Finite amplitude waves and phase transitions in solids

Yu. Ya. Boguslavskiĭ

Institute of High Pressure Physics, USSR Academy of Sciences
(Submitted 23 February 1978)
Zh. Eksp. Teor. Fiz. 75, 2155–2165 (December 1978)

A complete set of equations describing the average motion in a finite-amplitude wave in solids experiencing first-order phase transitions is obtained by taking temperature relaxation into account. Equations for quasi-simple waves are derived for two models of a finely dispersive two-phase system, i.e., one with a one-dimensional layer structure, and one with uniformly distributed spherical nuclei of the new phase. Expressions are found for massive transition of matter to the new phase in a shock wave. It is shown that the amount of matter transformed into the new phase is determined by the heat flow away from the boundaries between the adjacent phases. Some particular cases are considered.

PACS numbers: 64.60. – i, 62.50. + p

It is well known that many solid materials can exist in different crystalline modifications under different conditions. At certain values of the temperature and pressure, which are connected by a definite relation, transitions can occur that are accompanied by a discontinuity in the volume and (emission) absorption of latent heat, i.e., first-order phase transitions can take place. A mixture of different crystalline modifications can precede the formation of the new phase.[1)]

In the present paper, some regularities of the propagation of finite-amplitude waves in solids are investigated theoretically for solids undergoing first-order phase transitions. It is assumed that the characteristic dimension of the average motion in the wave is much greater than the dimensions of the inhomogeneities in the system. Not too strong shock waves are considered, in which the pressure is not very large; the increase in the entropy is small and the shock adiabat is close to isentropic. As is known, in a solid a shock wave even of 100 kbar is a weak one. Such a wave differs little from an acoustic one, since it propagates with a velocity that is close to the sound velocity and imparts to the material behind the front a velocity that is one tenth the velocity of propagation of the wave itself. At the same time, the pressure in the wave should be sufficiently large that one can neglect effects of rigidity, assuming the wave to be plastic (usually the limits of rigidity are ~1 kbar).

1. FUNDAMENTAL EQUATIONS

We shall assume the pressure to be hydrostatic and consider the following set of equations, which describe the motion of the medium in a finite-amplitude wave under conditions of a first-order phase transition:

$$\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\nabla)\mathbf{u}=-\frac{\nabla P}{\rho}+\frac{\eta}{\rho}\Delta\mathbf{u}+\frac{1}{\rho}\left(\xi+\frac{4}{3}\eta\right)\nabla(\nabla\mathbf{u}),$$
$$\frac{\partial\rho}{\partial t}+\nabla\rho\mathbf{u}=0,$$
$$\frac{\partial S}{\partial t}+(\mathbf{u}\nabla S)=\frac{\operatorname{div}(\varkappa\nabla T)}{\rho T}+\frac{\eta}{\rho T}\left(\frac{\partial u_i}{\partial z_k}+\frac{\partial u_k}{\partial z_i}+\frac{2}{3}\delta_{ik}\nabla u\right)^2+\frac{\xi}{\rho T}(\nabla u)^2,$$
$$\delta\rho=\delta\rho_s+\left(\frac{\partial\rho}{\partial S}\right)_P\delta S. \qquad (1)$$

The set (1) consists of the Navier-Stokes equation, the equation of continuity, the general equation of heat transfer, and the equation of state; $S=xS_2+(1-x)S_1$ is the entropy per unit mass of the mixture of phases, where S_1 and S_2 are the entropies per unit mass of the first and second phases; x is the mass fraction of the second (new) phase in the system:

$$\rho=\frac{1}{V}=\frac{1}{xV_2+(1-x)V_1}$$

is the density of the phase mixture; $V_1=1/\rho_1$, $V_2=1/\rho_2$ are the partial volumes of the first and second phases; $\delta\rho_s$ is change in the density at constant entropy of the mixture; u is the velocity of motion of the medium in the wave; P is the density in the wave; T is the temperature of the medium.

