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The problem of comparing the observation results with the theoretical data to check on relativistic effects in the solar system

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To develop a noncontradictory relativistic theory of the motion of celestial bodies, the dynamic effects must be determined (from the equations of motion of the bodies) and the measured quantities calculated (using the light-propagation equations) in a single coordinate frame. The choice of the frame itself is arbitrary. By way of illustration we consider complanar circular motions of the earth and of an inner planet in the sun's graviational field; this motion is described by an arbitrary parametrized metric in arbitrary quasi-Galilean coordinates. In radar observations, the measured quantitites are the signal-propagation time intervals, and in angle measurements these are the angles between the directions to the planet and to a remote immobile source (quasar) or between the directions to the planet and to the sun. These quantities can be calculated as functions of only the measured initial values of any form, irrespective of the employed coordinate frame. However, the relativistic corrections to the values of the physically measured quantities depend on the makeup of the initial measurements. Direct measurements of the angles uncover new possibilities of checking relativistic theories of gravitation.

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1. INTRODUCTION

The analysis of relativistic effects in the motion of celestial bodies, of light rays, and of radio waves in the solar system calls for the use of the metric of the gravitational field in the post-Newtonian approximation. Small perturbations of the metric are solutions of the field equation of a concrete gravitational theory and can be obtained by successive approximations accurate to four arbitrary functions, which determine the choice of some coordinate frame (coordinate conditions). It is obvious that this choice is quite arbitrary and in problems of relativistic celestial mechanics it can be restricted only by the requirement that the coordinate system be quasi-Galilean. The relativistic equations of the motion of celestial bodies, are expressed in the form of the equations of ordinary dynamics and their solutions therefore take different forms in different coordinate frames. As a result, in the general case the relativistic corrections to the Newtonian motion of celestial bodies (such as, e.g., the corrections to rectangular coordinates, semi-axes, etc.) likewise become coordinate-dependent and can therefore not be directly compared with the observation results. A qualitative way out of this situation becomes clear if one recalls that the dynamic theory is compared with the observations with the aid

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of light rays or radio waves, whose equations of motions are also coordinate-dependent. To construct a noncontradictory relativistic theory of motion of celestial bodies it is therefore necessary to calculate the measured quantities (with the aid of the equations of motion of light) in the coordinate frame that was used to derive the dynamic effects on the basis of the equations of motion of the bodies. In the course of this procedure, the coordinate-dependent terms should be eliminated from the physically measured effects.

This paper, which is an extension of Ref. 1 and of a paper delivered by us at the Interdepartmental Seminar on Gravitational Effects in General Relativity Theory (Minsk, September 1978), we examine the realization of such a procedure using as examples the two observation methods most widely used in astronomy, viz., radar (Sec. 3) and differential angle measurements e.g., with a radio-interferometer of ultralong base (Sec. 4). The high potential accuracy of these methods (of the order of 5×10^{-9} or 0.001″) makes the task of developing an invariant procedure for comparing calculated and measurement quantities important not only theoretically but also from the pure observational viewpoint, since the coordinate-dependent terms are of the same order as the dynamic effects. It must be noted that the published opinions on this subject are quite contradictory, including even naive attempts to separate a privileged coordinate system that "corresponds" to the astronomical observation (see, e.g., Ref. 2).

Coordinate-independent expressions pertaining to the considered types of measurements and expressed in terms of observable quantities make it possible to calculate on the basis of some assembly of initial measurements, for any instant of time t, the relativistic corrections to the time $T(t)$ of propagation of the radar signal for sounding an inner planet, and the corrections to the relative angle distances $\varphi(t)$ and $\psi(t)$ between the planet and the quasar respectively and between the planet and the sun. It is important to note that the magnitude of these relativistic effects depends substantially on the makeup of the initial measurements, in other words, on the concrete observation procedure.

2. GENERALIZED SCHWARZSCHILD METRIC

To investigate the problem of the coordinate conditions in the general case, we consider the generalized metric of a static spherically symmetrical field

$$ds^2=\left\{1-\frac{2m}{r}+2(A+\alpha(r))\left(\frac{m}{r}\right)^2\right\}(dx^0)^2 + 2\frac{m}{c}\frac{x^i}{r}\beta'(r)\,dx^0\,dx^i - \left\{\delta_{ij}+\frac{2m}{r}\left[\alpha(r)\delta_{ij}-\frac{x^ix^j}{r^2}(\alpha(r)-B-r\alpha'(r))\right]\right\}dx^i\,dx^j. \quad (1)$$

The constants A and B are determined here by gravitation theory [$A=0$ and $B=1$ for general relativity theory (GRT)], and the arbitrary functions $\beta(r)$ and $\alpha(r)$ specify the coordinate frame. In GRT, in particular, the values $\beta=0$ and $\alpha=0$ correspond to the standard Schwarzschild coordinates, $\beta=0$ and $\alpha=+1$ correspond to the harmonic (or isotropic) coordinates, and $\beta=0$ and $\alpha=-1$ correspond to the Painleve coordinates. The known Eddington-Robertson two-parameter (β,γ) metric follows from (1) at $\beta(r)=0$, $\alpha(r)=\gamma$, $A=B-\gamma$, $B=\gamma$. It is readily seen that the metric (1) is connected with the metric expressed in the standard coordinates $\bar{t}$ and $\bar{x}^i$ by the transformation

$$ct=c\bar{t}-\frac{m}{c}\beta(\bar{r}),\quad x^i=\bar{x}^i-m\frac{\bar{x}^i}{\bar{r}}\alpha(\bar{r}), \quad (2)$$

which preserves the quasi-Galilean character of the metric under the conditions

$$\beta'(r)\to0,\quad \alpha'(r)\to0,\quad \alpha(r)/r\to0,\quad r\to\infty. \quad (3)$$

In (1) and (2), $m=fM_\odot/c^2$ and the sun's gravitational parameter $fM_\odot$ and the speed of light c are assumed known.

