Stochastic oscillations of a parametrically excited nonlinear chain

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It is demonstrated experimentally that the onset of a turbulent state in a parametrically excited nonlinear dissipative medium may not be the result of the appearance of a large number of waves with different frequencies. When the amplitude of the external field is increased the parametric turbulence in a one-dimensional "medium" (a nonlinear chain) appears even after a small number of transitions and replaces jumpwise the regular stationary regime. The fact that the transitions observed in a resonator and in a matched chain, as well as the spatial structures of the field in the pre-turbulent regime (with discrete frequency spectrum) and of the field in the stochastic regime (with continuous spectrum) are close to each other, offer convincing evidence that the investigated turbulence is due to nonlinear interaction of a small number of modes and is not connected with enhancement of the fluctuations. The mathematical model of this turbulence is the strange attractor.

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1. INTRODUCTION

One of the most important problems, whose solution can bring us closer to the understanding of the nature of turbulence (including wave turbulence), is the identification of the mechanisms that produce the turbulence. The origin of the turbulent state produced via a large number of transitions characterized by progressive complication of the spectrum is not completely clear to this day, and can apparently be quite varied (a definite role in the maintenance of such a turbulent state is played by external fluctuations). Much clearer is the nature of the turbulence that sets in jumpwise as the result of a small number of transitions from a stationary state. Such a turbulence permits a finite-dimensional description and its mathematical representation is the strange attractor (an attracting set in finite-dimensional phase space of the system), on which all the trajectories are stable and have a very complicated behavior. Motion on a strange attractor is characterized by a continuous frequency spectrum (see, e.g., Ref. 1 and the literature there). By now we have several convincing physical experiments that confirm the "attractor" properties of turbulence in thermal convection[2] and in Cuette flow between cylinders.[3] The present paper is devoted to an experimental investigation of wave turbulence in an artificial nonlinear medium with dissipation when the medium is parametrically excited by homogeneous pumping.

Experimental investigations of the mechanisms that produce parametric turbulence, for example Langmuir waves in a laboratory plasma or spin waves in a ferromagnet, are known to be extremely difficult. The structure of the fields in such experiments can be assessed as a rule only by indirect attributes, particularly by stimulated-emission spectra, by the behavior of the nonlinear susceptibilities, and others.[4] In addition, the problem is made complicated by the inhomogeneity of the k-space of the parametrically excited waves. This makes it difficult to separate the elementary processes that lead to the onset of the continuous spectrum in pure form. In the present paper we investigate the mechanism of the onset of turbulence in an ensemble of parametrically excited waves in a one-dimensional nonlinear medium—a periodic lattice, whose spatial dispersion coincides in the long-wave approximation, for example, with the spectrum of ion-sound waves in a plasma.

2. DESCRIPTION AND RESULTS OF THE EXPERIMENT

We investigated the spatial and temporal spectra of the excitation of a nonlinear one-dimensional chain simulated by sixty LC line elements in which the dependence of the charge on the voltage u_n was of the form

$$Q_n(u_n)=C_0(u_n+\alpha_1 u_n^2+\alpha_2 u_n^3+\ldots), \qquad (1)$$

In the linear approximation, the considered one-dimensional "medium" is characterized in the investigated frequency region by a dispersion law (see Fig. 1)

$$\omega(k)=\omega_0 \sin(k/2). \qquad (2)$$

The linear and nonlinear properties of the medium were monitored against the spectra of the induced oscillations of the network, which was spatially homogeneously excited by a noise source. When the spectrum of the source spanned the entire transparency band of the medium, the observed spectrum of the induced oscillations in the frequency interval $0<\omega<\omega_0$ (see Fig. 1) duplicated the qualitatively distribution function of the number of oscillators over the spectrum $\rho(\omega)\sim dk/d\omega$ (cf. Figs. 1 and 2a). On the other hand, when the upper end point ω_{ep} of the spectrum of the exciting noise field was lower than ω_0, the spectrum took the form shown in Fig. 2b: owing to the nonlinear interaction of the modes, the energy is transferred to a frequency region in which

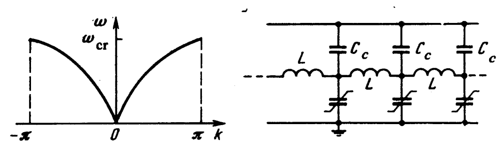

FIG. 1. Diagram of investigated LC network and its dispersion characteristic.