The change in the temperature on the boundary of adjoining phases is determined by the Clapeyron–Clausius equation:

$$\delta T=\frac{T(V_2-V_1)}{q}\delta P;$$

q is the latent heat of transition between phases. From the definition of the entropy of the mixture of phases and the definition of the density, it follows that

$$\delta\rho=-\rho^2\left[x\delta V_2+(1-x)\delta V_1-\frac{V_2-V_1}{S_2-S_1}x\delta S_2-\frac{V_2-V_1}{S_2-S_1}(1-x)\delta S_1\right]+\left(\frac{\partial\rho}{\partial S}\right)_P\delta S. \qquad (2)$$

We average the expression (2) over a volume small in comparison with the characteristic dimension of the mean motion in the wave, but containing a sufficient number of heterogeneous sections of both phases, so that the averaged characteristics of this volume are the same as that for the entire medium as a whole. By considering the changes in volume and entropy at each point as functions of the pressure and temperature, we get

 0038-5646/78/1201086-05$02.40

$$\langle\delta V_2\rangle=\left(\frac{\partial V_2}{\partial P}\right)_T\delta P+\left(\frac{\partial V_2}{\partial T}\right)_P\frac{1}{v_2}\int\limits_{v_2}T_2\,dv_2,$$
$$\langle\delta V_1\rangle=\left(\frac{\partial V_1}{\partial P}\right)_T\delta P+\left(\frac{\partial V_1}{\partial T}\right)_P\frac{1}{v_1}\int\limits_{v_1}T_1\,dv_1,$$
$$\langle\delta S_2\rangle=-\left(\frac{\partial V_2}{\partial T}\right)_P\delta P+\frac{c_{P_2}}{T}\frac{1}{v_2}\int\limits_{v_2}T_2\,dv_2, \qquad (3)$$
$$\langle\delta S_1\rangle=-\left(\frac{\partial V_1}{\partial T}\right)_P\delta P+\frac{c_{P_1}}{T}\frac{1}{v_1}\int\limits_{v_1}T_1\,dv_1,$$

where v_1 and v_2 are the volumes occupied by the phases; T_1 and T_2 are the temperature increases brought about by the varied pressure and thermal conductivity; c_{p1}, c_{p2} are the specific heats at constant pressure of the first and second phase; the symbol $\langle\ldots\rangle$ denotes averaging. We have used in (3) the fact that the pressure changes little at distances of the order of the dimension of the region of averaging. Consequently, the volume over which the averaging is performed is in a uniform pressure field.

The spatial distribution of the temperatures T_1 and T_2 is found from the corresponding general equation of heat transport for each phase, with account of the fact that the temperature change on the boundary of the adjoining phases is determined from the Clapeyron–Clausius equation. Substituting (3) in (2), we obtain an averaged equation of state of the medium. Consequently, the first three equations of the system (1) and the averaged pressure of the state (2) can be regarded as the complete set of equations describing the mean motion of the mixture of material in a wave of finite amplitude in first-order phase transitions.

Unfortunately, the investigation of the solutions of this set is accompanied by great difficulties in the general case. However, in the case of sufficiently small but finite amplitudes and small dissipative coefficients, we can obtain the equations of interest to us on the basis of the set (1) with accuracy to terms of second order inclusive, if we limit ourselves in the latter to nonlinear terms of second order and assume the dissipative coefficients to be small in first order. Then the linear dissipative terms will be small in second order, and the nonlinear dissipative terms can be neglected.[1]

We assume that the wave is propagated along the z axis. Then the first and third equations of the system (1) for the average motion take at the specified accuracy the form

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial z}+\left(\frac{4}{3}\eta+\xi\right)\frac{\partial^2 u}{\partial z^2}, \qquad (4)$$

$$\frac{\partial S}{\partial t}+u\frac{\partial S}{\partial z}=\frac{\varkappa}{\rho T}\frac{\partial^2 T}{\partial z^2}. \qquad (5)$$