For the metric (1), the Lagrangian of a nonrelativistic particle does not depend on $\beta(r)$ and takes the form

$$L_P=\frac{1}{2}\dot{\mathbf{r}}^2+\frac{fM_\odot}{r}+\frac{1}{8c^2}\dot{\mathbf{r}}^4+\frac{m}{2r}\left[(1-2A-2\alpha(r))\frac{fM_\odot}{r}+(2\alpha(r)+1)\dot{\mathbf{r}}^2 +2(B-\alpha(r)+r\alpha'(r))\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\right]. \quad (4)$$

Using (4), we can easily find that a planet in the field (1) moves in a circular orbit of radius $r=a$ with an angular velocity

$$n=\left(\frac{fM_\odot}{a^3}\right)^{1/2}\left\{1-\frac{m}{a}\left(A+\frac{3}{2}\alpha(a)\right)\right\} \quad (5)$$

and the sidereal period of the motion is $2\pi/n$.

The Lagrangian for the light in the field (1) is likewise independent of $\beta(r)$ and takes the form

$$L_R=\frac{1}{2}\dot{\mathbf{r}}^2+\frac{m}{r}\left[(\alpha(r)+1)\dot{\mathbf{r}}^2+(B-\alpha(r)+r\alpha'(r))\frac{(\mathbf{r}\dot{\mathbf{r}})^2}{r^2}\right]. \quad (6)$$

If at the instant $t=t_0$ the optical particle has coordinates $\mathbf{r}(t_0)=\mathbf{r}_0$ and if the direction of the light ray at $t=\infty$ and infinitely far from the sun is determined by a single vector $\boldsymbol{\nu}$, such that $\dot{\mathbf{r}}(\infty)=c\boldsymbol{\nu}$, and $\nu^2=1$, then the solution of the equations corresponding to the Lagrangian (6) is

$$\mathbf{r}(t)=\mathbf{r}_0+c(t-t_0)\mathbf{v}+m\left\{B\left(\frac{\mathbf{r}}{r}-\frac{\mathbf{r}_0}{r_0}\right)+\left(\alpha(r_0)\frac{\mathbf{r}_0}{r_0}-\alpha(r)\frac{\mathbf{r}}{r}\right) +(B+1)(r_0-r+c(t-t_0))\frac{[\mathbf{v}[\mathbf{r}_0\times\mathbf{v}]]}{[\mathbf{r}_0\times\mathbf{v}]^2}-(B+1)\mathbf{v}\ln\frac{r+\mathbf{r}_0\mathbf{v}+c(t-t_0)}{r_0+\mathbf{r}_0\mathbf{v}}\right\}, \quad (7)$$

$$\frac{1}{c}\dot{\mathbf{r}}=\mathbf{v}+m\left\{(\alpha(r)-B)\frac{[\mathbf{r}\times[\mathbf{r}_0\times\mathbf{v}]]}{r^3}-\alpha'(r)\frac{\mathbf{r}\mathbf{v}}{r^2}\mathbf{r}-\frac{B+1}{r}\mathbf{v} +\frac{B+1}{r}[r-\mathbf{r}_0\mathbf{v}-c(t-t_0)]\frac{[\mathbf{v}\times[\mathbf{r}_0\times\mathbf{v}]]}{[\mathbf{r}_0\times\mathbf{v}]^2}\right\}. \quad (8)$$

In the relativistic terms we must use here, naturally, the Newtonian value of $\mathbf{r}$ obtained from (7) at $m=0$.

3. RADAR MEASUREMENTS

We consider a model problem of complanar circular motion of the earth and of an inner planet (Mercury or Venus) and suggest a procedure of radar measurements in the framework of this problem. It suffices for this purpose to duplicate the procedure developed by Shapiro[3] for the Eddington-Robertson metric.

An observer on earth E sends at the instant t a radio signal to the planet P and measures, at the instant when the signal returns to the earth, the time of signal propagation $T(t)$ at the two ends. This interval is minimal at each conjunction. Therefore the instants of the suceeding inferior conjunction can always be recorded, and it is possible to measure the synoidic period T_s of the planet P and the earth. By the same token this yields the difference between the mean motions of the planet and the earth:

$$n_P - n_E = 2\pi / T_s. \tag{9}$$

1. Initial measurements: T_s, T_0, T_1

Assume that the time intervals $T_0 = T(0)$ and $T_1 = T(\frac{1}{4}T_s)$ have been measured, with the initial instant $t = 0$ taken to be the instant of the inferior conjunction. The three quantities T_0, T_1, and T_s make it possible to determine completely the motion in the model problem and calculate the interval $T(t)$ for any instant of time t.

