Integration of the Einstein equations by means of the inverse scattering problem technique and construction of exact soliton solutions

V. A. Belinskiĭ and V. E. Zakharov

L. D. Landau Theoretical Physics Institute, USSR Academy of Sciences, Moscow
(Submitted 13 July 1978)
Zh. Eksp. Teor. Fiz. 75, 1955–1971 (December 1978)

The inverse scattering problem technique is used to integrate the equations of gravity for the case when the metric tensor depends only on two coordinates. The simplest soliton solutions are explicitly calculated.

PACS numbers: 04.20.Jb

§1. INTRODUCTION

The purpose of the present paper is to describe a practical method (equivalent to the inverse scattering problem technique), allowing one to obtain explicitly large classes of new exact solutions of the vacuum Einstein equations for the case when the metric tensor depends only on two variables, if simple particular solutions of the equations are known. Moreover, if developed further, the method allows one in principle to approach the problem of finding, in a certain sense, "all the solutions" of the equations of gravity for the two-dimensional case under consideration, and may lead to a solution of the corresponding Cauchy problem.

For definiteness we assume that the metric tensor depends on time and on one spacelike variable; this corresponds to wavelike and cosmological solutions of the gravitational equations. The case when both variables are spacelike (corresponding to stationary gravitational fields) will not be considered separately, since the corresponding solutions can also be obtained from the analysis given here by imposing certain boundary conditions and carrying out the required complex transformations. Moreover, we limit ourselves to that special (albeit quite widespread) case of two-dimensional metrics where the interval has the form[1)]:

$$-ds^2=f(-dt^2+dz^2)+g_{ab}dx^a dx^b. \qquad (1.1)$$

Here the functions f and g_{ab} depend only on the variables t and z. For the coordinates we adopt the notation $(x^0, x^1, x^2, x^3)=(t, x, y, z)$. In this paper the Latin indices a, b, c, d take on the values 1 and 2 and refer to the variables x and y. We study this metric for the case of a vacuum gravitational field, when the Einstein equations reduce to the vanishing of the Ricci tensor.

A metric of this kind was first considered by Einstein and Rosen[1] for a diagonal matrix g_{ab}, when the Einstein equations actually reduce to one linear equation in cylindrical coordinates. The inclusion of the off-diagonal component[2)] g_{12} changes the situation radically, and converts the Einstein equations into a complicated essentially nonlinear problem. Equations for such a metric were first considered by Kompaneets,[2] who noted some of their general properties. In the past twenty years various authors, using different simplifying assumptions, have obtained a number of exact nontrivial solutions for a metric of the type (1.1) or its stationary analog (a large fraction of these results is listed in the review articles[3,4]), but no regular integration method has been found.

From the physical point of view the metric (1.1) and its stationary analog have many applications in gravitation theory. Suffice it to say that to this class belong the solutions for the Robinson-Bondi plane waves, cylindrical-wave solutions, homogeneous cosmological models of Bianchi types I through VII, the Schwarzschild and Kerr solutions and their NUT-generalizations, Weyl's axially symmetric solution, etc. As applied to cosmology the metric (1.1) was discussed in a paper by Khalatnikov and one of the present authors,[5] where it was shown that such a two-dimensional metric describes a general cosmological solution of the Einstein equations with a physical singularity on portions of the so-called "long eras." In the paper of Gowdy[6] the metric (1.1) was used to find new vacuum solutions representing closed cosmological models. Recently there has been considerable interest in inhomogeneous cosmological models containing singularities having simultaneously a spacelike and a timelike character. Such models have recently been discussed on the basis of the metric form (1.1) in the paper of Tomita.[7] All this shows that, in spite of its relative simplicity, a metric of the type (1.1) encompasses a wide variety of physical cases, and that a method for integrating the corresponding Einstein equations could significantly move forward our understanding of various aspects of gravitation theory.

It turns out that this case can be successfully treated by means of the inverse scattering problem technique in its modified form.[8,9] Moreover, Mikhaĭlov and one of the authors[9] have given a detailed exposition of this new method of integrating nonlinear differential equations, applied to a system which is quite close to the one to which the matrix $g_{ab}(t, z)$ is subject in the present paper. We explain the relation. The Einstein equations for the metric (1.1) are most conveniently investigated in light-cone coordinates ζ, η defined by the transformation

$$t=\zeta-\eta, \quad z=\zeta+\eta. \qquad 1.$$

In the sequel we shall always denote by g the two-dimensional matrix with elements g_{ab} [the two-dimensional block of the metric tensor (1.1)] and for the determinant we adopt the notation

$$\det g=\alpha^2. \qquad (1.3)$$

 0038-5646/78/120985-10$02.40

The complete system of Einstein equations (in vacuum) for the metric (1.1) decomposes into two groups of equations (cf., e.g., Ref. 5). The first group determines the matrix g and can be written in the form of a single matrix equation:

$$(\alpha g_{,\zeta} g^{-1})_{,\eta} + (\alpha g_{,\eta} g^{-1})_{,\zeta} = 0. \tag{1.4}$$

The second group expresses the metric coefficient $f(t, z)$ by quadratures in terms of a given solution of Eq. (1.4) via the relations

$$\begin{aligned} (\ln f)_{,\zeta} &= \frac{(\ln \alpha)_{,\zeta\zeta}}{(\ln \alpha)_{,\zeta}} + \frac{1}{4\alpha\alpha_{,\zeta}} \operatorname{Sp} A^2, \\ (\ln f)_{,\eta} &= \frac{(\ln \alpha)_{,\eta\eta}}{(\ln \alpha)_{,\eta}} + \frac{1}{4\alpha\alpha_{,\eta}} \operatorname{Sp} B^2, \end{aligned} \tag{1.5}$$

where the matrices A and B (introduced for the convenience of the subsequent analysis) are defined as follows:

$$A = -\alpha g_{,\zeta} g^{-1}; \quad B = \alpha g_{,\eta} g^{-1}. \tag{1.6}$$

It is easy to establish (cf. Ref. 5) that the integrability conditions for the equations (1.5) are automatically guaranteed if g is subject to Eqs. (1.3) and (1.4).

If one does not consider Eq. (1.5), the equation (1.4) has formally nontrivial solutions even if $\alpha \equiv 1$. That was the system of equations for a matrix g (in general complex and nonsymmetric) which was investigated in Ref. 9 where its integrability was proved and a procedure was described for the determination of the soliton solutions. Physically, such solutions are related to two-dimensional classical relativistic models of the theory of chiral fields. However, this case ($\alpha \equiv 1$) is not nontrivial when applied to a gravitational field described by the metric (1.1). It is easy to show (cf. Ref. 5) that the presence of the additional field component $f(t, z)$ related to the matrix g via the relations (1.5) leads for $\alpha \equiv 1$ only to the trivial solution, i.e., the Minkowski metric if one requires that the metric be real and have a physical signature.

In connection with this circumstance, the technique developed in Refs. 8, 9 requires some generalization, since one cannot apply it literally to the problem considered here. As will be seen in the sequel, the general idea of the method remains the same: it is based on a study of the analytic structure of the eigenvalues of some operators (as functions of a complex spectral parameter λ), operators which can be associated according to a definite law to the system (1.3), (1.4) (the so-called L-A pair). In particular, for solitonic solutions Eqs. (1.3) and (1.4) a fundamental role is played by the structure of the poles of the corresponding functions in the λ plane. For an α different from a constant the equations (1.3) and (1.4) require the introduction of generalized differential operators thus entering into the L-A pair, depend on the function $\alpha(\zeta, \eta)$, and contain differentiations also with respect to the spectral parameter. For soliton solutions this leads to "floating" poles of the eigenfunctions, and instead of stationary poles $\lambda_n = \text{const}$ (as was the case in Ref. 9) we shall have pole trajectories $\lambda_n(\zeta, \eta)$.

We try to develop our analysis in such a manner that the reading of this article should not require turning to all previous papers, if one is interested mainly in the results of the described method.