Discarding the small third-order term in the equation of continuity, we get

$$\frac{\partial \rho_s}{\partial t}+u\frac{\partial \rho_s}{\partial z}+\rho\frac{\partial u}{\partial z}=-\frac{\varkappa}{\rho T}\left(\frac{\partial\rho}{\partial S}\right)_P\frac{\partial^2 T}{\partial z^2}=-\frac{\varkappa}{\rho T}\left(\frac{\partial\rho}{\partial S}\right)_P\left(\frac{\partial T}{\partial P}\right)_s\frac{\partial^2 P}{\partial z^2}, \qquad (6)$$

$$\delta\rho_s=-\rho^2\left[x\left(\langle\delta V_2\rangle-\frac{V_2-V_1}{S_2-S_1}\langle\delta S_2\rangle\right)+(1-x)\left(\langle\delta V_1\rangle-\frac{V_2-V_1}{S_2-S_1}\langle\delta S_1\rangle\right)\right]. \qquad (7)$$

The distribution of the temperatures T_1 and T_2 in each phase is found from the corresponding equation of heat conduction:

$$\frac{\partial T}{\partial t}-\frac{\varkappa}{\rho c_P}\Delta T-\frac{T(\partial V/\partial T)_P}{c_P}\frac{\partial P}{\partial t}=0 \qquad (8)$$

Equations (4)–(7) comprise the complete set of equations for one-dimensional average motion, with accuracy to terms of second order inclusive. To obtain the final expression for the equation of state, we must consider the specific structure of the two-phase region.

2. ONE-DIMENSIONAL LAYERED STRUCTURE

We first consider a layered structure, assuming that the phases are arranged in alternate layers of thickness h—the second phase, and H—the first phase. The thicknesses of the layers are assumed to be small in comparison with the dimension of the average motion in the wave. If we take the center of a layer of the second phase as the plane $z=0$, then the spatial distribution of the temperature in the layer is obtained from the equation of heat conduction

$$\frac{\partial T_2}{\partial t}-\frac{\varkappa_2}{\rho_2 c_{P2}}\frac{\partial^2 T_2}{\partial z^2}-\frac{T(\partial V_2/\partial T)_P}{c_{P2}}\frac{\partial P}{\partial t}=0 \qquad (9)$$

with the boundary conditions

$$T_2|_{z=\pm h/2}=\frac{T(V_2-V_1)}{q}\delta P \qquad (10)$$

and with the initial condition

$$T_2|_{t=0}=0. \qquad (11)$$

The solution of Eq. (9) with the conditions (10) and (11) is

$$T_2=\frac{T(\partial V_2/\partial T)_P}{c_{P2}}\delta P-\frac{\theta_2}{h}\int\limits_0^t\frac{\partial}{\partial z}Q\left[\frac{z+h/2}{2h},\frac{\chi_2(t-\tau)}{h^2}\right]\delta P(\tau)\,d\tau+\frac{\theta_2}{h}\int\limits_0^t\frac{\partial}{\partial z}Q\left[\frac{h/2-z}{2h},\frac{\chi_2(t-\tau)}{h^2}\right]\delta P(\tau)\,d\tau, \qquad (12)$$

where

$$\theta_2=\frac{T(V_2-V_1)}{q}-\frac{T(\partial V_2/\partial T)_P}{c_{P2}},\qquad Q(z,t)=\sum_{n=-\infty}^{\infty}\exp(2ni\pi z-n^2\pi^2 t)$$

is the Jacobi function.

Averaging of T_2 over the layer h yields

$$\langle T_2\rangle=\frac{1}{h}\int\limits_{-h/2}^{h/2}T_2\,dz=\frac{T(\partial V_2/\partial T)_P}{c_{P2}}\delta P+\frac{8\chi_2}{h^2}\theta_2\int\limits_0^t\sum_{n=0}^{\infty}\exp\left[-\frac{\pi^2\chi_2(2n+1)^2\tau}{h^2}\right]\delta P(t-\tau)\,d\tau. \qquad (13)$$

Carrying out a similar calculation for the first phase of thickness H, we obtain

$$\langle T_1\rangle=\frac{T(\partial V_1/\partial T)_P}{c}\delta P+\frac{8\chi_1}{H^2}\theta_1\int\limits_0^t\sum_{n=0}^{\infty}\exp\left[-\frac{\pi^2\chi_1(2n+1)^2\tau}{H^2}\right]\delta P(t-\tau)\,d\tau, \qquad (14)$$

where χ_1 and χ_2 are the coefficients of thermal diffusivity of the first and second phases,

$$\theta_1=\frac{T(V_2-V_1)}{q}-\frac{T(\partial V_1/\partial T)_P}{c_{P1}}.$$

The quantities $\langle T_2\rangle$ and $\langle T_1\rangle$ will be respectively the same for all alternating layers, since the spatial distribution of the temperature in the layers will be repeated periodically.