Introducing the vector

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0, \tag{10}$$

we get

$$T(t) = \frac{2}{c} R(t) + \frac{2m}{c} \left\{ (B+1) \ln \frac{a_E + a_P + R(t)}{a_E + a_P - R(t)} - \frac{1}{2} B \frac{(a_E + a_P)[R^2 - (a_E - a_P)^2]}{a_E a_P R(t)} - \frac{1}{2R} \left[\alpha(a_P) \frac{a_E^2 - a_P^2 - R^2}{a_P} - \alpha(a_E) \frac{a_E^2 - a_P^2 + R^2}{a_E} \right] \right\}, \tag{11}$$

with

$$R(t) = \left(a_E^2 + a_P^2 - 2 a_E a_P \cos \frac{2\pi t}{T_s} \right)^{1/2}. \tag{12}$$

In the derivation of (11) we have neglected the motions of the earth and of the planet P during the radar sounding time. It was also assumed that the times of light propagation from E to P and from P to E are equal. These simplifications suffice for our purposes. A rigorous iteration solution for the GRT harmonic coordinates, which can be readily extended to the general case, is given in Tausner's paper.[4]

By assumption, the measured quantities are $T_0 = T(0)$ and $T_1 = T(\frac{1}{4}T_s)$, from which we can determine the radii a_E and a_P of the earth's and planet's circular orbits in the form

$$a_E = a_E^N + \Delta a_E, \quad a_P = a_P^N + \Delta a_P, \tag{13}$$

where

$$a_P^N = {}^1/_4 c[(2T_1^2 - T_0^2)^{1/2} - T_0], \quad a_E^N = {}^1/_4 c[(2T_1^2 - T_0^2)^{1/2} + T_0], \tag{14}$$

$$\Delta a_P = m \left[B - (B+1) \frac{(a_E^2 + a_P^2)^{1/2}}{a_E + a_P} \ln \frac{a_E + a_P + (a_E^2 + a_P^2)^{1/2}}{a_E + a_P - (a_E^2 + a_P^2)^{1/2}} + (B+1) \frac{a_E}{a_E + a_P} \ln \frac{a_E}{a_P} - \alpha(a_P) \right], \tag{15}$$

$$\Delta a_E = m \left[B - (B+1) \frac{(a_E^2 + a_P^2)^{1/2}}{a_E + a_P} \ln \frac{a_E + a_P + (a_E^2 + a_P^2)^{1/2}}{a_E + a_P - (a_E^2 + a_P^2)^{1/2}} - (B+1) \frac{a_P}{a_E + a_P} \ln \frac{a_E}{a_P} - \alpha(a_E) \right]. \tag{16}$$

The quantities a_P and a_E in the right-hand sides of (15) and (16) should be taken to mean their Newtonian values (14).

We now express the interval $T(t)$ at arbitrary t only in terms of measurable quantities, i.e., in terms of T_0, T_1, and T_s. We transform expression (12) for the relative distance between the planets in the form

$$R(t) = R^N(t) + \Delta R, \tag{17}$$

where

$$R^N(t) = \left[(a_E^N)^2 + (a_P^N)^2 - 2 a_E^N a_P^N \cos \frac{2\pi t}{T_s} \right]^{1/2}, \tag{18}$$

$$\Delta R = \frac{m}{R} \left\{ \left(\cos \frac{2\pi t}{T_s} - 1 \right) \left[(B+1)(a_E^2 + a_P^2)^{1/2} \times \ln \frac{a_E + a_P + (a_E^2 + a_P^2)^{1/2}}{a_E + a_P - (a_E^2 + a_P^2)^{1/2}} - B(a_E + a_P) \right] + (B+1)(a_P - a_E) \ln \frac{a_E}{a_P} \cos \frac{2\pi t}{T_s} + [a_P \alpha(a_E) + a_E \alpha(a_P)] \cos \frac{2\pi t}{T_s} - a_E \alpha(a_E) - a_P \alpha(a_P) \right\}. \tag{19}$$

Substituting this expression in (11) we easily verify that the coordinate-dependent expressions cancel each other and we get

$$T(t) = T^N(t) + \Delta T, \tag{20}$$

$$T^N(t) = \frac{2}{c} R^N(t),$$

$$\Delta T = \frac{2m}{c} (B+1) \left[\ln \frac{a_E + a_P + R}{a_E + a_P - R} + \cos \frac{2\pi t}{T_s} \frac{a_P - a_E}{R} \ln \frac{a_E}{a_P} \right. \tag{21}$$

$$\left. + \left(\cos \frac{2\pi t}{T_s} - 1 \right) \frac{(a_E^2 + a_P^2)^{1/2}}{R} \ln \frac{a_E + a_P + (a_E^2 + a_P^2)^{1/2}}{a_E + a_P - (a_E^2 + a_P^2)^{1/2}} \right]. \tag{22}$$

Thus, the signal propagation time $T(t)$ is expressed, independently of the employed coordinate frame, in terms of the measurable synodic period T_s and the values $T(t)$ for two initial instants of time (the choice of $t = 0$ and $t = T_s/4$ is of no fundamental significance and does not alter the final result). The values of $T(t)$ are the ephemeridal data for the radar measurements.

The measured quantities T_0 and T_1 enable us to determine the average motions n_P and n_E, and by the same token also the sidereal periods of the planets. In accord with (5) we have

$$n = n^N + \Delta n, \tag{23}$$

$$n^N = [fM_\odot / (a^N)^3]^{1/2}, \tag{24}$$

$$\Delta n = n \left\{ -\frac{3}{2} \frac{\Delta a}{a} - \frac{m}{a} \left(A + \frac{3}{2} \alpha(a) \right) \right\}. \tag{25}$$

Using (15) and (16), we get

$$\frac{\Delta n_P}{n_P} = \frac{3}{2} \frac{m}{a_P} \left\{ (B+1) \frac{(a_E^2 + a_P^2)^{1/2}}{a_E + a_P} \ln \frac{a_E + a_P + (a_E^2 + a_P^2)^{1/2}}{a_E + a_P - (a_E^2 + a_P^2)^{1/2}} - \left(\frac{2}{3} A + B \right) - (B+1) \frac{a_E}{a_E + a_P} \ln \frac{a_E}{a_P} \right\}, \tag{26}$$

$$\frac{\Delta n_E}{n_E} = \frac{3}{2} \frac{m}{a_E} \left\{ (B+1) \frac{(a_E^2 + a_P^2)^{1/2}}{a_E + a_P} \ln \frac{a_E + a_P + (a_E^2 + a_P^2)^{1/2}}{a_E + a_P - (a_E^2 + a_P^2)^{1/2}} - \left(\frac{2}{3} A + B \right) + (B+1) \frac{a_P}{a_E + a_P} \ln \frac{a_E}{a_P} \right\}. \tag{27}$$

We recall once more that a_E and a_P should be taken to be those given in (14). Thus, regardless of the employed coordinate system, the sidereal periods are expressed as functions of only the measured quantities.