§2. THE INTEGRATION SCHEME

We now pass to a systematic investigation of Eqs. (1.3) and (1.4). Taking the trace of Eq. (1.4) with account of the condition (1.3) yields

$$\alpha_{,\zeta\eta} = 0. \tag{2.1}$$

Thus, the square root of the determinant of the matrix g satisfies a wave equation (this result was already noted in Refs. 1, 2) with a solution

$$\alpha = a(\zeta) + b(\eta), \tag{2.2}$$

where $a(\zeta)$ and $b(\eta)$ are arbitrary functions. For the sequel we shall need a second independent solution of Eq. (2.1), which we denote by $\beta(\zeta, \eta)$ and choose in the form

$$\beta = a(\zeta) - b(\eta). \tag{2.3}$$

It should be understood that the metric (1.1) admits in addition arbitrary coordinate transformations $z' = f_1(z + t) + f_2(z - t)$, $t' = f_1(z + t) - f_2(z - t)$ which do not affect the conformally flat form of the metric $f(-dt^2 + dz^2)$ in (1.1). By an appropriate choice of the functions f_1 and f_2 one can bring the functions $a(\zeta)$ and $b(\eta)$ in (2.2) to a prescribed form. If, for instance, the variable $\alpha(\zeta, \eta)$ is timelike (corresponding to solutions of cosmological type[5]) the coordinates can be chosen in such a manner that $\alpha = t$, $\beta = z$. It is however more convenient to carry through the analysis in a general form, without specifying the functions $a(\zeta)$ and $b(\eta)$ in advance, and turning to special cases as the necessity arises.

It is easy to see that Eq. (1.4) is equivalent to a system consisting of the relations (1.6) and two first-order matrix equations that define the matrices A and B. From Eqs. (1.6) and (1.4) follows the first obvious equation for A and B:

$$A_{,\eta} - B_{,\zeta} = 0. \tag{2.4}$$

The second one is easily derived as an integrability condition for the relations (1.6) with respect to g. We obtain in this manner

$$A_{,\eta} + B_{,\zeta} + \alpha^{-1}[A, B] - \alpha_{,\eta}\alpha^{-1}A - \alpha_{,\zeta}\alpha^{-1}B = 0 \tag{2.5}$$

(here and in the sequel the square brackets denote the commutator).

The main step now consists in representing (2.4) and (2.5) in the form of compatibility conditions of a more general overdetermined system of matrix equations related to an eigenvalue–eigenfunction problem for some linear differential operators. Such a system will depend on a complex spectral parameter (which we denote by λ), and the solutions of the original equations for the matrices g, A, and B will be determined by the possible types of analytic structure of the eignevalues in the λ plane. At present there does not exist a general algorithm for the determination of such systems, but for the concrete case of Eqs. (1.3) and (1.4) this can be done. For this purpose we introduce the following differential operators:

$$D_1=\partial_\xi-\frac{2\alpha_{,\xi}\lambda}{\lambda-\alpha}\partial_\lambda,\quad D_2=\partial_\eta+\frac{2\alpha_{,\eta}\lambda}{\lambda+\alpha}\partial_\lambda, \tag{2.6}$$

where the symbol ∂ with a subscript denotes partial differentiation with respect to the corresponding variable and λ is a complex parameter independent of the coordinates ξ and η. It is easy to verify that the commutator of the operators D_1 and D_2 vanishes exactly when α satisfies the wave equation. Thus, taking (2.1) into account we have

$$[D_1, D_2]=0. \tag{2.7}$$

We now introduce the complex matrix function $\psi(\lambda, \xi, \eta)$ and consider the system of equations:

$$D_1\psi=\frac{A}{\lambda-\alpha}\psi,\quad D_2\psi=\frac{B}{\lambda+\alpha}\psi, \tag{2.8}$$

in which the matrices A and B do not depend on the parameter λ and are real (the same requirements are satisfied, of course, by the real function α). It then turns out that the compatibility conditions for the equations (2.8) coincide exactly with the equations (2.4) and (2.5). In order to see this it is necessary to operate with D_2 on the first of the equations (2.8) and with D_1 on the second one, and then subtract the results. On account of the commutativity of D_1 and D_2 we obtain zero in the left-hand side. In the right-hand side we get a rational function of λ which vanishes if and only if the conditions (2.4), (2.5) are satisfied. It is easy to see that a solution of the system (2.8) guarantees not only that the equations satisfied by the matrices A and B are true, but also yields a solution of the relations (1.6), i.e., directly the sought matrix $g(\xi, \eta)$ that satisfies the original equations (1.3) and (1.4). The matrix $g(\xi, \eta)$ is nothing else but the value of the matrix function $\psi(\lambda, \xi, \eta)$ at the point $\lambda = 0$:

$$g(\xi, \eta)=\psi(0, \xi, \eta). \tag{2.9}$$

Indeed, in this case the equations (2.8) for $\lambda = 0$ (for solutions which are regular in the neighborhood of $\lambda = 0$) duplicate exactly the relations (1.6). The matrix $g(\xi, \eta)$ must, of course, be real and symmetric. Below we shall formulate for the selection of the solutions of the equations (2.8) additional restrictions that guarantee this requirement.

The procedure of integration of the equations under consideration assumes the knowledge of at least one particular solution. Let $g_0(\xi, \eta)$ be such a particular solution of the Einstein equations (1.3), (1.4) in terms of which by means of Eq. (1.6) one can determine the matrices $A_0(\xi, \eta)$ and $B_0(\xi, \eta)$, and with the help of (2.8) one can obtain the corresponding function $\psi_0(\lambda, \xi, \eta)$. We now make in the equations (2.8) the substitution

$$\psi=\chi\psi_0. \tag{2.10}$$

Taking into account the fact that ψ_0 satisfies the system (2.8), we obtain the following equations for the matrix $\chi(\lambda, \xi, \eta)$:

$$D_1\chi=\frac{1}{\lambda-\alpha}(A\chi-\chi A_0),\quad D_2\chi=\frac{1}{\lambda+\alpha}(B\chi-\chi B_0). \tag{2.11}$$

We now indicate additional conditions which need to be imposed on the matrix χ in order to assure the reality and symmetry of the matrix g. The first consists in requiring the reality of χ on the real axis of the λ plane (the matrix ψ must also satisfy this condition). This implies

$$\bar{\chi}(\bar{\lambda})=\chi(\lambda),\quad \bar{\psi}(\bar{\lambda})=\psi(\lambda). \tag{2.12}$$

(Here and in the sequel the bar denotes complex conjugation. For the sake of brevity we often do not indicate the arguments ξ and η of the functions.) The second condition is less trivial and is related to the following invariance property of the solutions of the system (2.11). Assume that the matrix $\chi(\lambda)$ satisfies the equations (2.11). Replacing in it the argument λ by α^2/λ we form the new matrix $\chi'(\lambda)$:

$$\chi'(\lambda)=g\tilde{\chi}^{-1}(\alpha^2/\lambda)g_0^{-1}$$

(the tilde denotes transportation of a matrix). A direct verification convinces us that the new matrix $\chi'(\lambda)$ also satisfies the equations (2.12) if g is symmetric. We shall assume $\chi'(\lambda) = \chi(\lambda)$ which guarantees the symmetry of the matrix g. Thus, this condition takes the form

$$g=\chi(\alpha^2/\lambda)g_0\tilde{\chi}(\lambda). \tag{2.13}$$

Moreover, it is necessary to require that for $\lambda\to\infty$ the matrix $\chi(\lambda, \xi, \eta)$ tend to the unit matrix

$$\chi(\infty)=I. \tag{2.14}$$

These relations imply

$$g=\chi(0)g_0, \tag{2.15}$$

a result which also follows from the conditions (2.9)–(2.10).

