{\rm out})=P(\eta_{\rm in})\exp\left[-2i\int_{\eta_{\rm in}}^{\eta_{\rm out}}G(\eta)d\eta\right]$$
$$=P(\zeta_{\rm in})\exp\left[-i\int_{\zeta_{\rm in}}^{\zeta_{\rm out}}(n_1-n_2)d\zeta\right] \tag{2.8}$$

and the transformation matrix (1.16) becomes diagonal

$$\hat{F}=\begin{pmatrix}\exp\left(-i\int_{\zeta_{\rm in}}^{\zeta_{\rm out}}n_1 d\zeta\right) & 0\\ 0 & \exp\left(-i\int_{\zeta_{\rm in}}^{\zeta_{\rm out}}n_2 d\zeta\right)\end{pmatrix}. \tag{2.9}$$

This variant corresponds to geometrical-optics wave propagation

$$Q=0,\quad 2\gamma=\int_{\zeta_{\rm in}}^{\zeta_{\rm out}}(n_1-n_2)d\zeta.$$

When the interaction appears, the transformation coefficient Q becomes different from zero, and 2γ becomes different from the geometrical-optics "Faraday" phase difference

$$\int_{\rm in}^{\rm out}(n_1-n_2)d\zeta.$$

For large but finite values of the functions $G(\eta)$ along the beam ($|G(\eta)|\gg1$, an approximate solution of (2.1) can be obtained in the form of an expansion in the small parameter $1/a$, if we represent $G(\eta)$ as $ag(\eta)$ where $a\gg1$ and $|g(\eta)|\gtrsim1$. Then

$$P(\eta_{\rm out})=\left[P(\eta_{\rm in})-i\int_{\eta_{\rm in}}^{\eta_{\rm out}}\exp\left(2i\int_{\eta_{\rm in}}^{\eta}G(\eta)d\eta\right)d\eta\right]\exp\left(-2i\int_{\eta_{\rm in}}^{\eta_{\rm out}}G(\eta)d\eta\right)$$
$$\approx-\frac{1}{2G(\eta_{\rm out})}+\left[P(\eta_{\rm in})+\frac{1}{2G(\eta_{\rm in})}\right]\exp\left(-2i\int_{\eta_{\rm in}}^{\eta_{\rm out}}G(\eta)d\eta\right) \tag{2.10}$$

(in the transition to the last equation, the integral is calculated by the stationary-phase method). This result differs from the geometrical-optics solution in that it contains small terms of order of $1/a$.

The transformation coefficient is not equal to zero but remains small compared with unity (weak interaction):

$$Q\approx\left|\int_{\eta_{\rm in}}^{\eta_{\rm out}}\exp\left[2i\int_{\eta_{\rm in}}^{\eta}G(\eta)d\eta\right]d\eta\right|^2\leqslant G_{\min}^{-2}\ll1, \tag{2.11}$$

where $G_{\min}$ is the smallest value of $G(\eta)$ on the interval from $\eta_{\rm in}$ to $\eta_{\rm out}$. The phase shift 2γ differs little from the geometrical-optics phase difference:

$$\left|2\gamma-\int_{\zeta_{in}}^{\zeta_{out}}(n_1-n_2)\,d\zeta\right|\approx\left|\int_{\eta_{in}}^{\eta_{out}}d\eta/G(\eta)\right.$$

$$\left.-\sin\left[\int_{\zeta_{in}}^{\zeta_{out}}(n_1-n_2)\,d\zeta\right]\Big/2G(\eta_{in})G(\eta_{out})\right|\ll 1. \qquad (2.12)$$

In the other limiting case $G(\eta)=0$ (when $n_1=n_2$ but η changes along the beam), it follows from (2.1) that

$$P(\eta_{out})=-i\,\mathrm{tg}\,\{\eta_{out}-\eta_{in}+\mathrm{arctg}[iP(\eta_{in})]\}. \qquad (2.13)$$

