Motion of particles in the field of a naked singularity of Kasner type with complex or equal exponents

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Two exact solutions (metrics) of the gravitational equations are investigated. It is shown that one is axisymmetric and can be obtained from the Weyl metric, and that the other describes a strong gravitational wave of zero frequency in vacuum or around an infinitely long and thin naked singularity of Kasner type. This wave attracts a particle to the singularity or repels it from the singularity, depending on the sign of one of the terms of its metric. The motion of particles in the fields of the two metrics is investigated.

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There exist three exact vacuum solutions of the gravitational equations that depend on a single spatial coordinate x (Ref. 1, p. 492):

$$ds^2=-dx^2+x^{2p_1}dt^2-x^{2p_2}dy^2-x^{2p_3}dz^2,\qquad (1)$$
$$p_1+p_2+p_3=p_1^2+p_2^2+p_3^2=1;$$

$$ds^2=-dx^2+x^{2p'}[(dx_2^2-dx_1^2)\cos\varphi-2dx_1dx_2\sin\varphi]-x^{2p_3}dy^2,\qquad (2)$$
$$\varphi=2p''\ln(x/a),\quad 2p'+p_3=2p'^2-2p''^2+p_3^2=1;$$

$$ds^2=-dx^2+2x^{2p_1}dx_1dx_2\pm x^{2p_1}\ln(x/a)\,dx_2^2-x^{2p_3}dy^2,\qquad (3)$$
$$(p_1,p_3)=(0,1)\quad\text{or}\quad(2/3,-1/3).$$

The first of them is analogous to the Kasner metric; the second and the third, which do not have analogs among the time-dependent metrics, are obtained when the Kasner exponents become complex, $p_{1,2}=p'\pm ip''$, in the case of the metric (2) or equal, $p_1=p_2$, in the case of (3).

The metric (1) was investigated in Ref. 2, in which it was shown that this metric describes the field around a massive linear naked singularity. In the present paper, we investigate the motion of particles in the fields (2) and (3) and the physical meaning of these metrics.

The determinants of (2) and (3), which are both equal to $g=-x^2$, vanish for $x=0$. Since the curvature invariants of the metrics (2) [except the case $(p',p'',p_3)=(0,0,1)$] and (3) for $(p_1,p_3)=(\frac{2}{3},-\frac{1}{3})$ become infinite, these are true singularities, i.e., they cannot be eliminated. For $(p_1,p_3)=(0,1)$, the metric (3) has vanishing invariants and is of Petrov type N, i.e., it describes a strong gravitational wave of zero frequency. Of course, there is some licence in calling such a field a wave, just as there is in saying that an electromagnetic field with $\mathbf{E}\cdot\mathbf{H}=0$ and $|\mathbf{E}|=|\mathbf{H}|=\text{const}$ is a plane wave of zero frequency.

We show that the metric (3) for $(\frac{2}{3},-\frac{1}{3})$ also describes a wave of this kind, but one around a massive linear singularity. For this, we note that the metric

$$ds^2=-dx^2+2x^{2p_1}dx_1dx_2+F(x_2)\ln(x/a)x^{2p_1}dx_2^2-x^{2p_3}dy^2,\qquad (4)$$
$$(p_1,p_3)=(0,1)\quad\text{or}\quad(2/3,-1/3)$$

is also an exact vacuum solution. The curvature invariants of (4) do not depend on F and are exactly the same as for $F=0$. In this last case, the metric (4) reduces to the already investigated metric (1) with $(p_1,p_2,p_3)=(0,0,1)$ or $(\frac{2}{3},\frac{2}{3},-\frac{1}{3})$. The first case corresponds to flat space-time expressed in cylindrical coordinates. In this case

$$x=\rho,\quad x_1=\frac{t+z}{\sqrt{2}},\quad x_2=\frac{t-z}{\sqrt{2}},\quad y=\varphi.$$

In the second case, we have a field around a linear singularity, and

$$x=\rho,\quad x_1=\frac{t+\varphi}{\sqrt{2}},\quad x_2=\frac{t-\varphi}{\sqrt{2}},\quad y=z.$$

The singularity with these Kasner exponents has an interesting property: Massless particles moving in the plane perpendicular to the axis of the singularity can rotate at any distance from the singularity with the same angular momentum, since in this case the gravitational and centrifugal forces balance exactly.