Substituting (3) in (7), we obtain, with the help of (13) and (14), an equation of state which takes into account the temperature relaxation for the layered structure of a two-phase medium:

$$\delta\rho_s=x\left[-\rho^2\left(\frac{\partial V_2}{\partial P}\right)_{s2}\delta P+\frac{8\chi_2}{h^2}\rho^2D_2\int_0^t\sum_{n=0}^{\infty}\exp\left\{-\frac{\pi^2}{h^2}\chi_2(2n+1)^2\tau\right\}\delta P(t-\tau)d\tau\right]$$
$$+(1-x)\left[-\rho^2\left(\frac{\partial V_1}{\partial P}\right)_{s1}\delta P+\frac{8\chi_1}{H^2}\rho^2D_1\int_0^t\sum_{n=0}^{\infty}\exp\left\{-\frac{\pi^2}{H^2}\chi_1(2n+1)^2\tau\right\}\delta P(t-\tau)d\tau\right], \quad (15)$$

Here

$$D_2=\frac{T(V_2-V_1)^2}{q^2}\frac{c_{P\sigma2}^2}{c_{P2}},\quad D_1=\frac{T(V_2-V_1)^2}{q^2}\frac{c_{P\sigma1}^2}{c_{P1}};$$

$c_{p\sigma1}$ and $c_{p\sigma2}$ are the specific heats of the first and second phases along the phase equilibrium curve:

$$c_{P\sigma}=c_P-\left(\frac{\partial V}{\partial T}\right)_P\frac{q}{V_2-V_1}.$$

Equations (4)–(6) and (15) comprise the complete set of equations describing the motion of the material in a wave of finite amplitude in the case of a layered structure of a two-phase medium.

We now consider the case in which the relaxation time is sufficiently small in comparison with the characteristic period of the wave, i.e., $t_p\gg H^2/\chi_1$; then the quantity $\delta P(t-\tau)$ in Eq. (15) can be expanded in powers of τ. We limit ourselves to the firm term of the expansion and, taking it into account that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8},\quad \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}=\frac{\pi^4}{96},$$

we can write

$$\delta\rho_s=x\left[B_2\delta P+\frac{\rho^2h^2}{12\chi_2}D_2\frac{\partial P}{\partial t}\right]+(1-x)\times\left[B_1\delta P+\frac{\rho^2H^2}{12\chi_1}D_1\frac{\partial P}{\partial t}\right],$$
$$B_2=-\rho^2\left(\frac{\partial V_2}{\partial P}\right)+\rho^2D_2,\quad B_1=-\rho^2\left(\frac{\partial V_1}{\partial P}\right)+\rho^2D_1. \quad (16)$$

Further, using the usual method of Ref. 1, it is not difficult to obtain from (4), (6) and (16) Burgers equation for a quasi-simple wave:

$$\frac{\partial u}{\partial t}+(C_0+\gamma u)\frac{\partial u}{\partial z}=\mu\frac{\partial^2u}{\partial z^2};$$
$$\mu=\left[\beta+x\rho^2\frac{C_0^4h^2}{24\chi_2}D_2+(1-x)\rho^2\frac{C_0^4H^2}{24\chi_1}D_1\right],$$
$$\beta=\frac{1}{2\rho}\left[\frac{4}{3}\eta+\xi+\varkappa\left(\frac{1}{c_v}-\frac{1}{c_P}\right)\right], \quad (17)$$
$$\gamma=\frac{C_0^4}{2V^3}\left(\frac{\partial^2V}{\partial P^2}\right)_s,$$

where

$$C_0=[xB_2+(1-x)B_1]^{-1/2}$$

is the sound velocity in the mixture of phases in the case of low frequencies, i.e., as $\omega\to0$. We note that this expression for the velocity is identical in accuracy with the formula obtained by Landau and Lifshitz[2]: $\varkappa$ is the mean thermal conductivity of the mixture; c_p, c_v are the mean specific heats of the mixture.

