rents.[17]

[3)]Enhancement of the T-parity nonconservation effects in molecules in the presence of an electron dipole moment was pointed out by Sushkov and Flambaum.[14]

[4)]In the case of heavy molecules the spin—orbit interaction is important and it mixes the Π and Σ states, as allowed for in the estimates of $\langle V_{PT}\rangle$.

[1]A. N. Moskalev, R. M. Ryndin, and I. B. Khriplovich, Usp. Fiz. Nauk **118**, 409 (1976) [Sov. Phys. Usp. **19**, 220 (1976)].

[2]V. A. Alekseev, B. Ya. Zel'dovich, and I. I. Sobel'man, Usp. Fiz. Nauk **118**, 385 (1976) [Sov. Phys. Usp. **19**, 207 (1976)].

[3]V. G. Gorshkov, Preprint No. 268, Institute of Nuclear Physics, Leningrad, 1976.

[4]L. N. Labzovskiĭ, Izv. Akad. Nauk SSSR Ser. Fiz. **41**, 2491 (1977).

[5]P. E. G. Baird, M. W. S. M. Brimicombe, R. G. Hunt, G. J. Roberts, P. G. H. Sandars, and D. N. Stacey, Phys. Rev. Lett. **39**, 798 (1977).

[6]L. L. Lewis, J. H. Hollister, D. C. Soreide, E. G. Lindahl, and E. N. Fortson, Phys. Rev. Lett. **39**, 795 (1977).

[7]L. M. Barkov and M. S. Zolotorev, Pis'ma Zh. Eksp. Teor. Fiz. **27**, 379 (1978) [JETP Lett. **27**, 367 (1978)].

[8]L. N. Labzovskiĭ, Zh. Eksp. Teor. Fiz. **73**, 1623 (1977) [Sov. Phys. JETP **46**, 853 (1977)].

[9]Yu. I. Neronov and A. E. Barzakh, Zh. Eksp. Teor. Fiz. **75**, 1521 (1978) [Sov. Phys. JETP **48**, 769 (1978)].

[10]P. G. H. Sandars, Phys. Lett. **14**, 194 (1965); Phys. Rev. Lett. **19**, 1396 (1967); G. E. Harrison, P. G. H. Sandars, and S. J. Wright, Phys. Rev. Lett. **22**, 1263 (1969); C. Bouchiat, Phys. Lett. B **57**, 284 (1975); M. A. Player and P. G. H. Sandars, J. Phys. B **3**, 1620 (1970).

[11]I. B. Khriplovich, Zh. Eksp. Teor. Fiz. **71**, 51 (1976) [Sov. Phys. JETP **44**, 25 (1976)].

[12]V. V. Flambaum, Yad. Fiz. **24**, 383 (1976) [Sov. J. Nucl. Phys. **24**, 199 (1976)].

[13]E. A. Hinds, C. E. Loving, and P. G. H. Sandars, Phys. Lett. B **62**, 97 (1976).

[14]O. P. Sushkov and V. V. Flambaum, Zh. Eksp. Teor. Fiz. **75**, 1208 (1978) [Sov. Phys. JETP **48**, 608 (1978)].

[15]L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika, Nauka, M., 1974 (Quantum Mechanics: Non-Relativistic Theory, Pergamon Press, 3rd ed., Oxford, 1977).

[16]G. Herzberg, Molecular Spectra and Molecular Structure, Vol. 1, Spectra of Diatomic Molecules, Van Nostrand, New York, 1950 (Russ. Transl., IIL, M., 1949).

[17]V. G. Gorshkov and L. N. Labzovskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 19 (1975) [JETP Lett. **21**, 8 (1975)].

[18]L. I. Schiff, Phys. Rev. **132**, 2194 (1963).

Translated by A. Tybulewicz

Bohr-Sommerfeld quantization of *n*-dimensional neutral and charged pulsons

I. L. Bogolyubskiĭ

Joint Institute for Nuclear Research

(Submitted 2 July 1978)
Zh. Eksp. Teor. Fiz. **76**, 422–429 (February 1979)

The Klein-Gordon equation with logarithmic nonlinearity is used as an example to show that a scalar field can form n-dimensional oscillating field bunches held together by self-action forces and having apparently a Lyapunov stability. The Bohr-Sommerfeld quantization condition is used to obtain within the framework of this model the mass spectra of n-dimensional pulsons (localized oscillating extended solutions) that are either neutral or have an elementary charge $Q = 1$.

