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Disordered Ising model at low temperatures

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An Ising model with randomly distributed ferro- and anti-ferromagnetic bonds is treated. A procedure is derived for systematic expansion of the thermodynamic potential at low temperatures, in an arbitrary magnetic field, as a power series in the concentration of antiferro- or ferromagnetic bonds. It is shown that the susceptibility diverges as $1/T$ at $T \rightarrow 0$. The ground-state energy, the magnetic moment, and the residual entropy are calculated as power series in the concentration.

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1. INTRODUCTION

This paper treats the low-temperature behavior of an Ising model with randomly distributed ferro- and antiferromagnetic bonds, equal in absolute value. The treatment is carried out in the case of a square lattice. The method of calculation for a cubic lattice is completely equivalent, and the physical behavior is the same as in the two-dimensional case. The calculation procedure suggested enables one to obtain, in a systematic manner, an expansion of the thermodynamic potential as a power series in, for example, the concentration c of antiferromagnetic bonds in a ferromagnetic matrix:

$$\Phi(J, T, h, c) = \Phi_0 + \Phi_1(J, T, h)c + \Phi_2(J, T, h)c^2 + \dots \quad (1)$$

Here J is the absolute value of the interaction constant, T is the temperature, and h is the magnetic field. The functions Φ_1 , Φ_2 , etc. are calculated to within terms of order $e^{-2J/T}$, i.e., for the case of low temperatures.

In order to obtain the result, one first performs a dual transformation of the partition function in an arbitrary magnetic field. Then, by a procedure that reduces to the enumeration of a certain number of graphs, one calculates successively the functions Φ_1 , Φ_2 , etc. This enables one to calculate the energy of the ground state, the entropy, the magnetic moment, and the susceptibility.

2. DUAL TRANSFORMATION

The Hamiltonian of the model under consideration, for a square lattice, has the form

$$H = - \sum_{i,j} \{ J_{i+\eta_j, j} \sigma_{i,j} \sigma_{i+\eta_j, j} + J_{i, j+\eta_i} \sigma_{i,j} \sigma_{i, j+\eta_i} + h \sigma_{i,j} \}. \quad (2)$$

Here the indices i and j enumerate the sites of the square lattice along the horizontal and vertical directions respectively; $\sigma_{i,j}$ is the spin variable ($\sigma_{i,j} = \pm 1$); and $J_{i+\eta_j, j}$ is the interaction constant. It is assigned on the edges of the lattice cells and is numbered according to the coordinates of their centers. If c is, for example, the concentration of antiferromagnetic bonds in a ferromagnetic matrix, then the interaction constant has the value $+J$ with probability $1-c$ and the value $-J$ with probability c .

The partition function corresponding to (2) can be represented in the following form (N is the number of lattice sites):

$$Z = ch^{2N} (J/T) \text{ch}^N (h/T) \sum_{\sigma=\pm 1} \prod_{i,j} (1 + \text{th}(J_{i+\eta_j, j}/T) \sigma_{i,j} \sigma_{i+\eta_j, j}) \times (1 + \text{th}(J_{i, j+\eta_i}/T) \sigma_{i,j} \sigma_{i, j+\eta_i}) (1 + \text{th}(h/T) \sigma_{i,j}). \quad (3)$$

After summation over all values of σ , there will remain on the right side of (3) a sum over all possible products of factors $\tanh(J_{i+\eta_j, j}/T)$ and $\tanh(h/T)$. It

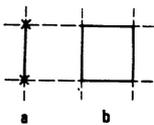


FIG. 1. Examples of graphs of the high-temperature expansion.

is convenient to describe these products by graphs, representing a factor $\tanh(J_{i+1/2,j}/T)$ by a line on the appropriate lattice edge, and a factor $\tanh(h/T)$ by a cross at the appropriate site. Thus, for example, a term of the form

$$\tanh(h/T) \tanh(J_{i-1,i+1/2}/T) \tanh(h/T)$$

corresponds to the graph of Fig. 1a, and a term

$$\tanh(J_{i,i+1/2}/T) \tanh(J_{i+1/2,i+1}/T) \tanh(J_{i+1,i+3/2}/T) \tanh(J_{i+3/2,i+2}/T)$$

to the graph of Fig. 1b (the lattice is represented by dashed lines). If the number of lines into any vertex is even, then it contains no cross. In fact a cross corresponds to the beginning or end of some line. In the absence of crosses, at $h=0$, there are only closed-line graphs. This expansion is convenient at high temperatures, when J/T is small, and is called the high-temperature expansion.¹

For investigation of the low-temperature behavior, it is convenient to perform a dual transformation with the partition function (3). In the case of an arbitrary magnetic field, such a transformation was suggested by Wegner² for a ferromagnetic lattice. The generalization of this transformation to the case of a disordered Ising model is trivial.