Let us consider the metric (4) with $F\neq 0$. In the region where $F\ln(x/a)\ll 1$, we can assume that the third term in it is a small correction which does not change the nature of the coordinates. For $(p_1,p_3)=(0,1)$, it describes a weak gravitational wave propagating along the axis of the cylindrical coordinate system, and for $(\frac{2}{3},-\frac{1}{3})$ a wave "rotating" around the singularity. In the general case, when $F\ln(x/a)$ is not necessarily small, both gravitational waves become strong. For the case $(0,1)$ this can be clearly seen, since the metric (4) reduces after it has been transformed to a Cartesian coordinate system to a special case of the Peres metric (Ref. 1, p. 446). Returning to the case $F(x_2)=\text{const}$ and using a coordinate transformation to make the wave amplitude F equal to ± 1, we obtain the metric (3). Thus, it describes a strong gravitational wave of zero frequency "propagating" along the axis of the coordinate system $(0,1)$ or "rotating" round the singularity $(\frac{2}{3},-\frac{1}{3})$; in both cases, its amplitude depends only on the distance to the axis.

The only circumstance that does not agree with this interpretation is that this axisymmetric solution cannot be obtained from the Weyl metric. However, the reason for this is that the Weyl metric is from the start sought in diagonal form (Ref. 1, p. 389), to which (3) cannot be transformed without violating its static nature. This indicates that axisymmetric fields cannot be completely described by means of the Weyl metric. The above in-

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terpretation makes it possible to explain why such waves are formed either in vacuum or around the singularity (1) with $(p_1, p_2, p_3) = (\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$. In all remaining cases, massless particles (including gravitons) feel forces of attraction or repulsion when they move along or around the singularity. For $(p_1, p_3) = (0, 1)$, these forces are absent because there is no singularity, and for $(p_1, p_3) = (\frac{2}{3}, -\frac{1}{3})$ they are absent because of the property mentioned above of such a central singularity.

In the presence of the wave, we cannot use the same coordinates t and z as in its absence, since there is a region where t becomes a spacelike and z a timelike coordinate. There are many ways in which one can introduce more appropriate coordinates that do not change their nature; for example

$$\tilde{t} = [x_1(\pm\ln(x/a) + [\ln^2(x/a)+4]^{1/2}) - x_2][\ln^2(x/a)+4]^{-1/4},$$
$$\tilde{z} = [x_1(\mp\ln(x/a) + [\ln^2(x/a)+4]^{1/2}) + x_2][\ln^2(x/a)+4]^{-1/4}, \tag{5}$$

in which the metric takes the form

$$ds^2 = -dx^2 + x^{2p_1}\left[d\tilde{t} - \frac{dx}{2xQ}\left(\pm(\tilde{z}+\tilde{t}) + \frac{1}{Q}\tilde{z}\ln\frac{x}{a}\right)\right]^2$$
$$-x^{2p_1}\left[d\tilde{z} - \frac{dx}{2xQ}\left(\mp(\tilde{z}+\tilde{t}) + \frac{1}{Q}\tilde{z}\ln\frac{x}{a}\right)\right]^2 - x^{2p_3}\,dy^2, \tag{6}$$
$$Q = \left[\ln^2\left(\frac{x}{a}\right) + 4\right]^{1/2}.$$

We recall that for $(p_1, p_3) = (0, 1)$ and $(p_1, p_3) = (\frac{2}{3}, -\frac{1}{3})$ the y and z coordinates, respectively, are angular coordinates.

In the metric (2), the coordinates x_1 and x_2 are alternately timelike and spacelike. Using the transformation

$$\tilde{t} = x_1\sin(\varphi/2) - x_2\cos(\varphi/2), \qquad \tilde{z} = x_1\cos(\varphi/2) + x_2\sin(\varphi/2), \tag{7}$$

we can reduce this metric to the form

$$ds^2 = -dx^2 + x^{2p'}(d\tilde{t} - p''\tilde{z}x^{-1}dx)^2 - x^{2p'}(d\tilde{z} + p''\tilde{t}x^{-1}dx)^2 - x^{2p_3}dy^2. \tag{8}$$

It can then be clearly seen that (8) and (6) are naked singularities, i.e., do not have an event horizon, since they admit the existence of particles at rest arbitrarily close to themselves.