It is seen from Eq. (17) that the propagation of the wave in the phase transition takes place in such fashion as if the body possessed an additional, large viscosity, determined by the last two terms in the expression for μ. We find the amount of matter going into the second phase from Eq. (5), with account of (3), (13) and (14).

At $t_p\gg H^2/\chi_1$, we obtain

$$\delta x=[xc_{P2}+(1-x)c_{P1}]T(V_1-V_2)\rho C_0u/q^2. \quad (18)$$

As is well known, the nonstationary Burgers equation has an exact solution.[3]

Under the initial conditions $u(z\to-\infty,0)\to\Delta u$ and $u(z\to\infty,0)\to0$, any excitation tends toward the stationary form:

$$u=\frac{\Delta u}{1+\exp(\gamma\Delta uz/2\mu)}. \quad (19)$$

The solution (19) is a shock wave with magnitude of jump Δu and with a width of the transition region $2\mu/\gamma\Delta u$. It follows from (18) and (19) that at $x\ll1$,

$$\delta x=c_{P1}\frac{T(V_1-V_2)}{q^2}\frac{\rho_1C_0\Delta u}{1+\exp(\gamma\Delta uz/2\mu)}. \quad (20)$$

Now let the relaxation time be large in comparison with the characteristic period of the wave $t_p\ll h^2/\chi_2\ll H^2/\chi_1$; we can then replace the sum in (15) by an integral. The expression (15) will consequently have the form

$$\delta\rho_s=x\left[-\rho^2\left(\frac{\partial V_2}{\partial P}\right)_{s2}\delta P+2\left(\frac{\chi_2}{\pi}\right)^{1/2}\rho^2\frac{D_2}{h}\int_0^t\frac{\delta P(t-\tau)}{\tau^{1/2}}d\tau\right]$$
$$+(1-x)\left[-\rho^2\left(\frac{\partial V_1}{\partial P}\right)_{s1}\delta P+\frac{2}{H}\left(\frac{\chi_1}{\pi}\right)^{1/2}\rho^2D_1\int_0^t\frac{\delta P(t-\tau)}{\tau^{1/2}}d\tau\right]. \quad (21)$$

In this case, we obtain the following nonlinear integro-differential equation from (4), (6), (21) at $x\ll1$ for the quasi-simple wave:

$$\frac{\partial u}{\partial t}+(C_\infty+\gamma_\infty u)\frac{\partial u}{\partial z}=\beta\frac{\partial^2u}{\partial z^2}$$
$$+\rho_1^2C_\infty^3\left[\frac{\rho_2}{\rho_1}\left(\frac{\chi_2}{\pi}\right)^{1/2}D_2+\left(\frac{\chi_1}{\pi}\right)^{1/2}D_1\right]\frac{1}{H}\int_0^t\frac{\partial u(t-\tau)}{\partial z}\frac{d\tau}{\tau^{1/2}}, \quad (22)$$

Here

$$\gamma_\infty=\frac{C_\infty^4}{2V_1^3}\left(\frac{\partial^2V_1}{\partial P^2}\right)_{s1},\quad C_\infty=\left[-\rho_1^2\left(\frac{\partial V_1}{\partial P}\right)_{s1}\right]^{-1/2}$$

is the sound velocity in the mixture of phases at $x\ll1$, and $\omega\to\infty$.

The expression for the amount of matter undergoing a transition to the second phase at $t_p\ll h^2/\chi_2\ll H^2/\chi_1$, will consequently have the form

$$\delta x=2T\left(\frac{V_1-V_2}{q^2}\right)\frac{\rho_1C_\infty}{H}\left[\frac{\rho_2}{\rho_1}\left(\frac{\chi_2}{\pi}\right)^{1/2}c_{P2}+\left(\frac{\chi_1}{\pi}\right)^{1/2}c_{P1}\right]\int_0^t\frac{u(t-\tau)}{\tau^{1/2}}d\tau. \quad (23)$$