Then the transformation matrix takes the form

$$\hat{F}=\exp\left(-i\int_{\zeta_{in}}^{\zeta_{out}}n\,d\zeta\right)\begin{pmatrix}\cos\Delta\eta & \sin\Delta\eta\\ -\sin\Delta\eta & \cos\Delta\eta\end{pmatrix}, \qquad (2.14)$$

where

$$\Delta\eta=\eta_{out}-\eta_{in}={}^1/_2(\mathrm{arctg}\,q_{out}-\mathrm{arctg}\,q_{in}). \qquad (2.15)$$

According to (2.14) the phase is $\varphi=0$ or π, the phase shift is $\gamma=0$, and the transformation coefficient is

$$Q=\sin^2\Delta\eta. \qquad (2.16)$$

It is clear from (2.5) and (2.13) that in the case under consideration the polarization ellipse of the total radiation does not vary along the beam: $K=\text{const}$. Therefore the matrix $\hat{F}$ (2.14) effects a simple transformation of two resolutions of this polarization ellipse: from a resolution into two orthogonal ellipses of the polarization of geometrical-optics waves at the input to the layer to a resolution into two orthogonal polarization ellipses of the polarization at the exit from the layer.

If the change of η in the plasma layer is appreciable ($|\Delta\eta|\sim1$, i.e., if the polarization of the geometrical-optics waves changes appreciably) the variant $G(\eta)=0$ corresponds to a strong wave interaction. The transformation coefficient is $Q=\frac{1}{2}$ if one transition from the quasitransverse propagation to quasilongitudinal propagation or vice versa takes place over the thickness of the plasma layer ($\Delta\eta=\pm\pi/4$). The quantity Q becomes equal to unity when $\Delta\eta=\pm\pi/2$, i.e., in the case of a double interchange of the character of the propagation in the plasma layer (for example, transition from quasitransverse to quasilongitudinal and then back to quasitransverse propagation).[8]

Formula (2.16) was obtained for a layer in which $G(\eta)=0$. Using the substitution $\tilde{P}=(1-P)/(1+P)$, which transforms Eq. (2.1) with the function $G(\eta)$ into an analogous Ricatti equation with a function $\tilde{G}(\eta)=G^{-1}(\eta)$, and using a solution of the type (2.10), we can easily write down the solution of Eq. (2.1) for the case when the function $G(\eta)$ remains small along the beam: $|G(\eta)|\ll1$. This solution introduces small corrections in the value of Q determined by formula (2.16). The phases φ and γ are also slightly altered compared with the variant with $G(\eta)=0$, by an amount of the order of $|G(\eta)|\ll1$.

It is clear from (2.16) that a small change of η along the beam ($|\Delta\eta|\ll1$) leads to the appearance of only a weak interaction: $Q\ll1$. The foregoing is valid not only in the case $|G(\eta)|\ll1$, but also for plasma layers with a function[2)] $|G(\eta)|\lesssim1$. [In accord with the foregoing, $Q\ll1$ also in the case $|G(\eta)|\gg1$, but now already at all increments $\Delta\eta$ along the beam.]

Summarizing the foregoing, we can state that an effective interaction with the transformation coefficient $Q\sim1$ can be realized only in layers in which the characteristic function $G(\eta)$ decreases to values $|G(\eta)|\lesssim1$ in the interval $|\Delta\eta|\sim1$.