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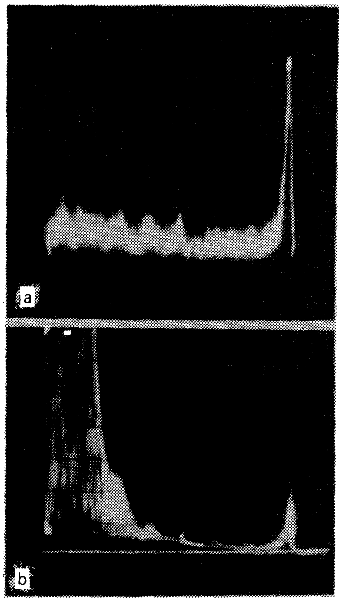

FIG. 2. Spectra of the oscillations of a nonlinear network excited by a noise source: a—width of source spectrum $0 \leqslant \omega \leqslant 6$ MHz, b—width of source spectrum $0 \leqslant \omega \leqslant 0.3$ MHz.

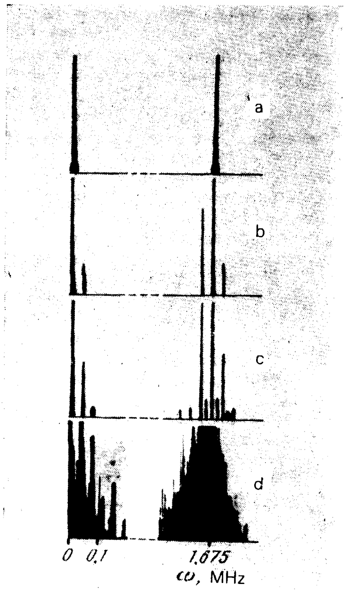

FIG. 3. Evolution of the spectrum of parametrically excited waves with increasing pump amplitude: a—mode synchronization regimes; b, c—appearance of satellites and subsatellites $\omega_p = 3.35$ MHz, $U_{p3}/U_{p1} \approx 2.38$, $U_{p4}/U_{p1} \approx 2.48$; d—jumpwise onset of a continuous spectrum ω_p 3.35 MHz, $U_{p5}/U_{p1} \approx 3.61$.

there is no source, and the steady-state distribution of the intensity over the spectrum in the interval $\omega_{rp} < \omega < \omega_0$ coincides qualitatively as before with $\rho(\omega)$.

In the presence of an intense spatially homogeneous pump, the field in the investigated model of the medium is described by an equation of the form

$$\partial^2 Q/\partial t^2 - (u_{n-1} - 2u_n + u_{n+1}) = C_{cn} L(U_p), \qquad (3)$$

where C_c is a coefficient that characterizes the coupling of the external field with the internal one, and L is a linlinear differential operator. In all the experiments we had $U_p = U_p \sin\omega_p t$, where $\omega_p/2 \lesssim \omega_0 = (2\pi)^{-1}/\sqrt{LC}$ (ω_0 is the critical frequency of the lattice).

We describe now the sequence of transitions observed when the amplitude was increased in a line matched at its terminals.[1)] The temporal spectrum of the excitations was determined with a spectrum analyzer, and the spatial spectrum was determined from direct measurements of the structure of the field along the lattice. When the parametric instability threshold was exceeded $U_p > U_{p1}$, a pair of opposing waves was excited with wave numbers k_1 and $-k_1$ having the same modulus. This corresponded to establishment of a regime that was spatially homogeneous in amplitude, at a frequency $\omega_p/2$. This single-mode regime existed up to $U_p \leqslant U_{p2}(\omega_p)$. The next transition at $U_p = U_{p2}$ corresponded to excitation of one or several more modes with wave numbers $k_1 \pm \Delta k$ and $k \gg \Delta k$, but the temporal spectrum (see Fig. 3a) remained single-frequency. The spatial structure of one of such regimes, observed at $U_p > U_{p2}$, is shown in Fig. 5a. The reason why the observed spatially inhomogeneous (non-single-mode) regime are single-frequency is that the modes are synchronized, an effect discussed in great greater detail below (see Sec. 3). The width of the region of the existence of the single-mode (spatially homogeneous) regime was strongly dependent on the pump frequency. At some frequencies, this regime was absent and hard excitation took place of spatially inhomogeneous regimes.