2. Initial measurements: T_s, T_E

Assume now that in place of the initial measurements of T_0 and T_1 we know from optical measurements, besides the synodic period T_s, also the earth's sidereal period T_E. By the same token we know also the sidereal period T_P of the planet. The major semi-axes are then determined from (13), but with the values

$$a_P^N = (fM_\odot / n_P^2)^{1/3}, \quad a_E^N = (fM_\odot / n_E^2)^{1/3}, \tag{28}$$

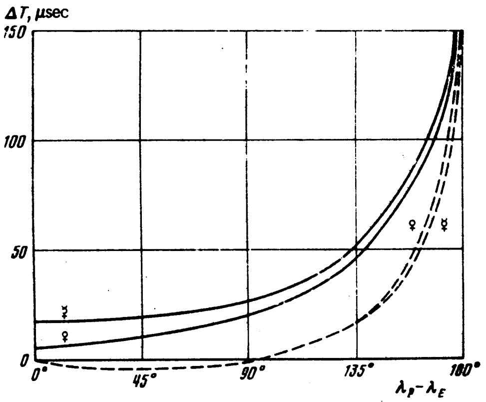

FIG. 1. The Shapiro effect for inner planets. Initial measurements: synodic period T_s and earth's sidereal period T_E—solid lines: T_s and radar data at the instants $t=0$ (inferior conjunction) and $t=T_s/4$—dashed curves.

$$\Delta a_{P,E}=-({}^2/_3A+\alpha(a_{P,E}))m. \tag{29}$$

It is now necessary to use the values (28) in (18), and in lieu of (19) we have

$$\Delta R=\frac{m}{R}\left\{\frac{2}{3}A(a_E+a_P)\left(\cos\frac{2\pi t}{T_s}-1\right)+[a_E\alpha(a_P)+a_P\alpha(a_E)]\cos\frac{2\pi t}{T_s}-a_E\alpha(a_E)-a_P\alpha(a_P)\right\}. \tag{30}$$

The Newtonian value (21) is now calculated from (18) and (28), and the relativistic effect ΔT is

$$\Delta T=\frac{2m}{c}\left\{\frac{2}{3}A(a_E+a_P)\left(\cos\frac{2\pi t}{T_s}-1\right)\frac{1}{R}+(B+1)\ln\frac{a_E+a_P+R}{a_E+a_P-R}+\frac{1}{2}B\frac{(a_E+a_P)(a_E-a_P+R)(a_E-a_P-R)}{a_Ea_PR}\right\}. \tag{31}$$

Comparison of (22) and (31) shows that, depending on the type of the initial measurements, the relativistic effect ΔT has a different structure and a different numerical value. The numerical values of the corrections (22) and (31) in GRT ($A=0, B=1$) for Mercury (☿) and Venus (♀) are shown in Fig. 1. It is interesting that, depending on the makeup of the initial measurements, radar measurements yield either only the constant B of the linear theory, or additionally the constant A of the post-Newtonian approximation of the relativistic theory. Thus, the dependence of the relativistic effect on the type of the initial measurements is not formal and makes it possible to plan the measurements in a way as to make them highly sensitive to different parameters of gravitation theory.

4. MEASUREMENT OF THE RELATIVE ANGLE DISTANCES

Within the framework of our problem, we adopt the following idealized procedure of measuring the relative angle distances between objects. The observer on earth fixes the instants of two successive conjunctions and determines thus the synodic period T_s. He chooses next some remote immobile source (quasar), from which he determines the sidereal period T_E of the earth or, equivalently, the average motion n_E. By virtue of (9), the average motion n_P can also be assumed known. Next, at the initial instant $t=0$, which for mathematical convenience is best chosen to be the instant of the inferior conjunction, the observer measures the geocentric angle $\varphi(0)=\varphi_0$ between the directions to the quasar and to the planet P. From the measured values of n_P, n_E, and φ_0 he must find the angle $\varphi=\varphi(t)$ between the directions to the quasar and to the planet P at any instant of time t.

Proceeding to realize this procedure, we turn to formula (8) for the velocity vector of the optical particle. At $t=-\infty$, at an infinite distance from the sun, the direction of the light is characterized by a single vector $\boldsymbol{\sigma}$, so that $\dot{\mathbf{r}}(-\infty)=c\boldsymbol{\sigma}$, and $\boldsymbol{\sigma}^2=1$. From (8) we get

$$\boldsymbol{\sigma}=\mathbf{v}+2m(B+1)\frac{[\mathbf{v}\times[\mathbf{r}_0\times\mathbf{v}]]}{[\mathbf{r}_0\times\mathbf{v}]^2}. \tag{32}$$

This expression, leads, incidentally, to the classical formula for the bending of alight ray in the sun's field. It is patently evident that this effect does not depend on the coordinate conditions.