Under the conditions of a cosmic plasma, the radio waves usually travel along routes that contain sections with quasitransverse ($q^2\gg1$) and quasilongitudinal ($q^2\ll1$) propagation and the transitions between them. On the route that includes one transition, the increment is $\Delta\eta=\pm\pi/4$; on a route containing two transitions the increment is $\Delta\eta=\pm\pi/2$. In these cases, as is clear from the foregoing, the interaction is weak ($Q\ll1$) if $|G(\eta)|\gg1$, along the entire beam. The interaction remains weak if the function $G(\eta)$ decreases to values $|G(\eta)|\lesssim1$ only in a small interval $|\Delta\eta|\ll1$, and retains large absolute values in the remainder of the route. However, the interaction becomes effective ($Q\sim1$) if $|G(\eta)|\lesssim1$ in a large interval $|\Delta\eta|\sim1$. Finally, the interaction is strong when $|G(\eta)|\ll1$ on the interval $|\Delta\eta|\sim1$; at $\Delta\eta=\pm\pi/4$ and $\Delta\eta=\pm\pi/2$ the transformation coefficient is close to its maximum values $\frac{1}{2}$ and 1, respectively. All these conclusions can be confirmed by analyzing the phase structure of the Ricatti equation (2.1).

It is clear therefore that in the case when the change of the function $G(\eta)$ in the interval $|\Delta\eta|\sim1$ is comparable with or less than the value of the function itself, the effectiveness of the interaction in the plasma layer can be characterized by the parameter $G=|G(\eta_0)|$. Here η_0 is a fixed value of the variable η from the interval $|\Delta\eta|\sim1$. As is clear from the definition (2.3) of η, the interval $|\Delta\eta|\sim1$ must include the region of values $q^2\sim1$ (since η changes little at $q^2\gg1$ and $q^2\ll1$). Therefore the interaction parameter can be defined by the expression[3)] (Ref. 1).

$$G=|G(\eta)|_{q^2\sim1}=\left|\frac{n_2-n_1}{2\Psi}\right|_{q^2\sim1}. \qquad (2.17)$$

We note that the value $q^2=1$, depending on the sign of q, corresponds to a value $\eta=3\pi/8$ or $\eta=\pi/8$ [see (2.3)].

This definition means the following. If the plasma contains a region with a substantial change of the polarization of the geometrical-optics waves $K_{1,2}$ (1.2) (i.e., $|\Delta\eta|\sim1$ along the beam), and if the relative change of the characteristic function $G(\eta)$ is smaller than or comparable with unity in this region, then the values of the interaction parameter make it possible to assess the effectiveness of the wave transformation:

$$\begin{aligned}&Q\ll1 && \text{if}\quad G\gg1,\\ &Q\sim1 && \text{if}\quad G\sim1,\\ &Q\approx Q_{max} && \text{if}\quad G\ll1.\end{aligned} \qquad (2.18)$$

We note further that according to (2.2)

$$2\int_{\eta_{in}}^{\eta_{out}}G(\eta)\,d\eta=\int_{\zeta_{in}}^{\zeta_{out}}(n_1-n_2)\,d\zeta.$$

It follows therefore that on the beam segment from

ζ_{in} to ζ_{out}, which coincides with the region of the effective interaction ($|\Delta\eta| \sim 1, |G(\eta)| \lesssim 1$), the absolute value of the integral

$$\int_{\zeta_{in}}^{\zeta_{out}} (n_1 - n_2)\, d\zeta$$

does not exceed unity. Since the condition $|\int (n_1 - n_2) d\zeta| \sim 1$ defines the spatial period of the beats between the geometrical-optics waves, it is clear that the effective interaction in the plasma is realized when a substantial change of the polarization of these waves takes place in the indicated period (i.e., when the period of the beats spans an interval $|\Delta\eta| \sim 1$). The interaction is connected with the fact that the polarization of the total radiation does not manage to follow the changes of the polarization of the geometrical-optics waves, so long as this change takes place over a distance shorter than or of the order of the indicated period of the beats.