Thus, the problem now consists in solving the system (2.11) and in determining the matrix χ satisfying the supplementary conditions (2.12), (2.14). It is necessary to note the following important circumstance. The solution $g(\xi, \eta)$ must also satisfy the requirement $\det g = \alpha^2$. We assume that the function $\alpha(\xi, \eta)$ is the same for the particular solution g_0 and for the generalized g [α is a given solution of the wave equation (2.1)], and that by definition the particular solution also satisfies the requirement $\det g_0 = \alpha^2$. Therefore, as follows from (2.15) one must impose on the matrix χ yet another restriction: $\det\chi(0) = 1$. It is more convenient not to worry about this condition during the calculations, and to use a simple renormalization of the final result in order to obtain the correct quantities. The latter will be called the physical quantities. It is easy to establish the legitmacy of this procedure from Eq. (1.4). If we had obtained a solution of that equation with $\det g \neq \alpha^2$, the trace of (1.4) indicates that $\det g$ satisfies the equation

$$[\alpha(\ln\det g)_{,\xi}]_{,\eta}+[\alpha(\ln\det g)_{,\eta}]_{,\xi}=0. \tag{2.16}$$

If one now forms the matrix g_{ph}:

$$g_{ph}=\alpha(\det g)^{-1/2}g, \tag{2.17}$$

it is easy to see that the latter again satisfies the equation (1.40) and moreover the condition $\det g_{ph} = \alpha^2$. The matrices A and B are also subject to appropriate transformations:

$$\begin{aligned}A_{ph}&=A-\alpha\{\ln[\alpha(\det g)^{-1/2}]\}_{,\xi}I,\\ B_{ph}&=B+\alpha\{\ln[\alpha(\det g)^{-1/2}]\}_{,\eta}I,\end{aligned} \tag{2.18}$$

where A and B are defined in terms of g according to (1.6) and A_{ph} and B_{ph} are defined by the same formulas but in terms of the matrix g_{ph}.

§3. CONSTRUCTION OF THE SOLITON SOLUTIONS

The solutions for the matrix $\chi(\lambda, \zeta, \eta)$ are constructed by means of the method described in Refs. 8 and 9. In the general case the determination of χ reduces to solving the Riemann problem of analytic-function theory, which in turn reduces to the solution of a linear integral equation. We shall return to this in § 6 and show there that the solution is determined by the analyticity properties of the matrix χ in the complex λ plane, and in general represents the sum of a soliton part and a nonsoliton part. In this section and in §§ 4, 5 we consider the purely solitonic solutions when the nonsoliton part is absent. This problem does not require the use of the Riemann problem (in fact it is a trivial special case of the Riemann problem) and can be explicitly solved to the end.

The existence of solutions of the soliton type is due to the presence in the λ plane of points of degeneracy (noninvertibility) of the matrix χ, i.e., points at which the determinant of χ vanishes in such a manner that the inverse matrix χ^{-1} has at these points simple poles. Thus, the purely solitonic solutions correspond to the case when χ^{-1} is representable by a rational matrix function of the parameter λ with a finite number of poles (we assume them to be simple) and which for $\lambda \to \infty$ tends to the unit matrix, as required by the condition (2.14). The matrix χ has the same properties, as can be easily seen from the supplementary condition (2.13). Indeed, (2.13) implies that if χ has n poles at the points $\lambda = \mu_k(\zeta, \eta)$ $(k = 1, \ldots, n)$ then χ^{-1} also has n poles at the points $\nu_k(\zeta, \eta)$ where $\nu_k = \alpha^2/\mu_k$. Moreover, it follows from (2.12) that the poles of the matrices χ and χ^{-1} are either on the real axis of the λ plane, or are paired: to each complex pole μ_k (or ν_k) corresponds the complex-conjugate pole $\bar{\mu}_k$ (or $\bar{\nu}_k$). For uniformity in our calculation we shall assume that the poles of the matrix χ are complex and that among them there are no coinciding ones (the equations for the case when the poles are on the real axis can be obtained by taking an appropriate limit).

It follows that the matrix χ has the form:

$$\chi = I + \sum_{k=1}^{n} \left(\frac{R_k}{\lambda - \mu_k} + \frac{\bar{R}_k}{\lambda - \bar{\mu}_k} \right),$$
$$\chi^{-1} = I + \sum_{k=1}^{n} \left(\frac{S_k}{\lambda - \nu_k} + \frac{\bar{S}_k}{\lambda - \bar{\nu}_k} \right), \quad (3.1)$$

where the matrices R_k and S_k (as well as the numerical functions μ_k and $\nu_k = \alpha^2/\mu_k$) no longer depend on λ. The matrices S_k can be expressed in terms of R_k by means of the obvious relation $\chi\chi^{-1} = I$. However, in the sequel we shall deal mainly with χ and the explicit expressions for S_k will not be needed.

It can be seen from (3.1) and (2.15) that the solution of the equations (1.4) for the matrix $g(\zeta, \eta)$ is

$$g(\zeta, \eta) = \left[I - \sum_{k=1} \left(\frac{R_k}{\mu_k} + \frac{\bar{R}_k}{\bar{\mu}_k} \right) \right] g_0. \quad (3.2)$$

We now determine the matrix R_k explicitly. For this it is necessary to substitute (3.1) in (2.11) and to satisfy these equations at the poles $\lambda = \mu_k(\zeta, \eta)$. First of all it can be seen that these equations determine explicitly the dependence of the position of the poles on the coordinates ζ and η, i.e., the functions $\mu_k(\zeta, \eta)$. Indeed, the right-hand sides of (2.11) at the points $\lambda = \mu_k$ have only first-order poles, whereas the left-hand sides $D_1\chi$ and $D_2\chi$ have second order poles. The requirements that the coefficient of the powers $(\lambda - \mu_k)^{-2}$ vanish in the left-hand sides yields the following equations for the pole trajectories $\mu_k(\zeta, \eta)$:

$$\mu_{k,\zeta} = \frac{2\alpha_{,\zeta}\mu_k}{\alpha - \mu_k}, \quad \mu_{k,\eta} = \frac{2\alpha_{,\eta}\mu_k}{\alpha + \mu_k}. \quad (3.3)$$

These equations are invariant with respect to the substitution $\mu_k \to \alpha^2/\mu_k$, i.e., the function $\nu_k = \alpha^2/\mu_k$ also satisfies (3.3). The solutions of (3.3) are roots of the quadratic equation (in λ).

$$\alpha^2/\lambda + 2\beta + \lambda = 2w_k, \quad (3.4)$$

where w_k are arbitrary complex constants. It is easy to see that for each given w_k Eq. (3.4) yields two solutions: a pole $\mu_k(\zeta, \eta)$ for the matrix χ and a pole $\nu_k = \alpha^2/\mu_k$ for the matrix χ^{-1}:

$$\mu_k = w_k - \beta - [(w_k - \beta)^2 - \alpha^2]^{1/2},$$
$$\nu_k = w_k - \beta + [(w_k - \beta)^2 - \alpha^2]^{1/2}. \quad (3.5)$$

Rewriting the equations (2.11) in the form

$$\frac{A}{\lambda - \alpha} = (D_1\chi)\chi^{-1} + \chi \frac{A_0}{\lambda - \alpha} \chi^{-1},$$
$$\frac{B}{\lambda + \alpha} = (D_2\chi)\chi^{-1} + \chi \frac{B_0}{\lambda + \alpha} \chi^{-1}, \quad (3.6)$$

we note that in order that they be satisfied at the poles $\lambda = \mu_k$ it is necessary that the residues at these poles vanish in the right-hand sides of (3.6), since the left-hand sides are holomorphic at the points $\lambda = \mu_k$. This requirement leads to the following equations for the matrices R_k:

$$R_{k,\zeta}\chi^{-1}(\mu_k) + R_k \frac{A_0}{\mu_k - \alpha} \chi^{-1}(\mu_k) = 0,$$
$$R_{k,\eta}\chi^{-1}(\mu_k) + R_k \frac{B_0}{\mu_k + \alpha} \chi^{-1}(\mu_k) = 0, \quad (3.7)$$

where use has been made of the relation

$$R_k\chi^{-1}(\mu_k) = 0, \quad (3.8)$$

following from the identity $\chi\chi^{-1} = I$ (considered at the poles $\lambda = \mu_k$). It can be seen from (3.8) that R_k and $\chi^{-1}(\mu_k)$ are degenerate matrices for which the elements can be written in the form

$$(R_k)_{ab} = n_a^{(k)} m_b^{(k)}, \quad [\chi^{-1}(\mu_k)]_{ab} = q_a^{(k)} p_b^{(k)} \quad (3.9)$$

then (3.8) signifies that

$$m_a^{(k)} q_a^{(k)} = 0. \quad (3.10)$$

Here and in the sequel summation will be understood over repeated vector and tensor indices a, b, c, d (they take the values 1, 2).