The physical meaning of the singularity (2) is not clear. The relations between its exponents have a solution only if $p_3 < -\frac{1}{3}$ or $p_3 > 1$. With regard to the coordinates in which it is expressed, we note that for $p'' = 0$ the metrics (3) and (8) go over into the metric (1) with $(p_1, p_2, p_3) = (0, 0, 1)$ or $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ and can therefore be expressed in a cylindrical coordinate system; we may therefore suppose that x and y are cylindrical coordinates and x_1 and x_2 are combinations of the remaining cylindrical coordinate and the time. In such a case, the angular coordinate for $p_3 < -\frac{1}{3}$ is z while for $p_3 > 1$ it is y.

To prove that the metric (2) is axisymmetric, we derive it from the Weyl metric. In this case, when the field depends only on ρ, the distance to the axis, the Weyl metric can be reduced to the form

$$ds^2 = -dx^2 + x^{2c/(1-c+c^2)}dt^2 - x^{2(1-c)/(1-c+c^2)}dz^2 - x^{2(c-c^2)/(1-c+c^2)}dy^2,$$

where $x = \rho^{1-c+c^2}$, and $c = \text{const}$ is the Lifshitz–Khalatnikov parameter u (Ref. 1, p. 495) taken with the opposite sign, as is shown in Ref. 2. If c is a complex number such that $1 - c + c^2$ is real, then after the substitution $x_1 = z + it, x_2 = t + iz$ we obtain the metric (2) with

$$p' = \mathrm{Re}[c/(1-c+c^2)], \quad p'' = \mathrm{Im}[c/(1-c+c^2)],$$
$$p_3 = [(c-c^2)/(1-c+c^2)].$$

However, because of the change of scale the angular variable is then defined in an interval $0 \leqslant \varphi < \varphi_{\max}$, where $\varphi_{\max}$ need not be equal to 2π.

We can find the mass of unit length of such a singularity from the formulas (Ref. 1, p. 425)

$$\int R_{x_1}^{x_1}(-g)^{1/2}\,dx\,dx_2\,dy = \oint (-g)^{1/2} g^{ix_1}\Gamma_{ix_1}^{x}\,dx_2\,dy,$$
$$\int R_{x_2}^{x_2}(-g)^{1/2}\,dx\,dx_1\,dy = \oint (-g)^{1/2} g^{ix_2}\Gamma_{ix_2}^{x}\,dx_1\,dy, \tag{9}$$

which hold because the metric does not depend on x_1 or x_2. From these expressions we obtain

$$R_{x_1}^{x_1} = R_{x_2}^{x_2} = {}^1/_4\,p'\delta(x^2). \tag{10}$$

And since for any x we can assume that one of the coordinates x_1 or x_2 is timelike and the other spacelike [if $\cos\varphi = 0$, we can use the symmetry (17)], we can assume that the mass of unit length of the singularity in the given coordinate system is

$$\mu = p'a/2, \quad a = \varphi_{max}/2\pi. \tag{11}$$

Let us consider the motion of test particles in the field with the metric (2). We use the Hamilton–Jacobi equation, which takes the form

$$-\left(\frac{\partial S}{\partial x}\right)^2 + x^{-2p'}\left[\cos\varphi\left(\left(\frac{\partial S}{\partial x_1}\right)^2 - \left(\frac{\partial S}{\partial x_2}\right)^2\right)\right.$$
$$\left.-2\sin\varphi\frac{\partial S}{\partial x_1}\frac{\partial S}{\partial x_2}\right] - x^{-2p_3}\left(\frac{\partial S}{\partial y}\right)^2 = m^2 \tag{12}$$

and can be readily integrated:

$$S = Ax_1 + Bx_2 + Cy + \int F^{1/2}\,dx,$$
$$F = x^{-2p'}[(A^2 - B^2)\cos\varphi - 2AB\sin\varphi] - C^2x^{-2p_3} - m^2. \tag{13}$$