Substituting (32) in (8), we obtain the velocity at the point $\mathbf{r}(t)$ of an optical particle emitted at $t=-\infty$ in the direction of $\boldsymbol{\sigma}$:

$$\frac{1}{c}\dot{\mathbf{r}}=\boldsymbol{\sigma}+\frac{m}{r}\left\{(\alpha(r)-B)\frac{[\mathbf{r}\times[\mathbf{r}\times\boldsymbol{\sigma}]]}{r^2}-(B+1)\left(\boldsymbol{\sigma}+\frac{[\boldsymbol{\sigma}\times[\mathbf{r}\times\boldsymbol{\sigma}]]}{r-\mathbf{r}\boldsymbol{\sigma}}\right)-\alpha'(r)\frac{\mathbf{r}\boldsymbol{\sigma}}{r^2}\mathbf{r}\right\}. \tag{33}$$

We consider now another optical particle that occupies at the instant t_0 the position $\mathbf{r}_0$ and arrives at the instant t at the point $\mathbf{r}(t)$. We obtain for the trajectory of this particle

$$\boldsymbol{\sigma}=\frac{\mathbf{R}}{R}+\frac{m}{R}\left\{(B+1)\frac{r-r_0+R}{[\mathbf{r}_0\times\mathbf{r}]^2}+\frac{r_0\alpha(r)-r\alpha(r_0)-B(r_0-r)}{R^2rr_0}\right\}[\mathbf{R}\times[\mathbf{r}_0\times\mathbf{r}]], \tag{34}$$

and the velocity of this optical particle at the point $\mathbf{r}(t)$ is given by

$$\frac{1}{c}\dot{\mathbf{r}}(t)=\frac{\mathbf{R}}{R}+\frac{m}{r}\times\left\{\left(\frac{r_0\alpha(r)-r\alpha(r_0)-B(r_0-r)}{R^3r_0}\mathbf{R}+\frac{\alpha(r)-B}{Rr^2}\mathbf{r}\right)[\mathbf{r}_0\times\mathbf{r}]-\frac{B+1}{R}\left(\mathbf{R}+\frac{[\mathbf{R}\times[\mathbf{r}_0\times\mathbf{r}]]}{rr_0+\mathbf{r}\mathbf{r}_0}\right)-\alpha'(r)\frac{\mathbf{r}\mathbf{R}}{rR}\mathbf{r}\right\}. \tag{35}$$

We obtain now the cosine of the angle φ between these two light rays at the point $\mathbf{r}(t)$. The velocity vector of the first ray, determined from (33), will be designated $\boldsymbol{\xi}$, and that of the second ray, determined from (35), by $\boldsymbol{\eta}$. Then

$$\cos\varphi=\frac{(\boldsymbol{\xi}\boldsymbol{\eta})_{rel}}{|\boldsymbol{\xi}|_{rel}|\boldsymbol{\eta}|_{rel}}, \tag{36}$$

where the subscript rel indicates that the scalar product of the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and their lengths must be calculated with the aid of the three-dimensional chronometrically invariant tensor of the metric (1):

$$\gamma_{ij}=\delta_{ij}+\frac{2m}{r}\left\{\alpha(r)\delta_{ij}-\frac{x^ix^j}{r^2}(\alpha(r)-B-r\alpha'(r))\right\}. \tag{37}$$

As a result we get

$$\cos\varphi=\frac{\mathbf{R}\boldsymbol{\sigma}}{R}+\frac{m}{r}\Big\{(B+1)\Big(\frac{[\mathbf{r}_0\times\mathbf{r}]}{rr_0+\mathbf{r}\mathbf{r}_0}-\frac{[\mathbf{r}\times\boldsymbol{\sigma}]}{r-\mathbf{r}\boldsymbol{\sigma}}\Big)+\frac{r\alpha(r_0)-r_0\alpha(r)-B(r-r_0)}{R^2r_0}[\mathbf{r}_0\times\mathbf{r}]\Big\}\frac{[\mathbf{R}\times\boldsymbol{\sigma}]}{R}. \tag{38}$$

In the collinear case, when $\mathbf{r}/r=\mathbf{r}_0/r_0=\mathbf{R}/r$, we have

$$\cos\varphi=\frac{\mathbf{r}\boldsymbol{\sigma}}{r}-\frac{m}{r}(B+1)\left(1+\frac{\boldsymbol{\sigma}\mathbf{r}}{r}\right) \tag{39}$$

(this expression determines the angle distance between the quasar and the sun).

We now apply Eqs. (38) and (39) to our model problem (Fig. 2). The x axis is chosen to be the line joining the sun, the planet P and the earth at the instant $t=0$ of the inferior conjunction. We neglect the planetary aberration here and below, since it is not connected with the coordinate-condition problem of interest to us. It is easy to take the planetary aberration into account by an iteration method.

At the instant $t=0$ we measure the angle φ_0 between the directions to the planet P and the quasar. This measurement determines the vector $\boldsymbol{\sigma}$. In fact, from (39) we have

$$\sigma_x=\cos\varphi_0+\frac{m}{a_E}(B+1)(1+\cos\varphi_0),$$
$$\sigma_Y=-\sin\varphi_0+\frac{m}{a_E}(B+1)(1+\cos\varphi_0)\operatorname{ctg}\varphi_0. \tag{40}$$

At an arbitrary instant t the coordinates of the planets are

$$\mathbf{r}_0=\mathbf{r}_P=(a_P\cos\lambda_P,\ a_P\sin\lambda_P),\quad \lambda_P=n_Pt,$$
$$\mathbf{r}=\mathbf{r}_E=(a_E\cos\lambda_E,\ a_E\sin\lambda_E),\quad \lambda_E=n_Et, \tag{41}$$

and the radii of the circular orbits are determined from (13), (28), and (39). With the aid of these relations we get from (38)

$$\cos\varphi=\frac{\mathbf{R}\boldsymbol{\sigma}}{R}-m\Big\{\frac{B+1}{a_E}\Big[\frac{\sin(\lambda_E-\lambda_P)}{1+\cos(\lambda_P-\lambda_E)}+\frac{\sin(\lambda_E+\varphi_0)}{1-\cos(\lambda_E+\varphi_0)}\Big]+\frac{a_E\alpha(a_P)-a_P\alpha(a_E)-B(a_E-a_P)}{R^2}\sin(\lambda_E-\lambda_P)\Big\}\sin\varphi. \tag{42}$$