Substituting (3.9) into (3.7) we obtain the equations which determine the evolution of the vectors $m_a^{(k)}$:

$$\left(m_{a,\zeta}^{(k)}+m_b^{(k)}\frac{(A_0)_{ba}}{\mu_k-\alpha}\right)q_a^{(k)}=0,$$
$$\left(m_{a,\eta}^{(k)}+m_b^{(k)}\frac{(B_0)_{ba}}{\mu_k+\alpha}\right)q_a^{(k)}=0. \tag{3.11}$$

A solution of these equations is easily expressed in terms of a given particular solution ψ_0 of the equations (2.8). Introducing the matrices

$$M_k=(\psi_0^{-1})_{\lambda=\mu_k}=\psi_0^{-1}(\mu_k,\zeta,\eta), \tag{3.12}$$

it is not hard to see that they satisfy the equations

$$M_{k,\zeta}+M_k\frac{A_0}{\mu_k-\alpha}=0,\quad M_{k,\eta}+M_k\frac{B_0}{\mu_k+\alpha}=0. \tag{3.13}$$

Thus, a solution of the equations (3.11) for the vectors $m_a^{(k)}$ will be[3)]

$$m_a^{(k)}=m_{0b}^{(k)}(M_k)_{ba}, \tag{3.14}$$

where the $m_{0b}^{(k)}$ are arbitrary complex constant vectors.

There remains the task of determining the vectors $n_a^{(k)}$ and thus the matrices R_k. This can be done by means of the supplementary condition (2.13) that must be satisfied by the matrix χ. Substituting (3.1) into (2.13) and considering the relation obtained in this manner at the poles of the matrix $\chi(\alpha^2/\lambda)$, i.e., at the points $\lambda=\nu_k=\alpha^2/\mu_k$, we reach the conclusion that the matrices R_k satisfy the following system consisting of n algebraic matrix equations:

$$R_k g_0\left[I+\sum_{l=1}^{n}\left(\frac{R_l}{\nu_k-\mu_l}+\frac{\bar{R}_l}{\nu_k-\bar{\mu}_l}\right)\right]=0, \tag{3.15}$$

where $k=1,\ldots,n$. Substituting the expression (3.9) for the matrices R_k we obtain a system of linear algebraic equations for the vectors $n_a^{(k)}$:

$$\sum_{l=1}^{n}\frac{m_b^{(l)}m_c^{(k)}(g_0)_{cb}}{\nu_k-\mu_l}n_a^{(l)}+\sum_{l=1}^{n}\frac{\bar{m}_b^{(l)}m_c^{(k)}(g_0)_{cb}}{\nu_k-\bar{\mu}_l}\bar{n}_a^{(l)}=-m_c^{(k)}(g_0)_{ca},$$
$$\sum_{l=1}^{n}\frac{m_b^{(l)}\bar{m}_c^{(k)}(g_0)_{cb}}{\bar{\nu}_k-\mu_l}n_a^{(l)}+\sum_{l=1}^{n}\frac{\bar{m}_b^{(l)}\bar{m}_c^{(k)}(g_0)_{cb}}{\bar{\nu}_k-\bar{\mu}_l}\bar{n}_a^{(l)}=-\bar{m}_c^{(k)}(g_0)_{ca}. \tag{3.16}$$

This completes the determination of the matrices R_k and from (3.2) one can now find a solution for the metric tensor $g(\zeta,\eta)$. We also note that from Eq. (3.6) one can obtain explicit expressions for the matrices A and B by equating the residues in the left-hand and right-hand sides of these equations at the poles $\lambda=\alpha$ and $\lambda=-\alpha$. As a result we obtain:

$$A=2\alpha\alpha_{,\zeta}\left\{\sum_{k=1}^{n}\left[\frac{R_k}{(\alpha-\mu_k)^2}+\frac{\bar{R}_k}{(\alpha-\bar{\mu}_k)^2}\right]\right\}\chi^{-1}(\alpha)+\chi(\alpha)A_0\chi^{-1}(\alpha),$$
$$B=2\alpha\alpha_{,\eta}\left\{\sum_{k=1}^{n}\left[\frac{R_k}{(\alpha+\mu_k)^2}+\frac{\bar{R}_k}{(\alpha+\bar{\mu}_k)^2}\right]\right\}\chi^{-1}(-\alpha)+\chi(-\alpha)B_0\chi^{-1}(-\alpha). \tag{3.17}$$

Calculating the traces $Tr\,A^2$ and $Tr\,B^2$ we obtain from (1.5) the component $f(\zeta,\eta)$ of the metric tensor by quadratures. We note, however, that for those simplest solutions which we consider in the following sections the corresponding indefinite integrals encountered in the calculation of f can be evaluated explicitly and the solution can be expressed in terms of the particular solution g_0, f_0, as well as the quantities μ_k, $m_a^{(k)}$ in algebraic form.

§4. SIMPLE SOLITONS

In this section we consider soliton solutions for the simplest case: when the matrix χ has only one pole. If there is only one pole it can be situated only on the real λ axis (a complex pole has always a complex-conjugate partner).

All the results are easily obtained from the preceding general analysis. The position of the pole is determined by the equation $\lambda=\mu(\zeta,\eta)$, where μ is real and is expressed in terms of α and β according to Eq. (3.5):

$$\mu=w-\beta-[(w-\beta)^2-\alpha^2]^{1/2}; \tag{4.1}$$

here w is a real arbitrary constant. For μ to be real the functions α and β must satisfy the inequality

$$(w-\beta)^2\geqslant\alpha^2, \tag{4.2}$$

the sense of which will become clear later. The matrix χ has the form

$$\chi=I+2R/(\lambda-\mu),\quad R_{ab}=n_a m_b, \tag{4.3}$$

where the vectors m_a and n_a are real. As follows from Eq. (3.12) and (3.14), the vector m_a is determined by the equations

$$m_a=m_{0b}M_{ba},\quad M=(\psi_0^{-1})_{\lambda=\mu}, \tag{4.4}$$

in which the arbitrary constant vector m_{0b} must be taken to be real and the matrix M will automatically be real on account of the conditions (2.12) and the reality of μ. The vector n_a is easily obtained from (3.16) (assuming that all the quantities in them are real and taking into account the fact that there is only one pole):

$$n_a=(\mu^2-\alpha^2)m_b(g_0)_{ba}/2\mu m_c m_d(g_0)_{cd}. \tag{4.5}$$

Furthermore it is convenient to introduce the matrix P with the elements:

$$P_{ab}=m_c(g_0)_{ca}m_b/m_c m_d(g_0)_{cd}. \tag{4.6}$$

From this definition it is clear that P has the properties

$$P^2=P,\quad \det P=0,\quad \mathrm{Tr}\,P=1. \tag{4.7}$$

Now it is easy to express the matrices χ and χ^{-1} in terms of P:

$$\chi=I+\frac{\mu^2-\alpha^2}{\mu(\lambda-\mu)}P,\quad \chi^{-1}=I+\frac{\mu^2-\alpha^2}{\alpha^2-\lambda\mu}P. \tag{4.8}$$

The equation (2.15) yields the matrix g:

$$g=\left(I-\frac{\mu^2-\alpha^2}{\mu^2}P\right)g_0,$$

whence (taking account of $\det g_0=\alpha^2$) it follows that

$$\det g=\alpha^4/\mu^2.$$

Thus, our solution does not satisfy the necessary condition $\det g=\alpha^2$ and we must renormalize it, going over to the physical values in which we are interested, according to the procedure described at the end of § 2.

We will denote (as in § 2) all physical quantities which yield the final result by the subscript "*ph*". In agreement with Eq. (2.17) we have $g_{ph} = \mu\alpha^{-1}g$ and obtain for the metric tensor g_{ph}

$$g_{ph}=\left(\frac{\mu}{\alpha}I-\frac{\mu^2-\alpha^2}{\alpha\mu}P\right)g_0, \tag{4.9}$$

an expression which satisfies both original equations (1.3) and (1.4).