3. MODEL OF A SPHERICAL NUCLEUS

We now consider a finely dispersed medium, consisting of spherical nuclei of the second phase in a volume of the first phase. Calculation can now be carried out under the condition that thermal interaction is absent between the nuclei of the second phase. (The distance between the nuclei of the second phase is greater than the characteristic length of the temperature wave in the first phase.) Then the temperature distribution near the nuclei of the second phase will be the same as for an isolated nucleus of this phase in an infinite medium of the first phase. Since the nuclei of the second phase have spherical shape, the solution of Eq. (8) with the boundary condition

$$\delta T|_{r=R}=\frac{T(V_2-V_1)}{q}\delta P$$

has the form

$$T_2=\frac{T(\partial V_2/\partial T)_P}{c_{P2}}\delta P+\frac{\chi_2}{r}\theta_2\int_0^t\frac{\partial}{\partial r}Q\left[\frac{R-r}{2R},\frac{\chi_2(t-\tau)}{R^2}\right]\delta P(t-\tau)\,d\tau \qquad (24)$$

for the second phase, and

$$T_1=\frac{T(\partial V_1/\partial T)_P}{c_{P1}}\delta P+\frac{R(r-R)}{2r(\pi\chi_1)^{1/2}}\theta_1\int_0^t\exp\left[-\frac{(r-R)^2}{4\chi_1(t-\tau)}\right]\frac{\delta P(\tau)}{(t-\tau)^{3/2}}d\tau \qquad (25)$$

for the first phase. Here

$$Q=\sum_{n=-\infty}^{\infty}\exp\left\{2ni\pi\frac{R-r}{2R}-\frac{n^2\pi^2\chi_2(t-\tau)}{R^2}\right\}$$

is the Jacobi function, r is the distance from the center of the nucleus of the second phase, and R is the radius of the nucleus.

Averaging T_2 and T_1, we obtain

$$\langle T_2\rangle=\frac{T(\partial V_2/\partial T)_P}{c_{P2}}\delta P+\frac{6\chi_2}{R^2}\theta_2\int_0^t\sum_{n=0}^{\infty}\exp\left[-\frac{\pi^2(n+1)^2\chi_2\tau}{R^2}\right]\delta P(t-\tau)\,d\tau, \qquad (26)$$

$$\langle T_1\rangle=\frac{T(\partial V_1/\partial T)_P}{c_{P1}}\delta P+4(\pi\chi_1)^{1/2}NR^2\theta_1\int_0^t\frac{\delta P(t-\tau)}{\tau^{1/2}}d\tau+4\pi\chi_1NR\theta_1\int_0^t\delta P(\tau)\,d\tau, \qquad (27)$$

where N is the concentration of the nuclei of the second phase in the system. Substituting (26) and (27) in (3), we rewrite (7) in the form

$$\delta\rho_S=x\left[-\rho^2\left(\frac{\partial V_2}{\partial P}\right)_{S2}\delta P+\frac{6\rho^2\chi_2}{R^2}D_2\int_0^t\sum_{n=0}^{\infty}\exp\left[-\frac{(n+1)^2\pi^2\chi_2\tau}{R^2}\right]\times\delta P(t-\tau)\,d\tau\right]+(1-x)\left[-\rho^2\left(\frac{\partial V_1}{\partial P}\right)_{S1}\delta P+4(\pi\chi_1)^{1/2}\rho^2R^2ND_1\times\int_0^t\frac{\delta P(t-\tau)}{\tau^{1/2}}d\tau+4\pi N\rho^2R\chi_1D_1\int_0^t\delta P(\tau)\,d\tau\right]. \qquad (28)$$

The expression (28) is the equation of state for a two-phase medium with spherical nuclei in the presence of temperature relaxation.

Equations (4)–(6) and (28) comprise the complete set of equations describing the average motion of material in a wave of finite amplitude in a two-phase medium with spherical nuclei.