Hence

$$x_1 = \int x^{-2p'}F^{-1/2}(A\cos\varphi - B\sin\varphi)\,dx, \tag{14}$$

$$x_2 = \int x^{-2p'}F^{-1/2}(A\sin\varphi + B\cos\varphi)\,dx, \tag{15}$$

$$y = C\int x^{-2p_3}F^{-1/2}\,dx. \tag{16}$$

Here, the generalized momenta A, B, and C are integrals of the motion. We obtain expressions for $\tilde{z}$ and $\tilde{t}$

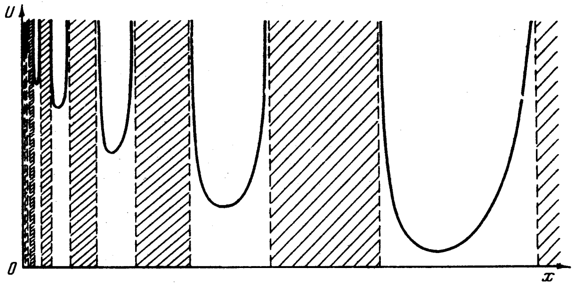

FIG. 1. Dependence of the "potential energy" U on x. The particle can be in one of the infinite number of potential wells, which accumulate at the singularity. In the hatched regions, no kinds of particle can be present; their boundaries are at $x = a\exp[(1+2k)\pi/4 - \tan^{-1}(B/A)]$. The raising of the bottoms of the wells as the singularity is approached occurs as shown in the figure when $p_3 > 1$; when $p_3 < -1/3$, the height of the bottom increases with increasing x.

from (7), (14), and (15). Using the fact the metric (2) is invariant under the transformation

$$x_1'=x_1\cos\alpha+x_2\sin\alpha,\quad x_2'=x_2\cos\alpha-x_1\sin\alpha,\quad \varphi'=\varphi-2\alpha, \tag{17}$$

which carries A and B into

$$A'=A\cos\alpha+B\sin\alpha,\quad B'=B\cos\alpha-A\sin\alpha, \tag{18}$$

we shall in what follows assume that this transformation has been made, and moreover in such a way that $B'=0$, and we shall drop the primes on A' and φ'.

A particle can move in the region where

$$F=A^2x^{-2p'}\cos\varphi-C^2x^{-2p_3}-m^2\geqslant 0. \tag{19}$$

It is clear that the sections where $\cos\varphi<0$ do not belong to this region. For $\cos\varphi>0$, we have the condition $A^2-U^2\geqslant 0$, where

$$U^2=(C^2x^{2(p'-p_3)}+m^2x^{2p'})\sec\varphi$$

can be regarded as a kind of potential. It consists of an infinite number of wells that accumulate at the singularity and are separated by a potential barrier of infinite height and finite width (Fig. 1). Therefore, no particle can go over into a neighboring well even by tunneling. It oscillates around the minimum at

$$\varphi_{min}=2p''\ln\frac{x}{a}=(2k+1)\pi-\text{arctg}\left[\frac{m^2p'+(p'-p_3)C^2x^{-2p_3}}{p''(m^2+C^2x^{-2p_3})}\right]. \tag{20}$$

As $x^{-p_3}\to 0$ or for $C=0$, we have

$$\varphi_{min}=(2k+1)\pi-\text{arctg}(p'/p''),\quad A^2_{min}=m^2x^{2p'}[1+(p'/p'')^2]^{1/2}. \tag{21}$$

As $x^{-p_3}\to\infty$ or for $m=0$,

$$\varphi_{min}=(2k+1)\pi-\text{arctg}[(p'-p_3)/p''],$$
$$A^2_{min}=C^2x^{2(p'-p_3)}[1+(p'-p_3)^2/p''^2]^{1/2}. \tag{22}$$

For $m=C=0$, the particle moves freely in the regions $\cos\varphi>0$, being reflected from the walls at $\varphi=\pi/2+k\pi$. As $p''\to 0$, the width of the well containing the point $x=a$ increases unboundedly, and this well is transformed into the potential of the metric (1).

In the case of motion in the field of the metric that generalizes (2) when A and B can vary, transitions from well to well when there is a change in sign of first one of these quantities and then the other are not ruled out.