From (4.3) and (4.8) we determine the matrices χ and χ^{-1} at the points $\lambda=\pm\alpha$ and, substituting in Eq. (3.17), we determine the matrices A and B. We next use (2.18) to determine their physical values of A_{ph} and B_{ph} that satisfy Eqs. (2.4) and (2.5) and the relations (1.6) (with g replaced by g_{ph}):

$$\begin{aligned} A_{ph}&=\alpha_{,\zeta}\frac{\alpha+\mu}{\alpha-\mu}(2P-I)+\left(I-\frac{\alpha+\mu}{\mu}P\right)A_0\left(I-\frac{\alpha+\mu}{\alpha}P\right),\\ B_{ph}&=\alpha_{,\eta}\frac{\mu-\alpha}{\mu+\alpha}(2P-I)+\left(I-\frac{\mu-\alpha}{\mu}P\right)B_0\left(I-\frac{\alpha-\mu}{\alpha}P\right). \end{aligned} \tag{4.10}$$

We now calculate the traces $\mathrm{Tr}\,A_{ph}^2$ and $\mathrm{Tr}\,B_{ph}^2$ and substitute the results into the equations (1.5), thus obtaining the physical value f_{ph} of the metric component f. These rather lengthy calculations lead to a simple result: the indefinite integrals which occur in the calculation of f_{ph} in (1.5) turn out to be trivial and are easily calculated, and the final result is

$$f_{ph}=\frac{C\mu m_a m_b (g_0)_{ab}}{[\alpha(w-2a)(w+2b)]^{1/2}}f_0. \tag{4.11}$$

Here C is an arbitrary integration constant, a and b are the arbitrary functions from (2.2), (2.3), and f_0 is the particular solution for the component f corresponding to the particular solution g_0 (the function $f_0(\zeta,\eta)$ satisfies (1.5), where A and B are replaced by A_0 and B_0).

The equations (4.1), (4.4), (4.6), (4.9), and (4.11) give the final solution of the Einstein equations for the case of simple solitons. In order to obtain concrete solutions one must substitute into these equations some concrete particular solutions. In order to illustrate the method we consider the simplest case when the particular solution of the problem is the Kasner solution. It is easy to see that the equations (1.3) and (1.5) have the following exact solution:

$$g_0=\begin{pmatrix}\alpha^{2s_1} & 0\\ 0 & \alpha^{2s_2}\end{pmatrix},\quad f_0=C_0\alpha_{,\zeta}\alpha_{,\eta}\alpha^{s_1^2+s_2^2-1}, \tag{4.12}$$

where C_0 is an arbitrary constant and s_1 and s_2 are constants satisfying the condition $s_1+s_2=1$, so that they can be expressed in terms of one arbitrary constant parameter q:

$$s_1={}^1/_2+q,\quad s_2={}^1/_2-q. \tag{4.13}$$

We now obtain from Eq. (2.8) the corresponding particular solution for the matrix ψ_0. One can choose for it the matrix

$$\psi_0=\begin{pmatrix}(\alpha^2+2\beta\lambda+\lambda^2)^{s_1} & 0\\ 0 & (\alpha^2+2\beta\lambda+\lambda^2)^{s_2}\end{pmatrix}. \tag{4.14}$$

Substituting (4.14) into (4.4) we obtain the vector m_a and then from (4.6), (4.9), and (4.11) we derive the explicit form of the solutions. We write out the final result for the special choice of coordinates when the arbitrary functions $a(\zeta)$ and $b(\eta)$ have the forms:

$$a(\zeta)=\zeta+w/2,\quad b(\eta)=-\eta-w/2. \tag{4.15}$$

This choice means that

$$\alpha=\zeta-\eta=t,\quad \beta=\zeta+\eta+w=z+w \tag{4.16}$$

(in these coordinates the solution (4.12), (4.13) takes on the usual Kasner form, and by means of a simple transformation of the time t it can be transformed to the standard synchronous form).

After simple calculations we obtain the final form of the metric

$$\begin{aligned} -ds^2=&\frac{C_1t^{2q^2}\operatorname{ch}(qr+C_2)}{[z^2-t^2]^{1/2}}(-dt^2+dz^2)+\frac{\operatorname{ch}(s_1r+C_2)}{\operatorname{ch}(qr+C_2)}t^{2s_1}dx^2\\ &+\frac{\operatorname{ch}(s_2r-C_2)}{\operatorname{ch}(qr+C_2)}t^{2s_2}dy^2-\frac{2\operatorname{sh}(r/2)}{\operatorname{ch}(qr+C_2)}t\,dx\,dy, \end{aligned} \tag{4.17}$$

where C_1 and C_2 are arbitrary constants, and the function r is defined in the following manner:

$$e^r=2\frac{z^2}{t^2}-1-2\left[\frac{z^2}{t^2}\left(\frac{z^2}{t^2}-1\right)\right]^{1/2}. \tag{4.18}$$

This is a solution of the cosmological type which cannot be called solitonic in the strict sense, since the velocity of the soliton here exceeds the speed of light. Indeed, let us consider, e.g., the field component g_{11} and determine the position of its extremum with respect to the spacelike variable z for various fixed instants of time t. It can be seen directly that for any t the extremum will correspond to the same constant value of the function $r=r_0=\text{const}$. Then Eq. (2.18) shows that the world line of the extremum has the equation $z=t\cosh(r_0/2)$, and therefore the speed of this localized disturbance exceeds unity.

Thus, we are simply dealing with the time evolution of a given initial state of the field. The situation changes however if one sets $C_1<0$ in (4.17). Then the variable t becomes spacelike and z takes on the meaning of a time. Such a solution is already connected with cylindrical waves and t is the radial coordinate. If one takes the case when the $t=0$ axis is free of singularities, i.e., if one chooses the Kasner indices in the form $s_1=0$, $s_2=1$ ($q=-1/2$), then the extremum of the component g_{11} in the radial variable t also corresponds to the constant value $r=r_0=2C_2$, the world line of the extremum has the same equation as in the preceding case, but now the velocity of the disturbance is smaller than one. Such a solution describes a cylindrical solitary wave incident on the axis and reflected from it.

In both cases the solution (4.17), (4.18) makes sense only for $z^2\geq t^2$. On the light cone $z^2=t^2$ the function r vanishes and the matrix g coincides with the unperturbed particular solution g_0. The solution for g can also be defined in the region $z^2<t^2$ using the following considerations, which have a general character and refer to all soliton solutions related to the real poles of the matrix $\chi(\lambda,\zeta,\eta)$. A real pole $\lambda=\mu$ is always given by the expression (4.1) with a real constant w. If, moving along the coordinate plane, we go from the region (4.2) into a region where $(w-\beta)^2<\alpha^2$, the quantity μ becomes complex and a continuation of the function g into this region will be the solution corresponding to the two-pole situation with $\lambda = \mu$ and $\lambda = \bar{\mu}$, where

$$\mu = w - \beta - i[\alpha^2 - (w-\beta)^2]^{1/2}.$$

However, for such a function we have $|\mu|^2 = \alpha^2$ and the poles are situated on the circle $|\lambda|^2 = \alpha^2$. As will be shown in the next section, the matrix χ is identically equal to the unit matrix if its poles are situated on this circle. This implies that in the region $(w-\beta)^2 < \alpha^2$ the solution g remains unperturbed and coincides identically with the particular solution g_0. The solution as a whole, while remaining itself continuous, suffers discontinuities of the first derivatives on the light cone $(w-\beta)^2 = \alpha^2$ (one can see from Eqs. (2.2), (2.3) that this equation yields a pair of straight lines ζ = const and η = const). This phenomenon requires, of course, additional investigation and appropriate interpretation. We note that such discontinuities do not occur in the solutions correspond to a χ matrix without poles on the real λ axis.