Further increase of the pump led to disintegration of the synchronization regime and to establishment, at $U_p \geqslant U_{p2}(\omega_p)$, of a three-frequency regime with satellites symmetrically located near the frequency $\omega_p/2$ (see Fig. 3b). The detection effect produced in the spectrum also a low-frequency component that did not correspond (because of the non-decay character of the spectrum) to the proper excitations of the lattice.

The structure of the fields[2)] on each of the spectral components at $U_p > U_{p3}$ was spatially inhomogeneous. The succeeding increase of the pump led to the appearance of subsatellites (see Fig. 3c), and at $U_p = U_{p4}(\omega_p)$ the spectral components on the screen of the spectrum analyzer turned out to be time-modulated, thus indicating the presence of subsatellites with even smaller $\Delta\omega$.

Finally, with further increase of pump intensity $[U_p = U_{p5}(\omega_p)]$, the nonstationary regime with the discrete spectrum was transformed abruptly and jumpwise ($\Delta U_p / U_p < 10^{-2}$) into a stochastic regime (see Fig. 4) characterized by the spectrum shown in Fig. 3d.

A diagram of the successive transitions to turbulence with increasing amplitude of the external field (Fig. 3) is highly reminiscent of the catastrophic occurrence of turbulence when the Taylor number is increased in Cuette flow between cylinders[3] and in convection in a closed cavity[5] when the Rayleigh number is increased.

The discrete character of the temporal and spatial spectra in the state directly preceding the jumplike transition to turbulence does not guarantee in general that the stochastic regime characterized by the continuous spectrum is also the result of an interaction of the same modes. In fact, since the frequency spectrum of

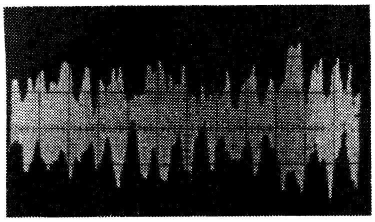

FIG. 4. Oscillogram of the stochastic pulsations whose spectrum is shown in Fig. 3d.

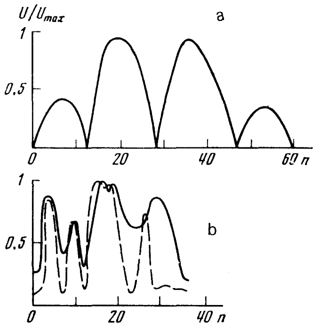

FIG. 5. a) Spatial structure of the field of a single-frequency regime with few modes at $U_p < U_{p3}$. b) Spatial distribution of the the field intensity in the pre-turbulent regime—dashed line. $U_p < U_{p5}$ and in the turbulent regime—solid line($U_p > U_{p5}$).

the excitations of a matched network is continuous, there still remains a possibility of simultaneous increase of the fluctuations in the band $\Delta\omega$, where the continuous spectrum attributed to the strange attractor is observed. Obviously, there is no room for this alternative if the spectrum of the excitations of the medium is made discrete, for example by placing the "medium" in a resonator. The corresponding investigations were made.

In the experiment, the resonator was already described open-ended line of 60 elements. The transitions observed in the resonator excited parametrically by a homogeneous pump, which corresponded to enrichment of the spectrum, and finally to the jumplike onset of the turbulent regime, agreed fully with the already discussed transitions in a matched line (see Fig. 3). The only difference was that these transitions occurred at lower pump fields. We add that an analogous picture of the transitions to a continuous spectrum was observed also in a resonator with half the density—an open-ended line of 30 elements.

Direct measurements of the spatial distribution of the intensity of the field along the resonator in the stage preceding the turbulence and during the turbulence $U_p \geqslant U_{p5}$ (see Fig. 5b) have shown that the spatial spectrum changes insignificantly during the transition. This confirms the statement that the onset of the turbulent regime is due not to the entry of a large number of newly produced modes in the interaction process, but to the loss of stability of the regular regime in a system having a small number of nonlinearly coupled modes that exist already in the pre-turbulent regime.