We consider a system of spins located at the centers of the edges of the original lattice (Fig. 2), with interaction Hamiltonian

$$H^* = -T^* \sum_{i,j} \{ K s_{i+1/2,j} s_{i,j-1/2} s_{i-1/2,j} s_{i,j+1/2} + B_{i+1/2,j} s_{i+1/2,j} + B_{i,j+1/2} s_{i,j+1/2} \}. \quad (4)$$

The partition function corresponding to H^* has the form

$$Z^* = \sum_{s=\pm 1} \exp \left\{ \sum_{i,j} (K s_{i+1/2,j} s_{i,j-1/2} s_{i-1/2,j} s_{i,j+1/2} + B_{i+1/2,j} s_{i+1/2,j} + B_{i,j+1/2} s_{i,j+1/2}) \right\}. \quad (5)$$

Here $s_{i+1/2,j}$ is the spin variable ($s_{i+1/2,j} = \pm 1$), and T^* is the temperature of the dual lattice. The constants K and $B_{i+1/2,j}$ are defined below. The four spins whose product occurs in (5) are marked by the dashed lines in Fig. 2.

We consider expansion of (5) as a series in inverted spins. The zero-order term corresponds to every $s=1$.

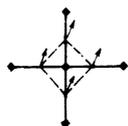


FIG. 2. Lattice of the dual model.

It is

$$Z_0 = \exp \left\{ NK + \sum_{i,j} (B_{i+1/2,j} + B_{i,j+1/2}) \right\}.$$

If a single spin is reversed, for example $s_{i-1/2,j+1/2}$, then the corresponding term in the partition function will have the form

$$Z_i = Z_0 \exp \{-2K - 2B_{i-1,i+1/2} - 2K\}.$$

We now consider four reversed spins, located on the edges of a square. Their contribution to the partition function is

$$Z_i = Z_0 \exp \{-2B_{i,i+1/2} - 2B_{i+1/2,i+1} - 2B_{i+1,i+3/2} - 2B_{i+3/2,i+2}\}.$$

The factor e^{-2K} is absent from Z_4 , since, for example, in the term

$$K s_{i+1/2,j} s_{i,j-1/2} s_{i-1/2,j} s_{i,j+1/2}$$

two spins change sign, and as a result the resultant sign does not change.

The contributions of the various spin configurations can be represented graphically if a reversed spin (factor $\exp(-2B_{i+1/2,j})$) is represented by a line going along the corresponding edge, and a factor e^{-2K} by a cross. Then if an even number of lines pass through a vertex, the cross is absent. It corresponds to the beginning or end of a line. It is easy to show that every graph in the expansion of (3) and Z can be put into unique correspondence with a graph of a spin configuration in (5). In the examples considered, Z_1 corresponds to the graph of Fig. 1a and Z_4 to that of Fig. 1b. If we impose on K and $B_{i+1/2,j}$ the conditions

$$\begin{aligned} \exp(-2K) &= \tanh(h/T), \\ \exp(-2B_{i+1/2,j}) &= \tanh(J_{i+1/2,j}/T), \end{aligned}$$

then the partition functions Z and Z^* will be proportional to each other.

By representing Z^* in a form analogous to the expression (3) for Z , one can obtain the partition function of the original lattice in the form

$$\begin{aligned} Z &= 2^{-2N} \exp \{ (2JN(1-2c) + |h|N)/T \} \\ &\times \sum_{s=\pm 1} \prod_{i,j} [1 + \exp(-2|h|/T) s_{i+1/2,j} s_{i,j-1/2} s_{i-1/2,j} s_{i,j+1/2}] \\ &\times [1 + \exp(-2J_{i+1/2,j}/T) s_{i+1/2,j}] [1 + \exp(-2J_{i,j+1/2}/T) s_{i,j+1/2}]. \quad (6) \end{aligned}$$

It is easily verified that Z is invariant to a change of sign of h ; therefore what occurs in (6) is $|h|$.

We remark that the transformation performed can be generalized directly to a cubic lattice.

3. EXPANSION IN POWERS OF THE CONCENTRATION

We introduce at each bond an operator $\hat{v}_{i+1/2,j}$ that takes two values, 0 and 1. If this is the operator of occupancy by antiferromagnetic bonds, then $\hat{v}'_{i+1/2,j} = 1 - \hat{v}_{i+1/2,j}$ is the operator of occupancy by ferromag-

netic bonds. If we denote by angular brackets $\langle \dots \rangle$ an average over all distributions of bonds on the lattice, then it is clear that $\langle \hat{\nu} \rangle = c$, $\langle \hat{\nu}' \rangle = 1 - c$. We shall consider some properties of the operator $\hat{\nu}$.

Suppose that it is necessary to calculate $f(\hat{\nu}\alpha)$. We represent it in the form

$$f(\hat{\nu}\alpha) = f_0 + \hat{\nu}f_1.$$

Since the equality must be satisfied on the discrete set of values $\hat{\nu} = (0, 1)$, this is the most general form for $f(\hat{\nu}\alpha)$. On setting $\nu = 0, 1$ we obtain $f_0 = f(0)$, $f_1 = f(\alpha) - f(0)$. Similarly one can write for the function $f(\hat{\nu}_1 + \hat{\nu}_2)\alpha$

$$f((\hat{\nu}_1 + \hat{\nu}_2)\alpha) = f_0 + (\hat{\nu}_1 + \hat{\nu}_2)f_1 + \hat{\nu}_1\hat{\nu}_2f_2.$$

The appearance of an f_2 term is due to the fact that on the basis of operators $\hat{\nu}_1$ and $\hat{\nu}_2$ one can construct also the operator $\hat{\nu}_1\hat{\nu}_2$. We set $\nu_1 = \nu_2 = 0$; this gives $f_0 = f(0)$. If we take $\nu_2 = 0$ and $\nu_1 = 1$, we can get $f_1 = f(\alpha) - f(0)$. Finally, setting $\nu_1 = \nu_2 = 1$, we get $f_2 = f(2\alpha) - f(0) - 2[f(\alpha) - f(0)]$.