PACS numbers: 03.70. + k

Ideas of constructing a quantum-field theory of elementary particles treated as excited states of a system of fundamental fields have been advanced numerous times. An example of such a theory is quantum chromodynamics, which is actively being developed at present and is called upon to explain hadron physics. It is known that its Lagrangian determines the interaction of spinor fields and nonlinear Yang-Mills fields. Next, one of the possible methods of representing the so-called "structureless" particles of nonzero mass is to assume that they are extended field "bunches" of finite dimensions (nonlinear-field quanta) held together by self-action forces. In the classical approach such modes are described by relativistically invariant (RIN) field equations, and the particle-like solutions of these equations can be regarded as classical proto-types of elementary particles. The development of methods of quantizing fields described by RIN equations is one of the pressing problems of the theoretical high-energy physics. At the present time, RIN models are quantized as a rule by quasiclassical methods (see, e.g., Refs. 1–3), which yield directly the energy spectrum of particle-like excitations if classically localized solutions (LS) of the RIN equations are known.

The most investigated LS of nonlinear wave equations are solitons (see the reviews[4,5]). These will be defined here as LS of the type $R(\mathbf{x})\exp(-i\omega t)$, having a finite energy, charge, etc; $R(\mathbf{x})$ can in general be a scalar, a vector, or a spinor. It is clear that there can exist LS of nonlinear equations, and in particular RIN equations, of more general type, e.g., periodic in time, $u(\mathbf{x}, t+T)=u(\mathbf{x}, t)$, and even more general ones, which determine time-periodic distributions of physical quan-

0038-5646/79/020213-05$02.40

tities (charge, energy density), $H(\mathbf{x}, t+T)=H(\mathbf{x}, t)$, $q(\mathbf{x}, t+T)=q(\mathbf{x}, t)$. For brevity we shall call these periodic solutions "pulsons."[6] The topological charge of the solitons and pulsons considered in the present paper is zero.

An example of a pulson in the case of one spatial coordinate ($n=1$) is an oscillating bound state of two solitons [the "bion" of the sine-Gordon (sG) equation]. Spherically symmetrical ($n=3$) scalar pulsons (weakly radiating and therefore only approximately periodic) were observed[1)] and tested for stability in Ref. 6. The importance of studying classical time-dependent LS of RIN equations when it comes to developing a nonlinear quantum field theory is demonstrated by the analogy indicated in Ref. 2b: "The Bohr radii of the hydrogen atom are not stationary...but periodic (in time)...solutions of the classical equations of motion." The pulson oscillations can be regarded as the "motion" corresponding to its internal degree of freedom; quantization of this motion results in the mass spectrum of extended particles.

Experience with the quasiclassical quantization (in the case $n=1$) of the *fully integrable* sG equation by the method of functional integration shows that the quasiclassical answer can coincide[1,2] with the exact quantum results[7] even at large values of the interaction constant. Bohr-Sommerfeld quantization (BSQ) of the bions of the sG equation leads to a mass spectrum that coincides with the one obtained by the method of functional integration.[1,2] One can therefore hope that the BSQ can give a reasonable mass spectrum even in those cases when it is impossible to quantize by the functional-integration method. The BSQ becomes particularly valuable in the realistic case of three spatial coordinates, inasmuch as so far we know of not a single RIN equation that is fully integrable at $n=3$ and has extended LS with finite energy, charge, etc.

It appears that the first example of an RIN equation that has at arbitrary n (including $n=3$) an exact analytic soliton solution was indicated in Ref. 8:

$$u_{\tau\tau}-\nabla_\xi^2 u+m^2u-l^{-2}u\ln(|u|^2a^{n-1})=0. \quad (1)$$

The present paper is devoted to the BSQ of the LS of this equation. In Sec. 1 are discussed the properties of the quasiclassical LS of Eq. (1), which were obtained in Refs. 8 and 9 and in the present paper, and an "improved" modification of the model (1). The BSQ of these LS has yielded the mass spectra of neutral "particles" (Sec. 2) and of "particles" having an elementary charge $Q=1$ (Sec. 3).