In a magnetoactive plasma, the interaction parameter (2.17) is determined by the expression

$$G^2 = [2(n_1 - n_2) k_0 \Lambda_q]^2 |_{q^2 \sim 1} = \frac{8uv^2(1-v)\cos^2\alpha}{(1-v-u)(1-v-u+uv\cos^2\alpha)} \times \left\{ \frac{d}{d\zeta}\left[\frac{u^{1/2}\sin^2\alpha}{2(1-v)\cos\alpha} \right] \right\}^{-2} \Bigg|_{q^2\sim 1}. \quad (2.19)$$

The curly brackets contain here the quantity $k_0\Lambda_q$, where

$$\Lambda_q = \left| \frac{1}{q}\frac{dq}{dl} \right|^{-1}_{q^2\sim 1}$$

is the scale of variation of the quantity q along the beam in the region $q^2 \sim 1$. It is important that the efficiency of the transformation is determined only by the character of the variation of $q(\zeta)$, while the derivatives of the refractive indices $n_{1,2}$ or of any other combinations of the plasma parameters along the beam are in fact of no importance. According to (2.19),

$$G \sim k_0 v \Lambda_q, \quad (2.20)$$

If $v \ll 1$ and $\alpha \sim u \sim |1-u| \sim 1$. An investigation of the general expression for G shows that the interaction parameter decreases substantially compared with (2.20) if one of the following conditions is satisfied:

$$u^{1/2} \ll 1-v, \quad \cos^2\alpha \ll 1, \quad v < 1 \quad (2.21)$$

or

$$u \gg 1, \quad \sin^2\alpha \ll 1, \quad v \ll u^{1/2} \quad (2.22)$$

[the inequality $v < 1$ ensures transparency of the plasma, and $v \ll u^{1/2}$ ensures mutual proximity of the dispersion curves $n_{1,2}(\zeta)$].

If the scale Λ_q is determined principally by the change of the magnetic field along the beam,[4)] then we can put in (2.19) $u = u(\zeta)$, $v = \text{const}$, $\alpha = \text{const}$. In the region (2.21) the interaction parameter is then

$$G_0 = 2^{5/2} k_0 v (1-v)^{1/2} \cos^2\alpha \left| \frac{u^{1/2}}{du^{1/2}/dl} \right|_{q^2\sim 1}, \quad (2.23)$$

and in the region (2.22)

$$G_\infty = 2^{1/2} \frac{k_0 v}{|1-v|} \sin^2\alpha \left| \frac{u^{1/2}}{du^{1/2}/dl} \right|_{q^2\sim 1}. \quad (2.24)$$

In a tenuous plasma ($v \ll 1$) the parameter (2.23) corresponds to the previously introduced[1] interaction parameter G_0, which describes, in particular, wave transformation in a neutral current sheath of a plasma.[8] The interaction defined by the parameter G_∞ (2.24) has not been investigated before.

Let now the main contribution to the scale Λ_q be made by the change of the direction of the magnetic field. Putting in (2.19) $\alpha = \alpha(\zeta)$, $u = \text{const}$, $v = \text{const}$ we find that in the region (2.21) the interaction parameter takes the form

$$G_\perp = \frac{\sqrt{2} k_0 v u}{(1-v)^{1/2}} \left| \frac{\cos\alpha}{d\cos\alpha/dl} \right|_{q^2\sim 1}. \quad (2.25)$$

This is a known parameter that characterizes the transformation occurring when a wave passes through a weak transverse magnetic field (see Refs. 2, 3, and 15). This transformation was investigated in Refs. 5 and 6. Next, in the region (2.22) we have

$$G_\parallel = \frac{\sqrt{2} k_0 v}{u^{1/2}} \left| \frac{\sin\alpha}{d\sin\alpha/dl} \right|_{q^2\sim 1}. \quad (2.26)$$

The parameter (2.26) was obtained in a preceding paper.[1] To our knowledge, no study was made of the transformation of waves under conditions when this parameter is valid.

It is easily seen that in a sufficiently tenuous plasma ($v \ll 1$) all four interaction parameters (2.23)–(2.26) contain small quantities which are not contained in the expressions for the parameter (2.20). Because of this circumstance, in the cases (2.23)–(2.26) the interaction effect sets in at larger values of Λ_q or in a less tenuous plasma than in the cases when formula (2.20) is valid.