Now let the relaxation time be small in comparison with the characteristic period of the wave $t_p\gg R^2/\chi_1\approx R^2/\chi_2$. Under this condition we can expand the quantity $\delta P(t-\tau)$ in the expression (28) in powers of τ. Limiting ourselves to the first term of the expansion and taking it into account that

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)^2}=\frac{\pi^2}{6},\quad \sum_{n=0}^{\infty}\frac{1}{(n+1)^4}=\frac{\pi^4}{90},$$

we obtain

$$\delta\rho_S\approx x\left\{B_2\delta P+\frac{\rho^2R^2}{15\chi_2}D_2\frac{\partial P}{\partial t}\right\}+(1-x)\left\{\left[-\rho^2\left(\frac{\partial V_1}{\partial P}\right)_{S1}+8NR^2\rho^2D_1(\pi\chi_1t_p)^{1/2}\right]\delta P+4\pi N\rho^2R\chi_1D_1\int_0^t\delta P(\tau)\,d\tau\right\}. \qquad (29)$$

Further, from (4), (6), and (29) we find the equation for the quasi-simple wave at $x\ll 1$:

$$\frac{\partial u}{\partial t}+(C+\gamma_\infty u)\frac{\partial u}{\partial z}+\Gamma u=\mu\frac{\partial^2u}{\partial z^2};$$

$$C\approx\left[-\rho_1^2\left(\frac{\partial V_1}{\partial P}\right)_{S1}+4NR^2\rho_1^2D_1(\pi\chi_1t_p)^{1/2}\right]^{-1/2},$$

$$\gamma_\infty=\frac{1}{2}\rho_1^3\left(\frac{\partial P}{\partial\rho_1}\right)_{S1}^2\left(\frac{\partial^2V_1}{\partial P^2}\right)_S,\quad \Gamma=2\pi NR\chi_1\rho_1^2\left(\frac{\partial P}{\partial\rho_1}\right)_{S1}D_1, \qquad (30)$$

$$\mu=\beta+\frac{2\pi}{45}NR^5\frac{\rho_2\rho_1^2}{\rho\chi_1}D_1\left(\frac{\partial P}{\partial\rho_1}\right)_{S1}.$$

From Eq. (5), with the help of (3), (26) and (27), and taking it into account that $x=(4/3)\pi N\rho_2R^3/\rho_1$, we determine the rate of growth of the nucleus of the second phase.

At $t_p\gg R^2/\chi_1$, we get

$$\frac{dR}{dt}=\frac{\chi_1}{R}c_{P1}\frac{\rho_1^2}{\rho_2}\frac{T(V_1-V_2)}{q^2}C_\infty u. \qquad (31)$$

We now consider the evolution of the wave described by Eq. (30) in the case in which we can neglect the high-frequency absorption. Omitting the term on the right side of Eq. (30) and transforming to the new variables

$$z'=z-Ct,\quad t'=t,\quad v=\gamma_\infty u,$$

we obtain

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial z}+\Gamma v=0 \qquad (32)$$

(here the primes are omitted on the independent variables). Equation (32) has an exact solution,[3] which can be written in the form

$$z=\varphi(v)+\frac{1-e^{-\Gamma t}}{\Gamma}v, \qquad (33)$$

where $\varphi(v)$ is an arbitrary function of the velocity.

The nonlinearity leads to a typical distortion of the wave profile, but the wave decays simultaneously. It also follows from (33) that the shock wave can be formed in this case, when $\partial v(z,0)/\partial z<-\Gamma$, i.e., the initial profile should have a sufficiently large negative curvature, otherwise the damping would prevent reversal. If the relaxation time is large in comparison with the characteristic period of the wave $t_p\ll R^2/\chi_1$, then, replacing the sum in (28) by an integral, we get

$$\delta\rho_s = x\left[-\rho^2\left(\frac{\partial V_2}{\partial P}\right)_{s2}\delta P + \frac{3}{R}\left(\frac{\chi_2}{\pi}\right)^{1/2}\rho^2 D_2 \int_0^t \frac{\delta P(t-\tau)}{\tau^{1/2}} d\tau\right]$$

$$+(1-x)\left[-\rho^2\left(\frac{\partial V_1}{\partial P}\right)_{s1}\delta P + 4(\pi\chi_1)^{1/2} N\rho^2 R^2 D_1 \int_0^t \frac{\delta P(t-\tau)}{\tau^{1/2}} d\tau\right]. \qquad (34)$$