1. SOLITONS AND PULSONS OF AN EQUATION WITH LOGARITHMIC NONLINEARITY

We introduce the dimensionless variables $t, \mathbf{x}$, and φ:

$$t=\tau l^{-1},\quad \mathbf{x}=\xi l^{-1},\quad u=l^{(1-n)/2}G\varphi,$$
$$G^2=(la^{-1})^{n-1}\exp[n+(lm)^2]. \quad (2)$$

Equation (1) reduces to the invariant form

$$\varphi_{tt}-\nabla_\mathbf{x}^2\varphi-n\varphi-\ln(|\varphi|^2)\varphi=0. \quad (3)$$

We put $U(\varphi)=(1-n)\varphi^2-\varphi^2\ln\varphi^2$. The initial equation (1) is obtained by varying the action with the Lagrangian density

$$\mathscr{L}=|u_\tau|^2-|\nabla_\xi u|^2-(m^2+l^{-2})|u|^2+l^{-2}|u|^2\ln(|u|^2a^{n-1})$$
$$=G^2l^{-(n+1)}[|\varphi_t|^2-|\nabla_\mathbf{x}\varphi|^2-U(|\varphi|)] \quad (4)$$

in the case of a charged complex field and

$$\mathscr{L}={}^1/_2[(u_\tau^2)-(\nabla_\xi u)^2-(m^2+l^{-2})u^2+l^{-2}u^2\ln(u^2a^{n-1})]$$
$$={}^1/_2G^2l^{-(n+1)}[(\varphi_t)^2-(\nabla_\mathbf{x}\varphi)^2-U(\varphi)] \quad (5)$$

in the case of a real uncharged field.

The invariants of Eq. (1) are written in the following form: the energy E_c of the complex field is

$$E_c=\int d^n\xi[|u_\tau|^2+|\nabla_\xi u|^2+(m^2+l^{-2})|u|^2-l^{-2}|u|^2\ln(|u|^2a^{n-1})]$$
$$=G^2l^{-1}\int d^nx[|\varphi_t|^2+|\nabla_\mathbf{x}\varphi|^2+U(|\varphi|)], \quad (6)$$

the energy E_r of the real field is

$$E_r=\frac{1}{2}G^2l^{-1}\int d^nx[(\varphi_t)^2+(\nabla_\mathbf{x}\varphi)^2+U(\varphi)], \quad (7)$$

and the charge is

$$Q=i\int d^n\xi(u_\tau u^*-u_\tau^*u)=iG^2\int d^nx(\varphi_t\varphi^*-\varphi_t^*\varphi). \quad (8)$$

The soliton solutions of (3) are given by[8]

$$\varphi(\mathbf{x}, t)=\exp(-\omega^2/2)\exp(-i\omega t)\exp(-\mathbf{x}^2/2). \quad (9)$$

The potential $U(\varphi)$ of the considered models is not analytic at $\varphi=0$: $d^2U/d\varphi^2|_{\varphi=0}=2m^2_{\rm eff}=\infty$. This is why the frequency ω in the solution (9) has no upper bound. Thus, this model provides an example wherein a finite mass exists at $m_{\rm eff}=\infty$.

It is noted in Ref. 9 that one can seek a solution of Eq. (1) or (3) in the factorized form:

$$\varphi(\mathbf{x}, t)=z(t)\exp(-\mathbf{x}^2/2),\quad z(t)=y(t)\exp[-i\psi(t)], \quad (10)$$

where $y(t)$ and $\psi(t)$ are real functions. It is easy to calculate for such solutions the charge

$$Q=2\pi^{n/2}G^2\gamma,\quad \gamma=y^2\psi_t=\text{const}, \quad (11)$$

the energy of the complex field

$$E_c=\pi^{n/2}G^2l^{-1}[y_t^2+y^2\psi_t^2+y^2(1-\ln y^2)],\quad y^2\psi_t^2=\gamma^2y^{-2}, \quad (12)$$

and the energy of the real field ($\gamma=0$)

$$E_r={}^1/_2\pi^{n/2}G^2l^{-1}[y_t^2+y^2(1-\ln y^2)]. \quad (13)$$

Formulas (12) and (13) state the energy conservation law for the motion of a material point (MP) in a potential relief (see Fig. 1):

$$U_\gamma(y)={}^1/_2[\gamma^2y^{-2}+y^2(1-\ln y^2)]. \quad (14)$$

The function $y(t)$ describes the change of the modulus of the radius vector of the MP when the latter moves