3. THE STANDARD PROBLEM

In the investigation of standard problems it is convenient to use the equation

$$d^2f/d\eta^2 + I(\eta) f = 0, \quad I(\eta) = 1 - i\, dG(\eta)/d\eta + G^2(\eta). \quad (3.1)$$

This equation can be obtained from the system (1.8) after eliminating the function f_1 and changing over to a new unknown

$$f = f_2 \exp\left(i \int n\, d\zeta \right)$$

and a new independent variable η (2.3). Exact solutions of (3.1), expressed in terms of well investigated functions, are known only for a limited number of characteristic functions $G(\eta)$ (see, e.g., Ref. 16). It must be noted, however, that each standard solution corresponding to a given form of the function $G(\eta)$ describes an entire class of plasma layers. This is clear from the definition (2.2) of $G(\eta)$, which imposes one differential constraint

$$\frac{d\eta}{d\zeta} = \frac{n_1(\zeta) - n_2(\zeta)}{2G(\zeta)} \quad (3.2)$$

on the choice of the functions $u(\zeta)$, $\alpha(\zeta)$, and $v(\zeta)$ that determine the variation of $n_{1,2}(\zeta)$ and $\eta(\zeta)$ along the beam.

Among the exact solutions of the linear interaction problem, the simplest is the case when $G(\eta)$ is constant, when the general solution of (2.1) can be expressed in terms of elementary functions:

$$P(\eta)=\frac{-1+\exp[2i(G^2+1)^{1/2}(\eta+C_0)]}{[G-(G^2+1)^{1/2}]-[G+(G^2+1)^{1/2}]\exp[2i(G^2+1)^{1/2}(\eta+C_0)]} \quad (3.3)$$

(the constant C_0 is determined by the value of P at the entrance of the layer). The transformation coefficient Q can be easily obtained from (3.3) with the aid of relation (2.5):

$$Q=\sin^2[(G^2+1)\Delta\eta]/(G^2+1). \quad (3.4)$$

Equation (3.4) goes over into (2.16) as $G\to 0$.

The dependence of the transformation coefficient on the parameters G and $\Delta\eta$ has an oscillating character, since the region of the interaction is not localized inside the layer that includes the entry to and the exit from the layer (the points q_{in} and q_{out}).

To illustrate the results of the qualitative analysis of the linear interaction, we consider in greater detail the standard problem with the function $G(\eta)$ in the form

$$G(\eta)=-\frac{p}{2^{3/2}}(q_{in}-q_{out})\frac{(1+\mathrm{ctg}^2 2\eta)^{1/2}}{(q_{in}+\mathrm{ctg}\,2\eta)(q_{out}+\mathrm{ctg}\,2\eta)} \quad (3.5)$$

where $q_{in}>q_{out}>0$. The role of the interaction parameter (2.17) in the case of a transition from the quasitransverse ($q_{in}\gg 1$) to the quasilongitudinal ($q_{out}\ll 1$) propagation is played here by the positive quantity $p=G(3\pi/8)=G(\eta)|_{q=1}$. At the entrance and exit, where

$$\eta_{in}=\pi/4+{}^1/_2\mathrm{arctg}\,q_{in},\quad \eta_{out}=\pi/4+{}^1/_2\mathrm{arctg}\,q_{out},$$

the function $G(\eta)$ assumes infinitely large positive values, and in the middle of the layer it reaches a minimum value $G_{min}>0$. In accordance with the qualitative picture of Sec. 2, we can state that the chosen form of $G(\eta)$ ensures validity of the geometrical-optics approximation on the edges of the interval, the interaction region $G(\eta)\lesssim 1$ can be located only in the central part of the interval $\Delta\eta=\eta_{out}-\eta_{in}$ (at $G_{min}\lesssim 1$). On the other hand, if $G_{min}\gg 1$, then geometrical optics is valid over the entire layer and there is no linear interaction.