We consider the motion of test particles in the field with the metric (3), in which the Hamilton–Jacobi equation can also be readily integrated:

$$S=Ax_1+Bx_2+Cy+\int F^{1/2}\,dx,$$
$$F=\mp x^{-2p_1}\ln(x/a)A^2+2x^{-2p_1}AB-x^{-2p_3}C^2-m^2. \tag{23}$$

Hence

$$x_1=\int x^{-2p_1}F^{-1/2}(\pm A\ln(x/a)-B)\,dx.$$
$$x_2=-A\int x^{-2p_1}F^{-1/2}\,dx,\quad y=C\int x^{-2p_3}F^{-1/2}\,dx, \tag{24}$$

where A, B, and C are, as before, generalized momenta along the axes x_1, x_2, y and are integrals of the motion. A particle can only move in the region $F\geqslant 0$. We sketch this region for $B=\text{const}>0$. If we choose the upper sign in the metric with $(p_1,p_3)=(0,1)$, then we obtain the region shown in Fig. 2. We see that the particle can reach $x=0$ only for $C=0$, i.e., in the case of radial motion, which is natural. For small x,

$$x_1\to x_1^{(0)}-x(-\ln(x/a))^{1/2},\quad x_2\to x_2^{(0)}+x(-\ln(x/a))^{1/2},$$

from which, using the variables introduced in (5), we obtain

$$\tilde{t}\to\tilde{t}^{(0)}-2x\ln(x/a),\quad \tilde{z}\to\tilde{z}^{(0)}+3x/\ln(x/a).$$

For massless particles when

$$B<B_{min}=Ca^{-1}(2e)^{-1/2},$$

where e is the base of natural logarithms, the allowed region for $A>0$ consists of two disconnected parts. For $B>B_{min}$, they merge and form a "tail", which extends to $A=0, x=\infty$. Therefore, for small A massless particles can penetrate into the region of large x for $x<x_{max}$. If

$$C\ll Ba\exp(2BA^{-1})$$

we obtain for x_{max} the expression

$$x_{max}=a\exp(2BA^{-1}).$$

No particles can reach infinity.

We have a similar picture for the metric (3) for $(p_1,p_3)=(\frac{2}{3},-\frac{1}{3})$ and the upper sign (Fig. 3). All particles can reach the singularity. At the same time

$$x_1\to x_1^{(0)}+3x^{1/3}[-\ln(x/a)]^{1/2},\quad x_2\to x_2^{(0)}+3x^{1/3}[-\ln(x/a)]^{-1/2},$$
$$y\to y^{(0)}+3Cx^{7/3}/7[-\ln(x/a)]^{1/2},\quad \tilde{t}\to\tilde{t}^{(0)}+6x^{1/3}\ln(x/a),$$
$$\tilde{z}\to\tilde{z}^{(0)}+9x^{1/3}/\ln(x/a).$$

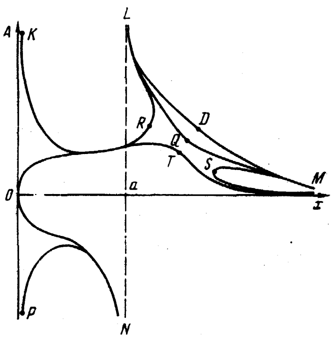

FIG. 2. The region in which a particle can move is shown for the metric (3) with $(p_1,p_3)=(0,1)$, the upper sign, and $B=\text{const}>0$. In the upper quadrant for $m\neq 0$, $C\neq 0$ the region lies within the curve *KRL*; for $C=0$, $m\neq 0$ within *KORL*; for $m=0$, $C\neq 0$, $B<B_{min}$ there are two regions: *KRL* and *MSM*, which for $m=0$, $C\neq 0$, $B>B_{min}$ merge into a "tail" with the particle moving within *KTMQL*; if $m=C=0$, it moves within *KOMDL*. In the lower quadrant for $C\neq 0$ the particle moves within *NP*, and for $C=0$ within *NOP*.