It is known[1,6,7] that the properties of the intrinsic turbulence of a dissipative nonlinear medium should depend little on the fluctuations. That the observed parametric turbulence has this property was verified in the following manner. A spatially homogeneous weak external field[3)] was produced to a linear network by a noise generator; the bandwidth of the external noise was approximately 3.5 times larger than the transparency band of the "medium" ($\omega_0 = 1.78$ MHz, $0 \leqslant \omega_{\text{noise}} \leqslant 6$ MHz). It was found that introduction of the noise does not influence qualitatively the character of the transitions or the spectrum of the parametric turbulence, and only lowered the threshold of the turbulence insignificantly. This confirms the assumption that the onset of the continuous spectrum in the investigated situation is indeed due to the complex dynamics of the system of several nonlinearly interacting modes.

To understand the mechanism of this interaction, which leads to stochasticity, it is necessary also to assess the role of the extraneous low-frequency oscillations (see Fig. 3), i.e., to ascertain whether they lead to enrichment of the spectrum of the high-frequency oscillations or, being purely induced, exert no influence on them whatever. The experiment that answered this question consisted of the following: we compared the singularities of the transitions and the spectra of the high-frequency oscillations in a high-Q resonator for all modes with corresponding transitions in a network that acts as a resonator only at the frequencies $0,8\omega_0 \leqslant \omega \leqslant \omega_0$. Lowering the Q by two orders of magnitude outside this band led to practical absence of low-frequency perturbations in the spectrum at pump levels corresponding to the onset of parametric turbulence. At the same time, the character of the transitions and the shape of the spectrum remained unchanged. The critical values of the pump fields corresponding to these transitions increased somewhat. This is probably due to the additional nonlinear damping of the parametrically excited modes because they transfer energy to the rapidly damped low-frequency modes. This accounts completely for the reaction of the low-frequency oscillations on the high-frequency ones.

3. DISCUSSION OF RESULTS

We presently know of several mathematical models of parametrically excited systems. Numerical experiments with these modes have demonstrated the possible existence, in a dynamic system, of stochasticity characterized by a continuous spectrum and corresponding to the strange attractor.[6,7] It is obviously easy to construct such a model, by proper choice of the parameters, also for our case. Now that such models have ceased to be exotic, however, it is no longer this which is most important. What comes to the forefront is the problem of realizability of a turbulence with a small number of modes in a system with a sufficiently rich mode spectrum (for example, a resonator with 60 modes). To answer such a question by performing a numerical experiment is no longer so simple a matter, because the experiment described above, while based on a model, is nevertheless physical and quite important. In such an experiment, the differences between the one-dimensional and two-dimensional media are not as important as the difference between a system with a small number of degrees of freedom and a system with a continuous spectrum. The urgency of the "one-dimensional experiment" is confirmed also by the fact that a typical situation for ferromagnets and antiferromagnets, for example, even in two-dimensional geometry, is one in which only two or even only one pair of waves is excited in the homogeneous pump field,[8] and the result is a stochasticity of the type investigated above but not connected with excitation of a large number of pairs of waves of the random phases—weak parametric turbulence.[4]

With respect to the mechanism that produces the stochasticity, we can make the following remarks. Recently A. L. Fabrikant and one of us (M.I.R.) in an analytic and numerical investigation of a decay process of the type $2\omega_0 = \omega_1 + \omega_2$, $2k_0 = k_1 + k_2$ in a nonlinear medium with amplification at frequency ω_0 and damping at frequencies $\omega_{1,2}$ have observed a very complicated dynamics characterized by a continuous temporal spectrum. This dynamics (and spectrum) is generated jumpwise from a simple dynamics (discrete spectrum) when one of the parameters goes through a critical value. It is possible that a similar modulation mechanism of the onset of the continuous spectrum, on account of interaction of the amplifying central mode and of the damp satellites, takes place also in our case, when the growth of the fundamental spectral component at the frequency ω_0 is due to parametric amplification.

We note in conclusion that so far it is possible to describe quantitatively, with sufficient degree of agreement with experiment, only one of the observed series of transitions (see Fig. 3), namely to determine, in the space of the parameters, the boundary of the transition from the single-mode to the multimode (spatially inhomogeneous) single-frequency regime. Let us discuss this in greater detail.