According to (3.2), the characteristics of the considered layers with $G(\eta)$ in the form (3.5) should be connected by the relation

$$q(\sigma)=\frac{q_{in}-q_{out}}{1+\exp(2^{5/2}\sigma/p)}+q_{out},\quad \sigma=\int\frac{n_1-n_2}{2(q^2+1)^{1/2}}d\zeta. \quad (3.6)$$

The function $q(\sigma)$ recalls the Epstein transition layer.[17] However, the function $q(\zeta)$ represents the Epstein layer only under the condition that σ is proportional to ζ, i.e., under the condition $n_1-n_2\propto(q^2+1)^{1/2}$. From the form (1.1) of $n_{1,2}$ it is clear that the last condition is realized if the quantity $C/\varepsilon^{1/2}$ remains constant along the beam (see (1.3) and (3.6)].

An example of a concrete realization of this standard problem is interaction in a strong magnetic field $u=\mathrm{const}\gg 1$, $v=\mathrm{const}\ll u^{1/2}$, $\alpha(\zeta)\ll 1$, when the inhomogeneity of the plasma layer along the beam is ensured by variation of the direction of the magnetic field: the angle $\alpha(\zeta)$ decreases monotonically along the beam from α_{in} to α_{out}. The interaction parameter p coincides in this case with $G_{\parallel}$ [Eq. (2.26)]. In the other limiting cases discussed at the end of Sec. 2, the parameter p is identical with G_0, G_∞, and $G_\perp$.

The solution of Eq. (3.1) with the function $G(\eta)$ (3.5) reduces to hypergeometric functions.[17] Using their asymptotic representations, we can find the transformation matrix (1.16). We present here only the expression for the transformation coefficient

$$Q=\frac{1}{1+\chi K^2_{2out}},\quad \chi=\frac{\Delta_{11}\Delta_{12}}{\Delta_{21}\Delta_{22}}\frac{\mathrm{sh}(2^{-5/2}\pi p\Delta_{12})\,\mathrm{sh}(2^{-5/2}\pi p\Delta_{21})}{\mathrm{sh}(2^{-5/2}\pi p\Delta_{11})\,\mathrm{sh}(2^{-5/2}\pi p\Delta_{22})}, \quad (3.7)$$

where $\Delta_{ij}=K_{i\,out}-K_{j\,in}$ $(i,j=1,2)$. According to (3.7), effective transformation ($Q\sim 1$) occurs under the conditions $p\lesssim 1$ and $|\Delta\eta|\sim 1$. In particular, for a layer containing a transition from quasitransverse propagation ($q^2_{in}\to\infty$) to quasilongitudinal propagation ($q^2_{out}\to 0$), the transformation coefficient takes the simple form

$$Q=[1+\exp(2^{-3/2}\pi p)]^{-1}. \quad (3.8)$$

In this case Q is a function of only the interaction parameter p, and in full correspondence with the results of the qualitative analysis of Sec. 2 the quantity $Q\sim 1$ at $p\lesssim 1$ and $Q\ll 1$ for $p\gg 1$.

It should be noted that according to (3.8) we have $Q\to\frac{1}{2}$ if $p\to 0$. This result becomes perfectly understandable if it is recognized that at $p=0$ (i.e., $G=0$) the radiation polzrization is not changed by propagation in the plasma layer (see the discussion of the variant $G(\eta)=0$ in Sec. 2). If a linearly polarized wave of one type (say, ordinary) is incident on the entry to the layer (where $q^2_{in}\gg 1$), then at the exit from the layer (where $q^2_{out}\ll 1$) such a polarization corresponds to superposition of two circularly polarized waves of equal intensity—ordinary and extraordinary. In accordance with the definition (1.17), this leads to a value $Q=\frac{1}{2}$.