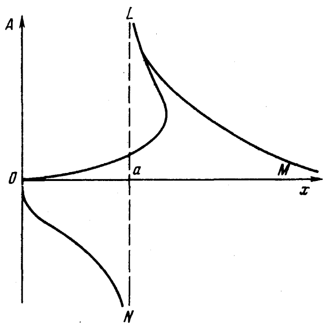

FIG. 3. The metric (3) with $(p_1,p_3)=(2/3,-1/3)$, upper sign, and $B=\text{const}>0$. For $m\neq 0$ or $C\neq 0$, the particle moves to the left of the line *LON*, and for $m=C=0$ to the left of the line *LMON*.

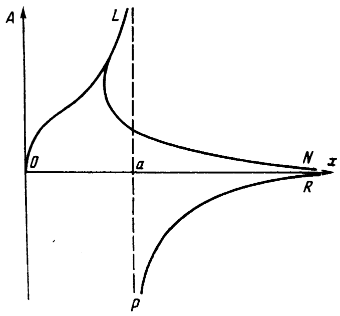

FIG. 4. The metric (3) with $(p_1, p_3) = (0, 1)$, lower sign, and $B = \text{const} > 0$. For $m \neq 0$ or $C \neq 0$, the particle moves to the right of the line $LNRP$, and for $m = C = 0$ to the right of the line $LORP$.

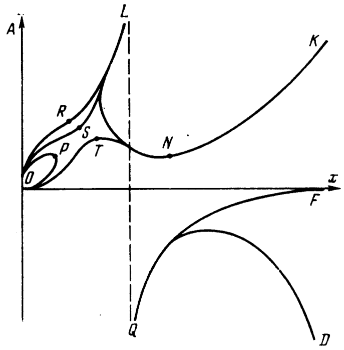

FIG. 5. The metric (3) with $(p_1, p_3) = (2/3, -1/3)$, lower sign, and $B = \text{const} > 0$. In the upper quadrant for $m \neq 0$ or $C \neq 0$ and $B < B_{\min}$ the particle may move within the regions OPO and LNK, which for $B > B_{\min}$ merge into the region $LSOTNK$. For $m = C = 0$, the region of motion is bounded by the curve $LROFQ$. In the lower quadrant for $m \neq 0$ or $C \neq 0$ the particle moves within QD.

In this case, there can be neither a second region nor a "tail", except for the variant $C = m = 0$, when a tail appears and the particle can reach

$$x_{max} = a \exp(2BA^{-1}).$$

From the graph of the allowed region for the metric (3) with $(p_1, p_3) = (0, 1)$ and the lower sign (Fig. 4), we see that particles can reach infinity but not the singularity. Only when $m = C = 0$ does there appear a tail, which reaches the point $A = x = 0$. In this case, the particle can be at

$$x > x_{min} = a \exp(-2BA^{-1}).$$

Particles in the field with the metric (3) for $(p_1, p_3) = (\frac{2}{3}, -\frac{1}{3})$ and the lower sign can reach neither the singularity nor infinity (Fig. 5). Infinity is reached only by particles with $m = C = 0$. For $B < B_{\min}$, the allowed region for $A > 0$ consists of two disconnected parts, while for $B > B_{\min}$ they merge into a tail. The particle can approach not nearer than the distance $x_{\min}$ to the singularity. When

$$C \ll Ba^{-1} \exp(2BA^{-1}), \quad m \ll Ba^{-2/3} \exp(4B/3A)$$

we have

$$x_{min} = a \exp(-BA^{-1}).$$

For $B = 0$, in Figs. 2–5, we should represent not the upper half but the symmetrically reflected lower half; for $m = C = 0$ the allowed region is $x < a$ for Figs. 2 and 3 and $x > a$ for Figs. 4 and 5. For $B < 0$, one must interchange the upper and lower parts in Figs. 2–5, reflecting the graphs symmetrically about the x axis.

Thus, we see that when the upper sign is taken in the metric (3), this corresponds to additional attraction toward the center due to the effect of the gravitational wave of zero frequency. Moreover, it does not permit a particle to move away to infinity. But if we choose the lower sign in the metric, the wave, acting on the particle, repels it from the center, which the particle cannot reach.

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[1] L. D. Landau and E. M. Lifshitz, Teoriya Polya, Nauka (1973); English translation: The Classical Theory of Fields, Pergamon, Oxford (1975).
[2] I. M. Khalatnikov and S. L. Parnovsky, Phys. Lett. A66, 466 (1978).

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