§ 5. TWO-SOLITON SOLUTIONS

In this section we consider the next-complicated case, when the matrix χ has a complex pole $\lambda = \mu$. On account of condition (2.12) it must also have the conjugate pole $\lambda = \bar{\mu}$; we thus deal with two poles. The matrix χ has the form

$$\chi = I + R/(\lambda-\mu) + \bar{R}/(\lambda-\bar{\mu}), \quad R_{ab} = n_a m_b. \tag{5.1}$$

According to (3.14) the vector m_a is

$$m_a = m_{0b} M_{ba}, \quad M = (\psi_0^{-1})_{\lambda=\mu}, \tag{5.2}$$

where m_{0b} is an arbitrary (now complex) vector. The matrix M is also complex. The vector n_a can be found from the equations (3.16), which are now two algebraic equations for n_a and $\bar{n}_a$ (as before, the index k takes on only one value). These equations have the following solution:

$$n_a = \frac{1}{\Delta}\left\{\frac{m_c \bar{m}_d (g_0)_{cd}}{\nu - \bar{\mu}} \bar{m}_b (g_0)_{ba} - \frac{\bar{m}_c \bar{m}_d (g_0)_{cd}}{\bar{\nu} - \bar{\mu}} m_b (g_0)_{ba}\right\},$$
$$\Delta = \frac{|m_a m_b (g_0)_{ab}|^2}{|\nu - \mu|^2} - \frac{|m_a \bar{m}_b (g_0)_{ab}|^2}{|\bar{\nu} - \mu|^2}, \tag{5.3}$$

where $\nu = \alpha^2/\mu$. Substituting (5.3) and (5.2) into the expression $R_{ab} = n_a m_b$ we obtain the matrix R and from Eq. (3.2) we obtain the metric tensor g. We can now calculate the determinant of g and obtain

$$\det g = \alpha^6/\rho^4, \tag{5.4}$$

where ρ is the modulus of μ expressed in the form

$$\mu = \rho e^{i\varphi}. \tag{5.5}$$

Thus, the physical solution g_{ph} of Eqs. (1.3) and (1.4) will be

$$g_{ph} = \rho^2 \alpha^{-2} g, \quad \det g_{ph} = \alpha^2. \tag{5.6}$$

The final expression for g_{ph} is:

$$(g_{ph})_{ab} = \frac{\rho^2}{\alpha^2}(g_0)_{ab} - \frac{\rho^2}{\alpha^2}\left(\frac{n_a m_c}{\mu} + \frac{\bar{n}_a \bar{m}_c}{\bar{\mu}}\right)(g_0)_{cb}, \tag{5.7}$$

where the vectors n_a and m_a are defined by Eqs. (5.2) and (5.3). The function μ is defined as before as the solution of the quadratic equation, in which w is now an arbitrary complex constant. Denoting

$$w = w_1 - i w_2, \tag{5.8}$$

we obtain for the modulus ρ and the phase φ from (3.4) the following system of equations:

$$\cos\varphi = \frac{(2w_1 - 2\beta)\rho}{\alpha^2 + \rho^2}, \quad \sin\varphi = \frac{2w_2\rho}{\alpha^2 - \rho^2}, \tag{5.9}$$

from which we can see that for $w_2 \neq 0$ the poles μ and $\bar{\mu}$ are either always inside the circle $|\lambda|^2 = \alpha^2$ ($\rho^2 < \alpha^2$), or outside it ($\rho^2 > \alpha^2$). For definiteness we shall consider that the poles are inside the circle and $\rho^2 \leq \alpha^2$. It can be seen from Eqs. (5.3) that as the poles tend to the circumference $\rho^2 \to \alpha^2$ the quantity $1/\Delta$ tends to zero like $(\rho^2 - \alpha^2)^2$ and the vector n_a vanishes like $\rho^2 - \alpha^2$. It then follows from Eq. (5.7) that $g_{ph} \to g_0$. Thus, if the poles of the matrix are situated on the circle $\rho^2 = \alpha^2$ the solution g_{ph} remains unperturbed and coincides with the solution g_0.

Having obtained the solutions for g and g_{ph} we can now (just as in the previous case) determine the matrices A and B from (3.17) and their physical values A_{ph}, B_{ph} from (3.18). Substituting the quantities Tr A^2_{ph} and Tr B^2_{ph} into the equations (1.5) we obtain the metric component f_{ph} by quadratures.

In order to illustrate the results we take again for the particular solution g_0, ψ_0, f_0 the Kasner solution (4.12)–(4.14) and consider only two special cases. The first is the isotropic case, when $s_1 = s_2 = 1/2$, and the second is flat space corresponding to $s_1 = 0$, $s_2 = 1$ ($q = -1/2$).

If $s_1 = s_2 = 1/2$ we obtain the following solution for the metric:

$$-ds^2 = C_1\alpha^{1/2}\sigma^{-1}Q(-dt^2+dz^2) + \alpha Q^{-1}\{[p_1^2 H - (1-\sigma)^2\cos 2\varphi$$
$$+ 2p_1(1-\sigma^2)\sin^2\varphi]dx^2 + [p_1^2 H - (1-\sigma)^2\cos 2\varphi - 2p_1(1-\sigma^2)\sin^2\varphi]dy^2$$
$$-2p_2(1-\sigma)^2 \sin 2\varphi\, dx\, dy\}. \tag{5.10}$$

Here we have introduced the notation:

$$Q = p_1^2 H - (1-\sigma)^2, \quad H = 1 + \sigma^2 - 2\sigma\cos 2\varphi, \quad \sigma = \rho^2\alpha^{-2}. \tag{5.11}$$

The quantities C_1, p_1, and p_2 are arbitrary constants restricted by the condition on p_1 and p_2

$$p_1^2 - p_2^2 = 1. \tag{5.12}$$

The functions ρ and φ are determined from the equations (5.9) which involve two other arbitrary constants: w_1 and w_2.

If one picks the coordinates in analogy with (4.15), i.e., in such a manner that $\alpha = t$ and $w_1 - \beta = z$, and if one analyzes the behavior of the field components g_{ab} as a function of the spacelike variable z at different times t, one can see that the solution (5.10)–(5.12) is of the two-soliton type and describes the interaction of two localized disturbances. For any fixed time t the matrix g will tend to the unperturbed solution $g_0 = \mathrm{diag}(t, t)$ at the infinities $z \to \pm\infty$. For all z we have $g_{11} > t$ and $g_{22} < t$. For sufficiently large values of t ($t \gg w_2$) each component g_{ab} has two extrema in the variable z, which are localized near the light cone $z^2 = t^2$. As t decreases these local disturbances start approaching one another, growing in amplitude. As $t \to 0$ (a singularity of cosmological character) both disturbances in the components g_{11} and g_{22} fuse into one concentrated near the origin $z=0$, reaching at this stage some finite amplitude. The disturbances in the component g_{12} do not fuse as $t \to 0$, but approach each other to a finite minimal distance equal to $2w_2$.

By amplitudes we mean the absolute values of the extrema (with respect to z) of the components of the matrix $(g-g_0)g_0^{-1}$. One can prove that as $t \to \infty$ the soliton amplitudes tend to zero, and as $t \to 0$ it is easy to calculate them from the asymptotic form of the matrix g corresponding to the solution (5.10)–(5.12). If $\alpha = t$ and $w_1 - \beta = z$, then as $t \to 0$ we get for g [here we have in mind everywhere the matrix g_{ph}, i.e., the one that appears directly in the physical solution (5.10)]

$$g = t\begin{pmatrix} \dfrac{z^2+s^2w_2^2}{z^2+w_2^2} & \dfrac{1-s^2}{s}\dfrac{zw_2}{z^2+w_2^2} \\ \dfrac{1-s^2}{s}\dfrac{zw_2}{z^2+w_2^2} & \dfrac{s^2z^2+w_2^2}{s^2(z^2+w_2^2)} \end{pmatrix}, \tag{5.13}$$

where $s=(1+p_1)/p_2$.

We now consider the case of solitons on a flat background, when $s_1=0$ and $s_2=1$ and when by means of a coordinate change the particular solution (4.12) can be reduced to the Minkowski metric. In this case the following choice of the functions $a(\zeta)$ and $b(\eta)$ turns out to be convenient:

$$2a = w_2 \operatorname{sh}(z+t) + w_1, \quad 2b = w_2 \operatorname{sh}(z-t) - w_1. \tag{5.14}$$

Whence, and from (2.2), we obtain:

$$\alpha = w_2 \operatorname{sh} z \operatorname{ch} t, \quad w_1 - \beta = -w_2 \operatorname{ch} z \operatorname{sh} t. \tag{5.15}$$

The equations (5.9) are simplest to solve for this choice of the functions α and β. For the modulus ρ and the phase φ we obtain:

$$\sin^2\varphi = \operatorname{ch}^{-2} t, \;\; \cos^2\varphi = \operatorname{th}^2 t, \;\; \rho^2 = \alpha^2 \operatorname{th}^2 (z/2). \tag{5.16}$$