The correctness of the qualitative analysis of the linear interaction presented in the present article is confirmed by results of an analysis of concrete problems on the propagation of electromagnetic waves through a quasitransverse magnetic field[5,6,7] and through a neutral current sheath,[8] and also by analysis of a number of standrrd problems concerning linear interaction of the limiting polarization type, which were investigated previously.[10,18]

[1)]In the quasi-isotropic approximation, the refractive indices are calculated only accurate to quantities of order $(n_1-n_2)^2/n^2\ll 1$, therefore the assumption $n_1+n_2\approx 2\varepsilon^{1/2}$ is valid.

[2)]Over the small interval $|\Delta\eta|\ll 1$ it is convenient to seek the solution of (2.1) in the form of a series in $\Delta\eta$. As a result we get $Q\approx(\Delta\eta)^2\ll 1$. The interaction therefore turns out to be weak for quasilongitudinal propagation ($q^2\ll 1$), quasitransverse propagation ($q^2\gg 1$), and in a tenuous ($v\ll 1$) inhomogeneous plasma with a homogeneous magnetic field ($u=\mathrm{const}$, $\alpha=\mathrm{const}$). In all these cases, as is clear from (1.4) and (2.3), the increment $\Delta\eta$ along the beam is small.

[3)]We note that in his article, Cohen[3] used the term interaction parameter for the function $G(\eta)$ rather than for the quantity G (2.17) introduced here and previously.[1,2]

[4)]The corresponding conditions for this can be easily found by turning to the definition given above for Λ_q in terms of the derivative of q (1.4) (see Ref. 1 in this connection).

[1]V. V. Zheleznyakov, Élektromagnitnye volny v komicheskoĭ plazme (Electromagnetic Waves in Cosmic Plasma), Nauka,

1977, Sec. 7.
[2]V. V. Zheleznyakov, Radioizluchenie Solntsa i planet (Radio Emission from the Sun and Planets), Nauka, 1964, Chap. VI.
[3]M. H. Cohen, Astrophys. J. **131**, 666 (1960).
[4]V. V. Zheleznyakov and E. Ya. Zlotnik, Izv. Vyssh. Ucheb. Zaved. Radiofizika **20**, 1444 (1977).
[5]V. V. Zheleznyakov, E. Ya. Zlotnik, Astron. Zh. **40**, 633 (1966) [*Sic!*].
[6]Yu. A. Kravtsov and O. N. Naĭda, Zh. Eksp. Teor. Fiz. **71**, 237 (1977) [Sov. Phys. JETP **44**, 122 (1977)].
[7]O. N. Naĭda, Radiotekhnika i élektronika **23**, 2489 (1978).
[8]V. V. Zheleznyakov, Zh. Eksp. Teor. Fiz. **73**, 560 (1977) [Sov. Phys. JETP **46**, 292 (1977)].
[9]V. V. Kocharovskiĭ and Vl. V. Kocharovskiĭ, Fiz. Plazmy 5, (1979) [Sov. J. Plasma Phys. **5**, (1979)].
[10]V. V. Kocharovskiĭ and Vl. Kocharovskiĭ, in: Proceedings of 11-Union Science Student Conference on Students and Scientific and Technical Progress, Novosibirsk State Univ., 1978, p. 78.
[11]V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in Plasma), Nauka, 1967, Secs. 10 and 11 [Pergamon].
[12]Yu. A. Kravtsov, Dokl. Akad. Nauk SSSR **183**, 74 (1968) [Sov. Phys. Dokl. **13**, 1125 (1969)].
[13]K. G. Budden, Radio Waves in the Ionosphere, Cambridge Univ. Press, 1961.
[14]F. T. Cheng and P. C. Fung, Astrophys. Space Sci. **49**, 367 (1977); **50**, 361 (1977).
[15]D. B. Melrose, Aust. J. Phys. **27**, 31, 43 (1974).
[16]E. Kamke, Differentialgleichungen, Vol. 1, Chelsea, 1974.
[17]L. M. Brekhovskikh, Volny v sloistykh sredakh (Waves in Layered Media), Nauka, 1973, Sec. 20 [Academic].
[18]N. G. Denisov, Izv. Vyssh. Ucheb. Zaved. **21**, 921 (1978); Preprint No. 118, Gor'kiĭ Radiophysics Inst., 1978.