We now put

$$\varphi=\varphi^N+\Delta\varphi. \tag{43}$$

Then

$$\cos\varphi^N=\mathbf{R}^N\boldsymbol{\sigma}^N/R^N, \tag{44}$$

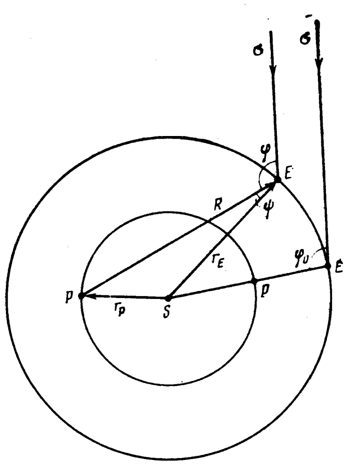

FIG. 2. Angle measurements in complanar circular motion of the earths and an inner planet.

where $\mathbf{R}^N$ is calculated from (10) and (41) with the Newtonian values (28), while $\boldsymbol{\sigma}^N$ denotes a vector with components ($\cos\varphi_0$ and $-\sin\varphi_0$). All the operations in (44) are carried out in the Euclidean sense. Next,

$$\Delta\varphi=-\frac{1}{\sin\varphi}\Delta\left(\frac{\mathbf{R}\boldsymbol{\sigma}}{R}\right)+m\Big\{\frac{B+1}{a_E}\Big[\frac{\sin(\lambda_E-\lambda_P)}{1+\cos(\lambda_P-\lambda_E)}+\frac{\sin(\lambda_E+\varphi_0)}{1-\cos(\lambda_E+\varphi_0)}\Big]+\frac{a_E\alpha(a_P)-a_P\alpha(a_E)-B(a_E-a_P)}{R^2}\sin(\lambda_E-\lambda_P)\Big\}.$$

It is easy to find that

$$\Delta\left(\frac{\mathbf{R}\boldsymbol{\sigma}}{R}\right)=\frac{1}{R^3}[\mathbf{R}\times\Delta\mathbf{R}][\mathbf{R}\times\boldsymbol{\sigma}]+\frac{1}{R}\mathbf{R}\Delta\boldsymbol{\sigma}.$$

The correction $\Delta\mathbf{R}$ is due to the corrections (29) to Δa_E and Δa_P, therefore

$$[\mathbf{R}\times\Delta\mathbf{R}]=\frac{a_E\Delta a_P-a_P\Delta a_E}{a_Ea_P}[\mathbf{r}_P\times\mathbf{r}_E].$$

The correction term $\Delta\boldsymbol{\sigma}$ is a vector whose components are equal to the relativistic terms in (40). Direct calculations yield

$$\frac{\mathbf{R}\Delta\boldsymbol{\sigma}}{R\sin\varphi}=\frac{m}{a_E}(B+1)\frac{1+\cos\varphi_0}{\sin\varphi_0}.$$

We ultimately get

$$\Delta\varphi=m\Big\{\frac{B+1}{a_E}\Big[\operatorname{tg}\frac{\lambda_E-\lambda_P}{2}+\operatorname{ctg}\frac{\lambda_E+\varphi_0}{2}-\operatorname{ctg}\frac{\varphi_0}{2}\Big]-\Big(\frac{2}{3}A+B\Big)\frac{a_E-a_P}{R^2}\sin(\lambda_E-\lambda_P)\Big\}. \tag{45}$$

Thus, from the known n_E, n_P, and φ_0 we can calculate φ independently of the employed coordinate system. The values of $\varphi(t)$ are the ephemeridal data for the differential angle measurements in our model problem.

We have dealt so far with differential measurements against the background of infinitely remote immobile objects. We consider now the differential angular measurements of a planet P, connected with its position relative to the sun (optical measurements—transit of planets across the solar disk, transit of the sun and of a planet through the same meridian; radio observation—interferometry in the near wave field when "beacons" are observed on planets). Ths sun's ray velocity vector at the point $\mathbf{r}(t)$ is equal to

$$\frac{1}{c}\dot{\mathbf{r}}(t)=\Big\{1-\frac{m}{r}(B+1+r\alpha'(r))\Big\}\frac{\mathbf{r}}{r}. \tag{46}$$

Identifying ξ with this vector, we obtain from (36) the cosine of the measured angle $\psi(t)$ between the directions to the sun and to the planet P:

$$\cos\psi=\frac{\mathbf{r}\mathbf{R}}{rR}-\frac{m}{r}\Big[\frac{B+1}{rr_0+\mathbf{r}\mathbf{r}_0}+\frac{r\alpha(r_0)-r_0\alpha(r)+B(r_0-r)}{R^2r_0}\Big]\frac{[\mathbf{r}\mathbf{R}][\mathbf{r}_0\mathbf{r}]}{Rr}. \tag{47}$$