The Hamiltonian of the investigated system can be written in the form

$$H=\sum_j \omega(k_j) a_{k_j} a^*_{k_j} + H_p + H_{int},$$
$$H_p = \frac{1}{2}\sum_j \left(hU_p \exp(i\omega_p t) a_{k_j} a_{-k_j} + \text{c. c.}\right), \tag{4}$$

where

$$H_{int} = \frac{1}{2}\sum_{12,34} T_{12,34} a^*_{k_1} a^*_{k_2} a_{k_3} a_{k_4} \delta(k_1+k_2-k_3-k_4).$$

Assuming the spatial spectrum to be discrete (a resonator), we find the initial equations of motion for the amplitudes and phases of the interacting waves:

$$\dot a_{k_j} + i\omega_{k_j} a_{k_j} + \gamma a_{k_j} + ihU_p \exp(i\omega_p t) a^*_{k_j}$$
$$+ i\sum_{1,2,3} T_{k_j 123} a^*_{k_1} a_{k_2} a_{k_3} \delta(k_j + k_1 - k_2 - k_3) = 0. \tag{5}$$

The stationary states of the lattice, characterized by a discrete frequency spectrum, correspond to the equilibrium states (5). The case when the number of excited modes with different $|k|$ exceeds the number of components of the temporal spectrum corresponds to total or partial synchronization of the frequencies of these modes. In experiments, such stationary excited states correspond to a regime with a spatially inhomogeneous field distribution.

Let us examine the effects of synchronization using as an example the interaction of modes with two different $|k|$:

$$\dot a_{\pm k_{1,2}} + i\omega_{k_{1,2}} a_{\pm k_{1,2}} + \gamma_{k_{1,2}} a_{\pm k_{1,2}}$$
$$+ i[hU_p \exp(i\omega_p t) + Ta_{\pm k_{1,2}} a_{\mp k_{2,1}}] a^*_{\mp k_{1,2}} + iTa_{\pm k_{1,2}}(|a_{\pm k_{1,2}}|^2 + 2|a_{\mp k_{1,2}}|^2)$$
$$+ 2iTa_{\pm k_{1,2}}(|a_{\pm k_{1,2}}|^2 + |a_{\mp k_{1,2}}|^2) = 0. \tag{6}$$

We assume for simplicity that the decrements are equal, $\gamma_{k_1} = \gamma_{k_2} = \gamma$. It is easy to note that Eqs. (6) have two integrals $\varphi_{k_{1,2}} - \varphi_{-k_{1,2}} = C_{1,2}$, and it follows from (6) that the intensities of the waves with identical $|k|$ become equalized within a time $\sim 1/\gamma$. The constants $C_{1,2}$ are determined here by the boundary conditions at the ends of the resonator: $C_{1,2} = 0$ if the line is open-ended, and $C_{1,2} = \pi$ when the line terminals are shorted. Using these integrals, we can write in place of (6) only two equations for the complex amplitudes of the waves in terms of the new variables ($T > 0$) $b_1 = a_{k_1}(T/\gamma)^{1/2}$, $b_2 = a_{k_2}(T/\gamma)^{1/2}$, $\tau = \gamma t$, $H = hU_p/\gamma$:

$$db_{1,2}/d\tau + b_{1,2} + ib_{1,2}(3|b_{1,2}|^2 + 4|b_{2,1}|^2)$$
$$+ i(H + b^2_{2,1}) b^*_{1,2} + i\Delta_{1,2} b_{1,2} = 0, \tag{7}$$

where $\Delta_{1,2} = (\omega_p - 2\omega(k_{1,2}))/\gamma$.

Assume that for the mode with $k = k_1$ the conditions of the resonance with the pump are exactly satisfied; it follows then from (7) that at $H_1 < H < H_2$, where $H_1 = 1$, $H_2 = (1 + \Delta^2/4)^{1/2}$, the single-mode regime is stable. At $H > H_2$, a second pair of waves is excited and a regime of pair synchronication by the pump—the stable equilibrium state (7)—is established. The two-frequency regime would correspond to a limit cycle in the phase space of the system (7). However, no such regimes were obtained for the two-mode model. The region of instability of one pair (or of the spatially homogeneous regime) is indicated in Fig. 6, which shows also the numerically constructed region of existence of the single-frequency spatially inhomogeneous regime (two pairs of waves are excited). Where the regions overlap, soft excitation of the inhomogeneous regime is possible, while outside the region of instability of the main pair (see Fig. 6), the inhomogeneous distribution of the field can be established only by hard excitation.