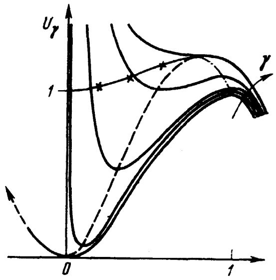

FIG. 1. Potential-relief curves $U_\gamma(y)$. The arrow indicates the direction of increasing γ; the crosses mark the curves that join the points $(U_\gamma(y_{min}(\gamma)), y_{min}(\gamma))$ at different γ; the dash-dot curve is a plot of $(U_\gamma(y_{max}(\gamma)), y_{max}(\gamma))$, and the dashed curve a plot of $(U_\gamma(y_s(\gamma)), y_s(\gamma))$.

in an orbit, and the orbital angular momentum conservation law $y^2\psi_t=\gamma=$ const determines in this case the angular displacement $\psi(t)$ of the MP.

The solitons (9) correspond to the point of minimum $U_\gamma(y)$ at fixed γ or, equivalently, at a fixed charge Q. The condition $dU_\gamma/dy=0, d^2U_\gamma/dy^2>0$ determines the region of stability of the solitons: $\omega>\omega_{cr}=1/\sqrt{2}$; correspondingly, the maximum amplitude of the stable solitons is $y_{s\,max}=\exp(-\omega_{cr}{}^2/2=e^{-1/4}$. It is natural to associate the soliton solution (9) with motion on a circular orbit $y=y_s=$ const, $\psi_t=\omega=$ const.

Corresponding to more general solutions[9]—complex pulsons— are radial oscillations of the MP relative to the soliton equilibrium position y_s in the potential relief $U_\gamma(y)$ between the turning points y_l and y_r. Their amplitude is limited by the requirement $y<y_{max}(\gamma)$, where $dU_\gamma/dy=0, d^2U_\gamma/dy^2<0$ at $y=y_{max}$, i.e., the MP must not jump out of the well of the potential relief $U_\gamma(y)$.

Real pulsons ($\gamma=0$) correspond to motion in the potential relief $U_0(y)$ between the points $y_r=y_0$ and $y_l=-y_0$. The maximum amplitude of these oscillations is $y_{max}=1$. There is no angular rotation in this case ($\Delta\psi=0$).

We now discuss the fundamentally important question of the Lyapunov stability of these pulson solutions. It is known (see, e.g., Ref. 10) that a scalar nonlinear field cannot produce stable stationary solitons $\varphi_s(\mathbf{x})$ at $n>2$ (the Derrick theorem). This theorem, however, does not forbid the existence of time-dependent stable LS. The real pulsons of Eq. (1) at any n are nonradiating and apparently stable objects. There is no analytic proof of their Lyapunov stability, but factorization of the solution (10) and the effective reduction of the problem to an investigation of the motion of a MP can apparently be regarded as an argument favoring the stability.

Numerical computer investigations point definitely to the stability of both real and complex pulsons at all admissible amplitudes. Thus, in the author's own numerical experiments at $n=1$ and 3, the pulson solution $\varphi(\mathbf{x},t)=z(t)\exp(-\mathbf{x}^2/2)$ was preserved with high accuracy throughout the calculation time (~10^3 pulson oscillations were traced); in particular, the pulson energy contained in the considered calculation region was conserved with accuracy ~10^{-5} (a figure that characterizes the computation error due to the nonconservative character of the difference scheme; the true fractions of the radiated energy is even smaller).