The calculations lead to the following interesting result:

$$\begin{aligned} -ds^2 &= \omega(-dt^2+dz^2) + \omega^{-1}(\gamma + a_1^2 \operatorname{sh}^2 z)\,dx^2 \\ &+ \omega^{-1}[\gamma(2b_1 \operatorname{ch} z - a_1 \operatorname{sh}^2 z)^2 + \operatorname{sh}^2 z\,(r^2+a_1^2+b_1^2)^2]\,dy^2 \\ &- 2\omega^{-1}[\gamma(2b_1 \operatorname{ch} z - a_1 \operatorname{sh}^2 z) + a_1 \operatorname{sh}^2 z\,(r^2+a_1^2+b_1^2)]\,dx\,dy, \end{aligned} \tag{5.17}$$

where we have used the notations

$$\begin{aligned} &\omega = r^2 + (b_1 - a_1 \operatorname{ch} z)^2, \;\; \gamma = (a_1^2 - b_1^2 - m_1^2) \operatorname{ch}^2 t, \\ &r = m_1 + [a_1^2 - b_1^2 - m_1^2]^{1/2} \operatorname{sh} t, \end{aligned} \tag{5.18}$$

and the quantities a_1, b_1, and m_1 are arbitrary constants satisfying the requirement $a_1^2 \geqslant b_1^2 + m_1^2$. We note that the constant w_2 is related to these variables by $w_2^2 = a_1^2 - b_1^2 - m_1^2$. This solution can be obtained from the known Kerr–NUT solution by means of a complex coordinate transformation:

$$\theta = iz, \quad r = m_1 + [a_1^2 - b_1^2 - m_1^2]^{1/2} \operatorname{sh} t, \quad \tau = x, \quad \varphi = y, \tag{5.19}$$

where θ, r, φ, and τ are the Boyer–Lindquist coordinates. For $b_1 = 0$ we obtain the Kerr solution in these coordinates with angle parameter a_1 and the mass m_1. The metric (5.17) then corresponds to the case $a_1 \geqq m_1$. This means that the Kerr solution can be obtained by means of the inverse scattering problem method discussed here, and also directly, by starting from the very outset not with the metric (1.1) but with its stationary analog, and by choosing for the particular or "background" solution the flat space in spherical coordinates. Then the Kerr solution will represent a double stationary soliton.

In conclusion we note that in the derivation of the metrics considered above we have also used linear transformations of the coordinates x, y (with constant coefficients). These have allowed us to remove some inessential constants and to simplify the solutions.

§ 6. ON THE CONSTRUCTION OF SOLUTIONS IN GENERAL

Here we describe briefly a procedure of construction of solutions in the general case, when in addition to solitons there is also a nonsoliton part of the solution.

We define the numerical function $w(\lambda, \zeta, \eta)$ by means of the formula

$$w = {}^1/_2(\alpha^2/\lambda + 2\beta + \lambda). \tag{6.1}$$

It is easy to see that, taking (2.2) and (2.3) into account,

$$D_1 w = 0, \quad D_2 w = 0, \tag{6.2}$$

and consequently for an arbitrary matrix $\Pi(w)$ we also have $D_1\Pi(w) = 0$ and $D_2\Pi(w) = 0$.

We now consider in the complex λ plane the circle $|\lambda|^2 = \alpha^2$ and define on it the matrix function $G_0(\lambda, \zeta, \eta)$, which in general does not admit of analytic continuation off the circle, and depends only on the combination w:

$$G_0 = G_0(w). \tag{6.3}$$

Putting $\lambda = \alpha e^{i\gamma}$ on the circle, verify that the argument of G_0 is real and varies from $-\infty$ to $+\infty$. We require

$$G_0(\infty) = G_0(-\infty) = I. \tag{6.4}$$

Moreover, we shall assume that the matrix G_0 is real and symmetric:

$$\bar{G}_0(\lambda) = G_0(\lambda), \quad G_0 = \tilde{G}_0. \tag{6.5}$$

Let ψ_0 be a particular solution of the equations (2.8). We define on the circle $|\lambda|^2 = \alpha^2$ the new matrix function $G(\lambda, \zeta, \eta)$:

$$G(\lambda, \zeta, \eta) = \psi_0 G_0 \psi_0^{-1}. \tag{6.6}$$

Since $D_{1,2}G_0(w) = 0$, we have the relations

$$D_1 G = \frac{1}{\lambda - \alpha}(A_0 G - G A_0), \quad D_2 G = \frac{1}{\lambda + \alpha}(B_0 G - G B_0). \tag{6.7}$$

One can now show that the determination of the matrix χ is closely related to finding the solution to the following problem (the Riemann problem) from analytic function theory. One is required to find the matrix function χ_1 holomorphic outside the circle $|\lambda|^2 = \alpha^2$, and the matrix function χ_2 holomorphic inside the circle, with the condition that the functions χ_1, χ_2 should satisfy on the circle the condition

$$\chi_1 = \chi_2 G. \tag{6.8}$$

Moreover, one can always require that the following normalization condition hold:

$$\chi_2(\infty)=I. \tag{6.9}$$

If the matrices χ_1 and χ_2 are nonsingular in their domains of analyticity (i.e., their determinants do not have zeroes there), and have no poles, then the solution of the Riemann problem is unique. Acting on (6.8) with the operators D_1 and D_2 and making use of (6.7), it is easy to derive the relations

$$\begin{aligned}\left(D_1\chi_1+\frac{1}{\lambda-\alpha}\chi_1A_0\right)\chi_1^{-1}&=\left(D_1\chi_2+\frac{1}{\lambda-\alpha}\chi_2A_0\right)\chi_2^{-1},\\ \left(D_2\chi_1+\frac{1}{\lambda+\alpha}\chi_1B_0\right)\chi_1^{-1}&=\left(D_2\chi_2+\frac{1}{\lambda+\alpha}\chi_2B_0\right)\chi_2^{-1}.\end{aligned} \tag{6.10}$$

Each of these four expressions is defined (by the way they were derived) on the circle $|\lambda|^2=\alpha^2$, but the equations (6.10) also determine their analytic continuations onto the whole complex λ plane. Since in their domains of analyticity the matrices χ_1, χ_2 are nonsingular and have no poles, the singularities exhibited by these expressions are obvious: the first two have a pole at $\lambda=\alpha$ the latter two have poles at $\lambda=-\alpha$. This implies that the quantities (6.10) have the form

$$\begin{aligned}\left(D_1\chi_1+\frac{1}{\lambda-\alpha}\chi_1A_0\right)\chi_1^{-1}&=\left(D_1\chi_2+\frac{1}{\lambda-\alpha}\chi_2A_0\right)=\frac{1}{\lambda-\alpha}A,\\ \left(D_2\chi_1+\frac{1}{\lambda+\alpha}\chi_1B_0\right)\chi_1^{-1}&=\left(D_2\chi_2+\frac{1}{\lambda+\alpha}\chi_2B_0\right)\chi_2^{-1}=\frac{1}{\lambda+\alpha}B,\end{aligned} \tag{6.11}$$

where A and B are matrices which do not depend on λ. But the equations (6.11) now coincide with the equations (2.11), and since the system (6.11) is compatible, the matrices A and B satisfy the equations (2.4), (2.5). The matrix χ introduced before equals χ_2 (it is holomorphic at the point $\lambda=0$ and tends to the unit matrix for $\lambda\to\infty$) and the matrix

$$g=\chi_2(0)g_0 \tag{6.12}$$

is the metric tensor satisfying the equation (1.4).

The matrices χ_1 and χ_2 must also satisfy some additional conditions similar to the conditions (2.12), (2.13) which follow from the symmetry and reality of the matrix G_0 and of the metric tensor g. These condition are now:

$$\bar{\chi}_{1,2}(\bar{\lambda})=\chi_{1,2}(\lambda),\qquad g=\chi_1(\alpha^2/\lambda)g_0\tilde{\chi}_2(\lambda). \tag{6.13}$$

Until now we have assumed that the matrices χ_1 and χ_2 are invertible in their domains of analyticity and have no poles there. The solution of this regular Riemann problem is reduced to a solution of a singular integral equation, as is well known. If one represents the inverse matrices χ_1^{-1} and χ_2^{-1} in the form

$$\begin{aligned}\chi_1^{-1}&=I+\frac{1}{\pi i}\int_\Gamma\frac{\rho(z)}{\lambda-z+i0}dz,\\ \chi_2^{-1}&=I+\frac{1}{\pi i}\int_\Gamma\frac{\rho(z)}{\lambda-z-i0}dz,\end{aligned} \tag{6.14}$$

where the contour Γ is the circle $|\lambda|^2=\alpha^2$, and then substitutes these expressions into (6.8), one can see easily that the matrix function $\rho(z)$ satisfies the equation

$$\rho(z)+T(z,\zeta,\eta)\left(I+\frac{1}{\pi i}\int_\Gamma\frac{\rho(z')}{z-z'}dz'\right)=0. \tag{6.15}$$

In this equation

$$T=(I-G)(I+G)^{-1} \tag{6.16}$$

is the Cayley transform of the matrix G; the points z and z' are situated on the circle of radius α and the integral is to be taken in the principal value sense.