Proceeding as before, we have

$$\psi=\psi^N+\Delta\psi,$$
$$\cos\psi^N=\frac{\mathbf{r}^N\mathbf{R}^N}{r^NR^N}=\frac{1}{R^N}\left(a_E^N-a_P^N\cos\frac{2\pi t}{T_s}\right), \tag{48}$$

$$\Delta\psi=-\frac{1}{\sin\psi}\Delta\left(\frac{\mathbf{r}\mathbf{R}}{rR}\right)+m\Big[\frac{B+1}{a_E}\operatorname{tg}\frac{\lambda_P-\lambda_E}{2}+\frac{a_E\alpha(a_P)-a_P\alpha(a_E)+B(a_P-a_E)}{R^2}\sin(\lambda_P-\lambda_E)\Big], \tag{49}$$

where the first correction term can be represented in the form

$$\Delta\left(\frac{\mathbf{rR}}{rR}\right)=\left(\frac{[\mathbf{R}\Delta\mathbf{R}]}{R^2}-\frac{[\mathbf{r}\Delta\mathbf{r}]}{r^2}\right)\frac{[\mathbf{Rr}]}{Rr}=\frac{a_P^2}{R^2}\sin(\lambda_P-\lambda_E)\sin\psi\cdot\Delta\left(\frac{a_E}{a_P}\right).$$

We obtain now an expression for $\psi(t)$ as a function of the measured initial quantities for three types of measurements:

Case 1—only relative measurements (relative to the sun). We can regard as known T_s and the value of the angle $\psi(t)$ at some instant, say $\psi_1=(\frac{1}{4}T_s)$. From (47) we have

$$\cos\psi_1=\frac{a_E}{(a_E^2+a_P^2)^{1/2}}-m\sin\psi_1\left[\frac{B+1}{a_E}+\frac{a_E\alpha(a_P)-a_P\alpha(a_E)+B(a_P-a_E)}{a_E^2+a_P^2}\right]. \tag{50}$$

This expression enables us to find the ratio of the orbit radii in the form

$$\frac{a_E}{a_P}=\left(\frac{a_E}{a_P}\right)^N+\Delta\left(\frac{a_E}{a_P}\right), \tag{51}$$

$$\left(\frac{a_E}{a_P}\right)^N=\operatorname{ctg}\psi_1, \tag{52}$$

$$\Delta\left(\frac{a_E}{a_P}\right)=m\left\{\frac{B+1}{a_E}\left(1+\frac{a_E^2}{a_P^2}\right)+\frac{a_E\alpha(a_P)-a_P\alpha(a_E)+B(a_P-a_E)}{a_P^2}\right\} \tag{53}$$

[to find the radii themselves we must use relation (9)]. Therefore

$$\Delta\psi=\frac{m}{a_E}(B+1)\left[\operatorname{tg}\frac{\lambda_P-\lambda_E}{2}-\frac{a_E^2+a_P^2}{R^2}\sin(\lambda_P-\lambda_E)\right]. \tag{54}$$

In the Newtonian term (48) we must use the value (52).

Case 2—absolute and relative measurements. We assume here that T_s and n_E (hence also n_P) are known. Using (29), we get

$$\Delta\left(\frac{\mathbf{rR}}{rR}\right)=\frac{m}{R^2}\sin(\lambda_P-\lambda_E)\sin\psi\left[a_E\alpha(a_P)-a_P\alpha(a_E)+\frac{2}{3}A(a_E-a_P)\right].$$

In this case therefore

$$\Delta\varphi=m\left\{\frac{B+1}{a_E}\operatorname{tg}\frac{\lambda_P-\lambda_E}{2}-\left(\frac{2}{3}A+B\right)\frac{a_E-a_P}{R^2}\sin(\lambda_P-\lambda_E)\right\}, \tag{55}$$

where we must use the values (28) in the Newtonian term (48).

Case 3—radar measurements. The measured quantities are T_s, T_0, and T_1. We carry out the calculations with the corrections (15) and (16), so that as a result we get

$$\Delta\psi=\frac{m}{a_E}(B+1)\left\{\operatorname{tg}\frac{\lambda_P-\lambda_E}{2}+\left[\frac{a_E^2+a_P^2}{a_E+a_P}\ln\frac{a_E}{a_P}-\frac{a_E-a_P}{a_E+a_P}(a_E^2+a_P^2)^{1/2}\ln\frac{a_E+a_P+(a_E^2+a_P^2)^{1/2}}{a_E+a_P-(a_E^2+a_P^2)^{1/2}}\right]\frac{a_E}{R^2}\sin(\lambda_P-\lambda_E)\right\}, \tag{56}$$

and in the Newtonian term (48) we must use the values (14).

Thus, depending on the type of the measured initial quantities, the corrections (54)–(56) for $\Delta\psi$ have different structures, but neither expression depends on the coordinate frame.

Figures 3–5 show the numerical values of the corrections (54)–(56) in GRT ($A=0, B=1$) for Mercury and

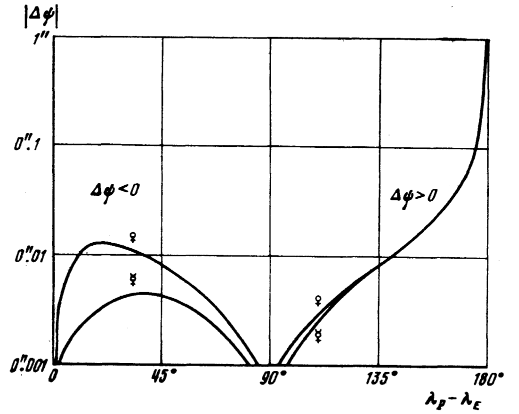

FIG. 3. Relativistic effect in the angule distance ψ of an inner planet from the sun. Initial measurements T_s (synodic period) and ψ $(T_s/4)$.