In another numerical experiment ($n=2$) the width of the Gaussian "bell" was increased at the initial instant by 25%: $\varphi(\mathbf{x},0)=y(0)\exp[-(0.8\mathbf{x})^2/2]$, $y(0)=0.5$. In this case almost-periodic (with a period T_1 equal approximately to four pulson periods T) compression—expansion cycles of the oscillating field bunch in space about an average position described by a distribution $\exp(-\mathbf{x}^2/2)$ were superimposed on the pulson-field oscillations. The energy radiated at infinity in one period T_1 of such a compression–expansion cycle turn out to be very small, so that the computer time needed to follow the formation of the unperturbed pulson to the end is very large. It is perfectly obvious, however, that the behavior of such a "broadened" pulson in the model (1) differs qualitatively from the behavior of the *unstable* "broadened" spherically symmetrical pulson in the model of the real Klein–Gordon equation with cubic nonlinearity[6b] (the latter spreads out "monotonically" in the course of time).

The foregoing arguments, in the author's opinion, lead to the conclusion that Eq. (1) has a unique property—it has nonradiating stable real and complex pulsons at all n, including $n=3$ (spherically symmetrical). Thus, the self-action forces can hold together an oscillating scalar field, even if it is not charged, in stable particle-like bunches in real four-dimensional space time (cf. the results of Ref. 6).

In concluding this section we note that the potential $U_\gamma(y)=\gamma^2y^{-2}+y^2(1-\ln y^2)$ is not positive-definite, $U_\gamma(y)\to-\infty$ as $y\to\infty$. When the Schrödinger equation with such a potential is solved, there are no stationary energy levels (only quasistationary levels with finite lifetimes are possible instead, and they can be set in correspondence in quantum field theory[2]) with unstable particles). It is possible to "improve" the model by replacing $\ln|u|^2$ in the Lagrangian and in the Hamiltonian by $-|\ln|u|^2|$. The Hamiltonian then becomes positive-definite and the quantum field-theoretical problem of the spectrum of the stationary levels is not *a priori* meaningless; the factorization of (10) and the corresponding classical solutions remain in force at $|\varphi|<1$. Equation (3) remains valid at $|\varphi|<1$ and goes over at $|\varphi|>1$ into

$$\varphi_{tt}-\nabla_x{}^2\varphi+(2-n)\varphi+\varphi\ln|\varphi|^2=0 \qquad (15)$$

[cf. (3)].

2. BOHR-SOMMERFELD QUANTIZATION OF NEUTRAL n-DIMENSIONAL PULSONS

The condition for the BSQ of real pulsons that oscillate with the period T of field systems with infinite number of degrees of freedom, will be written[11] in the form ($\tau_2-\tau_1=T$)

$$\int_{\tau_1}^{\tau_2} d\tau\int d^n\xi\frac{\partial\mathscr{L}}{\partial u_\tau}u_\tau=2\pi N \qquad (16)$$

(N is the number of the excited level, $N=1,2,3,\ldots$). Taking (5) and (10) into account, we get

$$G^2\int \exp\left(-\frac{x^2}{2}\right)\int_{t_1}^{t_2} y_t^2\,dt=2\pi N. \qquad (17)$$

Changing from integration over one-quarter of a period, substituting $\int \exp(-x^2/2)d^n x=\pi^{n/2}$, and expressing y_t^2 with the aid of the relation

$$y_t^2+2U_0(y)=y_t^2+y^2(1-\ln y^2)=2U_0(y_0),$$

we obtain finally the BSQ conditions for real pulsons:

$$N=2G^2\pi^{n/2-1}I(y_0),\quad I(y_0)=\int_0^{y_0}\{2[U_0(y_0)-U_0(y)]\}^{1/2}dy. \qquad (18)$$