Translated by J. G. Adashko

A soliton system subject to perturbation. Oscillatory shock waves.

V. I. Karpman

Institute for Earth Magnetism, the Ionosphere, and Radiowave Propagation, Academy of Sciences of the USSR
(Submitted 11 January 1979)
Zh. Eksp. Teor. Fiz. **77**, 114–123 (July 1979)

We study a system of solitons of the perturbed Korteweg–de Vries equation with nearly equal amplitudes. We show that in that case there may exist quasistationary systems with a large number of solitons for well-defined relations between the amplitudes. Such systems become stationary when there is a piston which compensates for the damping of the solitons and their mutual repulsion. Using such an approach we give a detailed description of the soliton structure of oscillatory shock waves.

PACS numbers: 02.30.Jr

1. In the present paper we study effects which occur when a permanent perturbation acts on a system of solitons, and we consider from that point of view the structure of shock waves in weakly dispersive media. To fix our ideas we consider here waves which are described by the perturbed Korteweg–de Vries (KdV) equation

$$u_t-6uu_x+u_{xxx}=\varepsilon R[u],\tag{1}$$

where R is a (generally speaking non-linear) operator acting upon the function $u(x,t)$. The general approach discussed here and several of the results turn out to be valid also in a number of other cases.

The evolution of a single perturbed KdV soliton is described by the equations[2-4]

$$\begin{gathered}u(x,t)=u_s(z,\varkappa(t))+\delta u(x,t),\\ u_s(z,\varkappa)=-2\varkappa^2\operatorname{sech}^2 z,\qquad z=\varkappa[x-\xi(t)],\end{gathered}\tag{2}$$

where

$$\frac{d\varkappa}{dt}=-\frac{\varepsilon}{4\varkappa}\int_{-\infty}^{\infty}R[u_s]\operatorname{sech}^2 z\,dz,\tag{3}$$

$$\frac{d\xi}{dt}=4\varkappa^2-\frac{\varepsilon}{4\varkappa^3}\int_{-\infty}^{\infty}R[u_s]\,(z\operatorname{sech}^2 z+\operatorname{th} z+\operatorname{th}^2 z)\,dz.\tag{4}$$

As far as the deformation $\delta u(x,t)$ is concerned we shall discuss here only its "tail" part which is described by the expressions[2,3]

$$\delta u_-=\lim_{x\to-\infty}\delta u=\varkappa^2\varepsilon q,$$

$$q=\frac{1}{4\varkappa^5}\int_{-\infty}^{\infty}R[u_s]\operatorname{th}^2 z\,dz.\tag{5}$$

Indeed, $\delta u(x,t)$ is transformed into a flat tail already at a few soliton lengths behind the soliton, and this can also be seen from numerical solutions[5] obtained for $R=\partial^2/\partial x^2$).

The characteristic time scale, defined by the perturbation, is[1] $t_p=t_s/\varepsilon q$, where $t_s=(2\varkappa)^{-3}$ is the characteristic time connected with the unperturbed soliton. If, therefore, there are two solitons with greatly different amplitudes ($\delta\varkappa=\varkappa_2-\varkappa_1\sim\varkappa_{1,2}$), the time it takes the larger soliton to pass through the smaller one is of the order of t_s. As $t_s\ll t_p$, the interaction of the solitons does not appreciably interfere with the effects of the perturbation.

However, this interference may turn out to be important if the solitons have almost the same amplitudes, i.e., $\varkappa_1\approx\varkappa_2\gg|\delta\varkappa|$. We shall therefore consider