A solution of the equation (6.15) yields the purely nonsoliton part of the solutions of the original equations (1.3), (1.4). In this case the meaning of the method consists in the fact that the equations (6.15) present considerably fewer difficulties than the original problem of integration of the equations (1.3), (1.4).

If the Riemann problem is not regular and the matrices χ_1 and χ_2 are degenerate (noninvertible) in their domains of analyticity, so that χ_1^{-1} and χ_2^{-1} have pole singularities there, the solutions will also involve solitons. The method exposed here also generalizes without difficulty to that case. In this case the right-hand sides of the expressions (6.14) for the matrices χ_1^{-1}, χ_2^{-1} will contain an additional term: the matrix $U(\lambda,\zeta,\eta)$ of the form

$$U=\sum_k\left(\frac{S_k}{\lambda-\nu_k}+\frac{\bar{S}_k}{\lambda-\bar{\nu}_k}\right), \tag{6.17}$$

which also enters as an additive term into the expression in parentheses in Eq. (6.15). In this case one has to add to the equation (6.15) a system of equations which determine the matrix $S_k(\zeta,\eta)$ (ν_k are the same functions as in the purely solitonic case), but this system contains (linearly) also the contour integrals which occur in (6.14) considered as functions of λ at the poles $\lambda=\nu_k$. The derivation of these equations is simple and is based on the same method as used for the determination of the matrix R_k in the soliton case described above. The form of this complete system of equations will not be given here. We only indicate that the equations which determine pure soliton solutions follow from it in the special case when the matrix G is identically equal to the unit matrix. If $G=I$ it follows from (6.16) and (6.15) that $T=0$, $\rho=0$.

We also note that the soliton of the general system of equations for $\beta\to\pm\infty$ tends to a purely solitonic one. Indeed, since the matrix G_0 is given on the circle $|\lambda|^2=\alpha^2$, we may set in its argument w $\lambda=\alpha e^{i\gamma}$. Then $w=\alpha\cos\gamma+\beta$ and for $\beta\to\pm\infty$ we obtain $w\to\pm\infty$, but on account of the condition (6.4) this implies $G_0\to I$ and from (6.6) it follows that $G\to I$. But according to what was said above, for $G\to I$ the solution goes over into a solitonic one. A similar phenomenon occurs for $\alpha\to\pm\infty$ also.

[1]We use a system of units where the speed of light is one. The four-dimensional metric is written in the form $-ds^2=g_{ik}dx^idx^k$, where g_{ik} has the signature $(-+++)$.

[2]In the language of weak gravitational waves this corresponds to the appearance of a second independent polarization state of the wave. For a stationary analog of the metric (1.1) such a generalization means (under reasonable boundary conditions) that rotation has been included.

[3]In reality, in the solution (3.14) for the vectors $m_a^{(k)}$ there may also be arbitrary complex factors depending on the index k and the coordinates ζ, η. However, such factors reduce to an inessential renormalization of the vectors $m_a^{(k)}$

and disappear from the final expression for the matrices R_k; we therefore set them equal to one.

[1]A. Einstein and N. Rosen, J. Franklin Inst. **223**, 43 (1937).
[2]A. S. Kompaneets, Zh. Eksp. Teor. Fiz. **34**, 953 (1958) [Sov. Phys. JETP **7**, 659 (1958)].
[3]V. P. Frolov, Trudy FIAN **96**, 72 (1977).
[4]G. A. Alekseev and V. I. Khlebnikov, Preprint No. 79, Inst. Appl. Math. USSR Acad. Sci., 1977.
[5]V. A. Belinskiĭ and I. M. Khalatnikov, Zh. Eksp. Teor. Fiz. **57**, 2163 (1969) [Sov. Phys. JETP **30**, 1174 (1970)].
[6]R. H. Gowdy, Ann. Phys. (N.Y.) **83**, 203 (1974).
[7]K. Tomita, Preprint RRK 77-8, Inst. Theor. Phys. Hiroshima University, 1977.
[8]V. E. Zakharov and A. B. Shabat, Fukts. Analiz i ego Prilozh. **12**, No. 3 (1978) [Functional Analysis and Appl. **12**, No. 3 (1978)].
[9]V. E. Zakharov and A. V. Mikhaĭlov, Zh. Eksp. Teor. Fiz. **74**, 1953 (1978) [Sov. Phys. JETP **47**, 1017 (1978)].

Translated by M. E. Mayer

Quasiparticle excitations in a rotating nucleus

I. M. Pavlichenkov

I. V. Kurchatov Institute of Atomic Energy

Zh. Eksp. Teor. Fiz. **75**, 1972–1988 (December 1978)

A method is developed for solving the Hartree-Fock-Bogolyubov equations for the rotational states of an axially deformed nucleus with large angular momentum. The method is based on the quasiclassical approximation and uses a one-dimensional realization of the group $SU(2)$. Rotational states of two intersecting bands (the ground-state band and a band based on a two-quasiparticle excitation from the subshell with maximal j on the Fermi surface) are found in the zeroth approximation in the interaction between them. The point of intersection of these bands corresponds to vanishing of the energy of the two-quasiparticle excitation. The energies of neutron quasiparticle excitations in the $i_{13/2}$ subshell are calculated in the model with rectangular potential well. The results of the calculations agree with the experiments.

PACS numbers: 21.60.Jz, 21.10.Re

§1. INTRODUCTION

Investigation of rotational excitations is an effective method for studying the structure of nuclei. For example, the existence of pairing correlations is most clearly manifested in the value of the moment of inertia of the nucleus.[1] Investigation of the lowest states (up to spin $I=10$) of rotational bands made it possible to establish the degree of adiabaticity of the rotational motion. It was shown that the distortion of the rotational spectrum in even–even nuclei is due to the interaction of the rotation with the quasiparticle degrees of freedom.[2] The parameter of this interaction is the ratio $\alpha = j_F\Omega/\Delta$ of the energy of the Coriolis interaction of a pair to the correlation energy Δ (Ω is the rotation frequency of the nucleus and j_F is the single-particle angular momentum of a nucleon on the Fermi surface). The parameter α is $A^{1/3}$ (A is the number of nucleons in the nucleus) times greater than the parameter of the interaction of the rotational motion with the vibrational motion.[3]

In experiments in recent years on the excitation of rotational levels in reactions with heavy ions in rotational bands there has been discovered an S-shaped dependence of the moment of inertia on Ω^2, this being observed at spins $I \sim 12-16$. This anomaly of the rotational spectrum is known in the English literature as backbending. The numerous attempts to explain this phenomenon reduce ultimately to two alternative hypotheses: 1) The backbending arises as a result of a phase transition at large angular momenta due to the vanishing of the pairing correlation[4] of the anisotropy along the directions of the symmetry axis of the axially deformed nucleus[5]; 2) the backbending is due to the intersection of the ground-state band with a band based on a two-quasiparticle excitation whose angular momentum is aligned along the rotation axis of the nucleus. In the literature, this band has been called the superband. The model was proposed by Stephens and Simon.[6]

Intersection of bands belonging to different phases also occurs in a phase transition. However, the upper parts of the intersecting bands are absolutely unstable and cannot exist in nuclei. Upper and lower levels of intersecting bands on both sides of the intersection point have now been found experimentally[7] in the nuclei Gd^{154}, Dy^{156}, and Er^{164}. The difficulty of detecting upper levels due to their being weakly populated in electromagnetic $E2$ transitions can be successfully overcome if the method of direct Coulomb excitation is used. As a result, it can now be regarded as a reliably established fact that there is no phase transition in the backbending region.

On the other hand, the nature of the superband has not yet been sufficiently well established. In the model of Stephens and Simon, it is a band based on an excitation whose angular momentum is completely decoupled from the deformation. Such bands really are observed in transition nuclei with small deformation. However, backbending also exists in strongly deformed nuclei. This forces us to look for a more general explanation of the phenomenon. We see such an explanation in the

0038-5646/78/120994-09$02.40