The integral $I(y_0)$ was calculated with a computer, using Simpson's rule, at $0<y_0<1$. The maximum possible N is directly proportional to G^2. By determining from the condition (18) the discrete values of $y_{0N}(G)$ ($N=1,2,3,\ldots,N_{max}$ we get [see (13)] the energy-level spectrum $E_N(G)$:

$$E_N(G)=\pi^{n/2}G^2l^{-1}U_0(y_{0N}). \qquad (19)$$

From (18) and (19) we have

$$E_N l/N=\pi U_0(y_{0N})/2I(y_{0N}). \qquad (20)$$

The universal function $U_0(I)$ obtained with the aid of the numerical calculations, i.e., $EG^{-2}l\pi^{-n/2}$ or $NG^{-2}\pi^{1-n/2}/2$ (N is treated here as a continuous variable) is shown in Fig. 2. A remarkable fact is that even though the potential is not analytic ($U''(0)=\infty$) we have $E\approx \text{const}\cdot N$ at small N. It is seen from Fig. 2 that $d^2E/dN^2<0$ at all $N\in(0,N_{max})$, i.e., with increasing N the energy levels come closer together. The inequality $d^2E/dN^2<0$ means, for example, that the N-th state cannot decay into the ($N-1$)-st and 1-st state.

We note that quasiclassical quantization of the bions of the sG equation also leads to a spectrum for which $d^2E/dN^2<0$ ($E_N=2M_0\sin(mN/2M_0), N\leqslant\pi M_0/m$).[1,2] A similar condensation of the levels is observed in the BSA of the motion of a MP in the following potential reliefs:

1) $U_1(y)=-U_0\,\text{ch}^{-2}(\alpha y),\ U_0>0,\ -\infty<y<+\infty;$
2) $U_2(y)=\alpha|y|^k,\ \alpha>0,\ k<2,\ -\infty<y<+\infty;$
3) $U_3(y)=y^2-\beta^2y^4,\ 0\leqslant|y|\leqslant(\sqrt{2}|\beta|)^{-1};$
4) $U_4(y)=1-\cos y,\ -\pi\leqslant y\leqslant+\pi.$

In these models $d^2E/dI^2<0$, where

$$I(y_0)=\int_0^{y_0}\{2[U(y_0)-U(y)]\}^{1/2}dy.$$

For U_1 and U_2 this fact can be easily verified analytically. In the case of U_3 and U_4 this fact is obtained by numerical integration.

3. "DOUBLE" QUANTIZATION OF CHARGED PULSONS

For complex pulsons, the condition (16) is modified to

$$\int_{\tau_1}^{\tau_2}d\tau\int d^n\xi\left(\frac{\partial\mathcal{L}}{\partial u_\tau^*}u_\tau+\frac{\partial\mathcal{L}}{\partial u_\tau}u_\tau^*\right)=2\pi N \qquad (16')$$

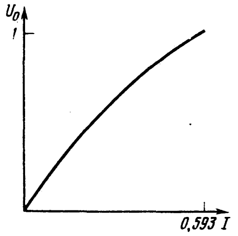

FIG. 2. The universal function $U_0(I)$ for Bohr-Sommerfeld quantization of real pulsons of Eq. (3).

or

$$\int_{\tau_1}^{\tau_2}d\tau\int d^n\xi|u_\tau|^2=\pi N. \qquad (21)$$

Changing to integration over a half-period of the motion in the potential relief (14) we get the BSQ condition:

$$N=2G^2\pi^{n/2-1}\int_{y_l}^{y_r}(y_t^2+\gamma^2y^{-2})y_t^{-1}dy,$$
$$y_t=\{2[U_\gamma(y_r)-U_\gamma(y)]\}^{1/2}. \qquad (22)$$

We impose additionally the "charge quantization" condition, i.e., we require that the pulsons in question have an elementary charge, $Q=1$ ("double" quantization). Depending on the value of G, which is determined by the constants m, l, and a of the initial equation, this condition singles out the quantity $\gamma=(2\pi^{n/2}G^2)^{-1}$ and the corresponding $U_\gamma(y)$ curve.