The experimentally observed single-frequency stationary regimes with a spatially inhomogeneous field distribution, which result from the synchronization of the fre-

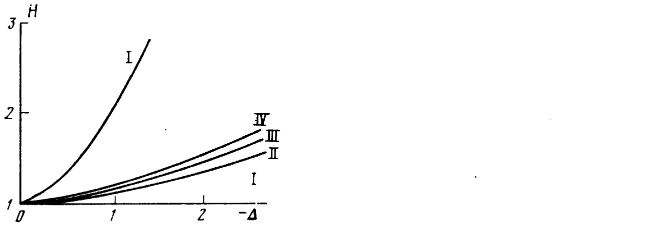

FIG. 6. Plane of the parameters H and Δ (H is the relative amplitude of the pump, Δ is the relative detuning): I—region of stability of single-mode regime (natural frequency of the mode $\omega = \omega_p/2$); II, III—regions of the existence of the two-mode regime; IV—stability region of single-mode regime (natural frequency of the mode $\omega = \omega_p/2 - \Delta\gamma$).

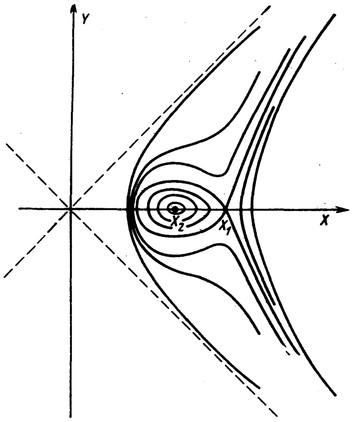

FIG. 7. Phase plane of the system (9): $X_{1,2} = \pm (H^2 - 1)^{1/2} + (H^2 - 1 + C)^{1/2}$.

quencies, generally speaking, of a large number of modes with close k, can be described by introducing the amplitudes of opposing waves that vary slowly along the network ($T > 0$):

$$\begin{aligned}\partial b_+/\partial x + b_+ + iHb_-^* + i\Delta b_+ + ib_+(|b_+|^2 + 2|b_-|^2) &= 0,\\ -\partial b_-/\partial x + b_- + iHb_+^* + i\Delta b_- + ib_-(|b_-|^2 + 2|b_+|^2) &= 0.\end{aligned} \quad (8)$$

In terms of the new variables $X = |b_+|^2 + |b_-|^2$, $Y = |b_-|^2 - |b_+|^2$, $Z = b_+b_- + b_+^*b_-^*$ we have

$$X' = 2Y, \ Y' = 2X + 2HS, \ Z' = YS; \quad S^2 = X^2 - Y^2 - Z^2. \quad (9)$$

By simple transformation we obtain one of the integrals of the system (9) $X^2 - Y^2 + 4HZ = C$. The phase plane is shown in Fig. 7.

Motions close to the separatrix correspond to the experimentally observed single-frequency multimode regimes ("blackout" solitons). On the other hand, motions close to the "center" correspond to regimes with few modes, one of which is shown in Fig. 5a.

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1) Such a line simulates an infinite one-dimensional medium with a continuous frequency spectrum.

2) The corresponding measurements were made with a frequency-selective voltmeter.

3) By "weakness" of the noise is meant satisfaction of the relation $I_\omega/I_c \lesssim 5\cdot 10^{-3}$, where I_c is the total excitation intensity in the parametric turbulence regime, and I_ω is the intensity of the noise in the band $0 < \omega < \omega_0$.

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Metallic screening in a Peierls-Frölich dielectric with pinning to impurities at finite temperatures

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The temperature dependence of the metallic properties in a one-dimensional system with a charge-density wave (CDW) in the presence of sparse impurities is investigated. The statistical correlation function of the CDW phase is found, making it possible to determine the effective number of free carriers participating in the Frölich conduction at finite temperatures. The correlation function of the order parameter of the system is also considered.

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1. INTRODUCTION

1. A significant number of quasi-one-dimensional compounds display anomalous electrical and optical properties. The exceptionally high values of the static and microwave permittivity ε,[1] amounting to $\varepsilon \sim 10^2-10^4$, and also the presence of the peak in the conductivity in the far infrared region,[2] confirmed recently in Ref. 3 by precise measurements, are attracting great attention. These phenomena are observed in a wide range of temperatures, from room temperatures to liquid-helium temperatures. Most of these substances (see Refs. 1 and 4) undergo a transition to the dielectric state, with activation energy

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