The equation for a quasi-simple wave propagating in a two-phase system is found in this case at $x \ll 1$ from (4), (6) and (34):

$$\frac{\partial u}{\partial t} + (C_\infty + \gamma_\infty u)\frac{\partial u}{\partial z} = \beta\frac{\partial^2 u}{\partial z^2} + 2NR^2\rho_1^2 C_\infty^2$$

$$\times\left[\frac{\rho_2}{\rho_1}(\pi\chi_2)^{1/2} D_2 + (\pi\chi_1)^{1/2} D_1\right]\int_0^t \frac{\partial u(t-\tau)}{\partial z}\frac{d\tau}{\tau^{1/2}}. \qquad (35)$$

It is seen that Eq. (35) differs from the usual Burgers equation by the presence of an integral term on the right side.

The rate of growth of the nucleus of the second phase at $t_p \ll R^2/\chi_1$ is

$$\frac{dR}{dt} = \left(\frac{\chi_1}{\pi}\right)^{1/2} c_{p1}\frac{\rho_1^2}{\rho_2}\frac{T(V_1 - V_2)}{q^2} C_\infty \int_0^t \frac{\partial u(t-\tau)}{\partial t}\frac{d\tau}{\tau^{1/2}}. \qquad (36)$$

Writing Eq. (35) in divergence form and integrating it over z from $-\infty$ to $+\infty$, we get

$$\int_{-\infty}^{\infty} u(z,t)\,dz = \int_{-\infty}^{\infty} u(z,0)\,dz,$$

i.e., the area bounded by the function $u(z, t)$ is an integral of the motion.

Linearizing next Eq. (35) and omit from it the term which describes the damping due to ordinary viscosity and thermal conductivity:

$$\frac{\partial u}{\partial t} + C_\infty\frac{\partial u}{\partial z} = -L\int_0^t \frac{\partial u(t-\tau)}{\partial t}\frac{d\tau}{\tau^{1/2}}, \qquad (37)$$

$$L = 2NR^2\rho_1^2 C_\infty^2\left[\frac{\rho_2}{\rho_1}(\pi\chi_2)^{1/2} D_2 + (\pi\chi_1)^{1/2} D_1\right].$$

(We can set $\partial u/\partial t = -C_\infty \partial u/\partial z$ in Eq. (35) with accuracy up to small terms of second order inclusive.)

The linearized Eq. (37) describes the evolution of a weakly damped sound wave with account of the real dispersion of the velocity in the linear approximation. As $t \to \infty$ solutions of the form

$$u = u_0 e^{-\Gamma z} e^{ik(z - Ct)};$$

$$\Gamma = \frac{L}{C_\infty}\left(\frac{\omega\pi}{2}\right)^{1/2}, \quad C = C_\infty\left(1 - L\left(\frac{\pi}{2\omega}\right)^{1/2}\right), \quad \omega = C_\infty k \qquad (38)$$

satisfy it. The fact that $\Gamma \sim \omega^{1/2}$ is entirely natural, since in this case the absorption takes place in a narrow region of order $d \approx (\chi_1 t_p)^{1/2} \ll R$ near the boundary between phases.

It follows from Eqs. (31) and (36) that the growth of the nucleus of the second phase is determined by the heat removal from the boundary of the adjoining phases.

If the temperature wavelength in the first phase is greater than the distance between nuclei of the second phase, i.e., $(\chi_1 t_p)^{1/2} \gg N^{-1/2}$, then the propagation of the quasi-simple wave will be described by Burgers equation, in which

$$\mu \approx \rho^2 C_0^4 N^{-2/3} D_1.$$

The sound velocity C_0 is determined by the expression obtained by Landau and Lifshitz.[2]

1) The process of formation of nuclei of the new phase is not considered here. It is assumed that they are formed instantaneously and in sufficient quantity.

[1]V. I. Karpman, Nelineĭnye volny v dispergiruyushchikha sredakh (Nonlinear waves in dispersive media) Nauka, 1973, pp. 62–64.

[2]L. D. Landau and E. M. Lifshitz, Mekhanika sploshnykh sred (Mechanics of continuous media) Gostekhizdat, 1954, p. 305 [Pergamon, 1958].

[3]G. B. Whitham, Linear and Nonlinear Waves, Russian translation, Mir, 1977, p. 105.

Translated by R. T. Beyer