Selecting with the aid of (22) the points $y_r(N,\gamma)$, for which $N=1,2,3,\ldots$, and the $E_N=E(N,\gamma)$ $=\pi^{n/2}G^2l^{-1}U_\gamma(y_r(N,\gamma))$ corresponding to them, we obtain the energy spectrum of the n-dimensional pulsons having a charge $Q=1$. In order not to confine ourselves to fixed values of m, l, and a we shall assume that the condition $Q=1$ is satisfied on each $U_\gamma(y)$ curve, and select G in accord with formula (11a), knowing γ. Then the condition (22) can be rewritten in the form

$$N=\frac{1}{\gamma\pi}\int_{y_l}^{y_r}\frac{y_t^2+\gamma^2y^{-2}}{y_t}dt=\frac{1}{\gamma\pi}\int_{y_l}^{y_r}\frac{2U_\gamma(y_r)-y^2(1-\ln y^2)}{\{2[U_\gamma(y_r)-U_\gamma(y)]\}^{1/2}}dy. \qquad (23)$$

In the case of small deviations y_r-y_s, the $U_\gamma(y)$ curve can be approximated by the parabola $U_\gamma(y)=U_\gamma(y_s)+1/2\,U_\gamma''(y_s)(y-y_s)^2$; it can be easily found that $U_\gamma''(y_s)$

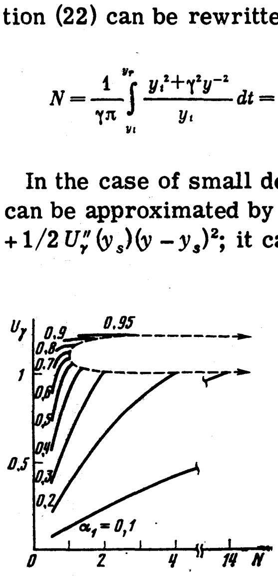

FIG 3. $U_\gamma(N)$ curves for "double" quantization of charged pulsons at different values of $\alpha=y_s(\gamma)/y_{s\,max}$.

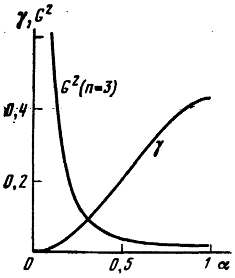

FIG. 4. Plots of $G^2(\alpha)$ at $n=3$ and the function $\gamma(\alpha)$.

$=8(\omega^2-1/2)$. In this case the integral in (23) can be explicitly calculated, and by regarding N as a continuous variable we find that $N \to N_s(\omega) = \omega(\omega^2 - 1/2)^{-1/2}/2$ at $y_r \to y_s(\omega)$. The limiting value is thus $N_s \neq Q = 1$, which contradicts the result of Ref. 12, where it is stated that the "charge quantization" $Q = N$ and the BSQ are equivalent for solitons of the type $R(\mathbf{x})e^{-i\omega t}$. The point is that for these solitons the classical densities of the physical quantities (of the Hamiltonian, charge, etc.) are constant in time because of the invariance of the field equations to gauge transformations of the first kind, and it is not clear how to choose a period T having a physical meaning. The period becomes definite only if arbitrarily small pulson oscillations $\delta y(t)$ about the soliton equilibrium position y_s are considered. Since the orbits are not closed in the central fields (except the potentials $U_1 = C_1 y^2$ and $U_2 = -C_2 y^{-1}, C_1, C_2 > 0$),[13] the angle ψ changes in the period T by an amount that differs from 2π when the solution (19) tends to the soliton solution, and consequently $T \neq 2\pi/\omega$. Therefore $N_s \neq Q = 1$. We have $N_s \to 1/2$ as $y_s \to 0, N_s \to \infty$ as $y_s \to y_{s\,\max}$, and $N_s = 1$ at $y_s = e^{-1/3}$.

For the general case the problem was solved numerically. The soliton amplitude was chosen

$$y_s(\omega) = \exp(-\omega^2/2), \quad 0 < y_s(\omega) < y_{s\,\max}(\omega_{cr}) = e^{-1/4},$$

and the corresponding value $\gamma = y_s^2(\omega)\omega$, was calculated, followed by the determination of the point $y_{\max}(\gamma)$ corresponding to the local maximum of the $U_\gamma(y)$ curve and hence to the maximum amplitude of the pulson oscillations at the given γ. The point y_r was next chosen such that $y_s < y_r < y_{\max}(\gamma)$, and the point y_l corresponding to it was obtained from the condition $U_\gamma(y_l) = U_\gamma(y_r)$. The integral (23) was calculated by Simpson's method, with the integration segment broken up into $2M$ parts ($M = 500$).