Venus as functions of the relative angle between the planet P and the earth. Near the superior conjunction all the curves practically coincide, for in this case the main contribution to (54)–(56) is made by one and the same first term. However, even far from the superior conjunction the relativistic corrections are appreciable and they must be taken into account in the case of high precision measurement of the angle distances (accurate to 0.001″). The correction (45) for $\Delta\varphi$ can be easily calculated if it is recognized that it consists of the correction (55) for $\Delta\psi$ taken with a negative sign (Fig. 1) and of a term that does not depend on the position of the planet P.

Just as in the case of radar measurements, the numerical values of the relativistic effects $\Delta\varphi$ and $\Delta\psi$ do not depend on the coordinate frame. At the same time, the magnitude of a relativistic effect depends substantially on the composition of the initial measurements. In particular, if the initial data include the earth's sidereal period, then the considered relativistic effects enable us to determine the constant A of the post-Newtonian approximation of the relativistic theory.

In real astronomical problems involving the reduction of observation data, to verify any particular gravitational theory we must construct a corresponding theory

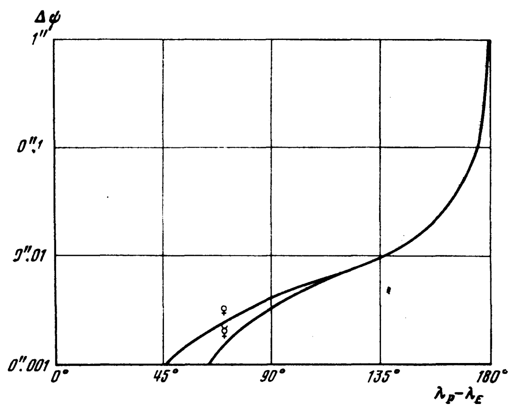

FIG. 4. Relativistic effect in angle distance ψ of an inner planet from the sun. Initial measurements: T_s (synodic period) and T_E (sidereal period of the earth).

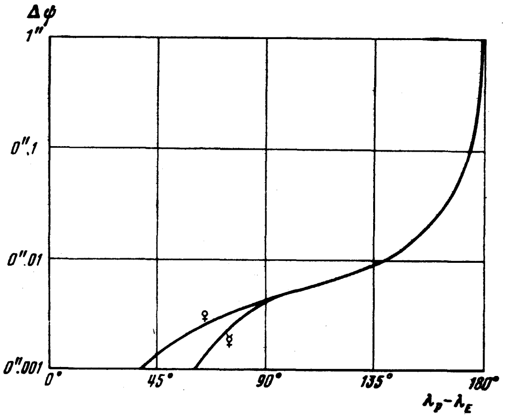

FIG. 5. Relativistic effect in the angle distance ψ of inner planet from the sun. Initial measurements: radar data at the instants $t=0$ (inferior conjunction) and $t=T_s/4$ (T_s is the synodic period).

of the motion of the celestial bodies, calculate the observational characteristics of the motion using formulas such as (11), (42), and (47) (with the same value of the function $\alpha(r)$ as used in the dynamic theory), determine the parameters of the motion by comparison with the observations, and then assess the reliability of the employed gravitation theory from the degree of agreement with subsequent observation.

The relativistic expressions obtained here for the measured angle distances φ and ψ add to the possibility of new tests of relativistic effects. These effects are of particular interest because the relativistic corrections to φ and ψ are necessitated by three factors: the dynamic theory of the motion of the bodies, the laws of light propagation, and the bending of space in the vicinity of the sun.

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Quantum mechanical analysis of the sensitivity of a gravitational antenna

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The quantum mechanical problem of the optimal estimate of the amplitude of an external force acting on a gravitational antenna of a given structure is solved. The optimal spectral operation for processing the output signal of the antenna which forms the observed variable is found. It is shown that there is no quantum sensitivity limit when the optimal procedures are followed. A practical possibility of attaining the resolution needed for second-generation antennas is illustrated.

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§1. INTRODUCTION

The realistic estimate for the intensity of bursts of gravitational radiation arriving at the Earth from outer-space covers the range $W\sim10^4-1$ erg/cm^2 with a duration $\hat{\tau}\sim10^{-3}-10^{-4}$ sec (see, for example, the reviews Ref. 1). For a gravitational detector of Weber type (measuring $l=10^2$ cm), such a pulse is equivalent to the action of an acceleration field $F/m\sim10^{-9}-10^{-11}$ cm/sec^2. Is it possible to detect such a weak disturbance? The answer to this question is crucial for modern gravitational-wave experiments.

Taking a quantum oscillator as a model of a gravitational detector in the limiting case of zero temperature, and assuming that it is in a coherent state (as the state nearest to a classical state), we can formulate a rule for detecting a force acting on the oscillator. It is natural to regard the force as detectable if it shifts the wave packet (or rather, its center) by an amount of the order of its width. In the coordinate representation, this shift is $\Delta k_{\rm qu}\approx(\hbar/2m\omega)^{1/2}$. Hence, for the quantum sensitivity limit we have

$$(F/m)_{\rm qu}\approx\hat{\tau}^{-1}(\hbar\omega/m)^{1/2}, \tag{1}$$

which for the typical parameters $m\approx10^4$, $\omega\sim10^4$, and $\hat{\tau}\sim2\times10^{-4}$ of a gravitational detector[1] gives $(F/m)_{\rm qu}$ $\sim10^{-10}$ cm/sec^2, which is in the middle of the range in which we are interested.

This and similar considerations[2] forces us to approach the problem of detecting gravitational bursts with more care. In reality, the temperature of an antenna is not zero but, in fact, corresponds to a high excitation level and, it would seem, there is no need to invoke quantum arguments. A classical analysis of the sensitivity of a

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