Figure 3 shows plots of $U = \pi^{-n/2} G^{-2} l E$ against the continuous variable N for different $U_\gamma(y)$ curves; we shall characterize these curves by the quantity $\alpha = y_s / y_{s\,\max} = y_s e^{1/4}$ (Fig. 4). It is seen that at small α there exist in the case of "double" quantization energy levels with large numbers N. With increasing α, the value of $N_{\max}$ decreases gradually; at $\alpha > \alpha_1 \approx 0.31$ only one level with $N=1$ remains. It is preserved, as indicated above, right up to $\alpha_2 = e^{-1/3} e^{1/4} = e^{-1/12}$; at $\alpha > \alpha_2$ we have $N_s > 1$. A level with $N=2$ appears anew at $\alpha = 0.92$, a level with $N=3$ at $=0.96$, and so on; $N_{\min} \to \infty$ and $N_{\max} \to \infty$ as $\alpha \to 1$. It is seen from Fig. 3 that as $\alpha \to 1$ these levels have practically equal energies, $U_N \to U_{\max} \approx 1.123$; $U_{\max} = U_\gamma(y_{s\,\max})$ at $\gamma = y_{s\,\max}^2 \omega_{cr}$. We note that at all $\alpha \in (0,1)$ there exists at least one level (with integer N). For all the $U_\gamma(N)$ curves, just as in the case of a real field, the inequality $d^2U/dN^2 < 0$ is satisfied.

At small α the "mass" ratio is $E_2/E_1 \approx 2$, and with increasing α this ratio decreases ($E_2/E_1 \approx 1.43$ at $\alpha = 0.3$). We have $U_{N\,\max} \to 1$ and $U_1 \to 0$ as $\alpha \to 0$, and therefore $E_{N\,\max}/E_1 \to \infty$ as $\alpha \to 0$.

Thus, "double" quantization of complex pulsons of the considered model gives rise to a quasiclassical discrete mass spectrum of "particles" having like charge ($Q=1$) and like spin ($S=0$).

The author thanks B. S. Getmanov, V. E. Korepin, N. V. Makhaldiani, V. G. Makhan'kov, I. S. Shapiro, and D. V. Shirkov for useful discussions.

1) Analogous objects at $n=1$ were first observed in Ref. 14.
2) This remark was made by N. V. Makhaldiani.

1. V. E. Korepin and L. D. Faddeev, Teor. Mat. Fiz. 25, 147 (1975).
2. R. F. Dashen, B. Hasslacher, and A. Neveu, a) Phys. Rev. D 10, 4114 (1974); b) Phys. Rev. D 11, 3424 (1975).
3. R. Rajaraman, Phys. Rep. 21, 227 (1975).
4. A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE 61, 1443 (1973).
5. V.G. Makhankov, Phys. Rep. 35, 1 (1978).
6. a) I. L. Bogolyubskiĭ and V. G. Makhan'kov, Pis'ma Zh. Eksp. Teor. Fiz. 24, 15 (1976); 25, 120 (1977) [JETP Lett. 24, 12 (1976); 25, 107 (1977)]; b) I. L. Bogolyubskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. 24, 579 (1976) [JETP Lett. 24, 544 (1976)]; c) I. L. Bogolubsky, Phys. Lett. A 61, 205 (1977).
7. S. Coleman, Phys. Rev. D 11, 2088 (1975).
8. I. Bialnynicki-Birula and J. Mycielski, Bull. Acad. Pol. Sci. Ser. Mat. 23, 461 (1975).
9. G. C. Marques and I. Ventura, IFUSP Preprint P-83, 1976.
10. G. H. Derrick, J. Math. Phys. 5, 1252 (1964).
11. R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).
12. I. Ventura and G. C. Marques, Phys. Lett. B 64, 43 (1976).
13. L. D. Landau and E. M. Lifshitz, Mekhanika (Mechanics), Nauka, 1965, p. 46 [Pergamon, 1968].
14. A. E. Kudryavtsev, Pis'ma Zh. Eksp. Teor. Fiz. 22, 178 (1975) [JETP Lett. 22, 82 (1975)].

Translated by J. G. Adashko