If we apply this procedure to $f((\hat{\nu}_1 + \hat{\nu}_2)\alpha + \hat{\nu}_1\hat{\nu}_2\beta)$, we easily get

$$f((\hat{\nu}_1 + \hat{\nu}_2)\alpha + \hat{\nu}_1\hat{\nu}_2\beta) = f(0) + (\hat{\nu}_1 + \hat{\nu}_2)[f(\alpha) - f(0)] + \hat{\nu}_1\hat{\nu}_2[f(2\alpha + \beta) - 2f(\alpha) + f(0)].$$

We shall call the number of different operators in any product of operators $\hat{\nu}$ the order of that product. It is clear that by setting certain of the $\hat{\nu}$'s equal to zero, one can always reduce to zero the values of operators of order larger than some prescribed order, though the operators of lower order will have nonzero values. This enables us to calculate the expansion coefficients successively, order by order, as was done in the examples given. Then the taking into account of operators of higher order does not change the expansion coefficients for operators of lower order. This fact enables us to carry out a systematic expansion as a power series in the concentration.

For an arbitrary distribution of ferro- and antiferromagnetic bonds, the partition function Z can be rewritten in terms of the operators $\hat{\nu}$ in the form

$$Z = Z' \exp \{ [2JN(1-2c) + |h|N] / T \},$$

$$Z' = 2^{-2N} \sum_{\nu_1, \nu_2} \prod_{i,j} \{ 1 + e^{-2|N|/T} s_{i-\nu_1, j} s_{i, j+\nu_2} s_{i+\nu_1, j} s_{i, j-\nu_2} \}$$

$$\times \{ 1 + [e^{-2J/T} + \hat{\nu}_{i-\nu_1, j} (e^{2J/T} - e^{-2J/T})] s_{i-\nu_1, j} \}$$

$$\times \{ 1 + [e^{-2J/T} + \hat{\nu}_{i, j-\nu_2} (e^{2J/T} - e^{-2J/T})] s_{i, j-\nu_2} \}.$$
(7)

For the value of Z' after averaging over s , one can obtain an expansion in powers of $e^{2J/T}$, analogous to the expansion (3). Since the first term in the sum for Z' is unity, in the low-temperature limit one can discard terms of order $e^{-2J/T}$ and below. It is convenient to represent the expansion graphically. A factor $e^{-2J/T}$ corresponds to a wavy line passing through the center of the corresponding bond and perpendicular to

it. A factor

$$\hat{\nu} (e^{2J/T} - e^{-2J/T}) \approx \hat{\nu} e^{2J/T}$$

corresponds to a solid line, also arranged perpendicular to the corresponding bond. Each lattice site surrounded by a contour of the graph corresponds to a factor $\exp(-2|h|/T)$.

From the rule of construction it is clear that the graphs of the expansion for Z' coincide, with respect to the form of their contour, with the graphs of the expansion (3) but are constructed from two types of lines. At $T \rightarrow 0$, only those graphs will be important in which the number of solid lines is not less than the number of wavy ones. Examples of the simplest graphs are represented in Fig. 3 (the lattice is shown by dashed lines). For example, the graph of Fig. 3a is obtained after averaging of a term

$$\sum_{i=-1}^1 \{ \hat{\nu}_{i-\nu_1, j} e^{2J/T} s_{i-\nu_1, j} e^{-2J/T} s_{i, j+\nu_2} e^{-2J/T} s_{i+\nu_1, j} s_{i, j-\nu_2} \}$$

$$\times \hat{\nu}_{i, j-\nu_2} e^{2J/T} s_{i, j-\nu_2} e^{-2|N|/T} s_{i-\nu_1, j} s_{i, j+\nu_2} s_{i+\nu_1, j} s_{i, j-\nu_2}.$$

The graph of Fig. 3c denotes the following term in Z' :

$$\exp \{ (4J - 2|h|) / T \} \hat{\nu}_{i, j-\nu_2} \hat{\nu}_{i-\nu_1, j} \hat{\nu}_{i+\nu_1, j}.$$

After averaging over s , operators occur that represent sums of different products of $\hat{\nu}$'s. Since only terms proportional to N occur in the expansion of the thermodynamic potential, operators proportional to powers of N higher than the first will make no contribution to the thermodynamics. Therefore the important operators have the form

$$\hat{\tau}_\alpha^{(n)} = \sum_{i,j} \prod_{(k,l)} \hat{\nu}_{i+k, j+l}. \quad (8)$$

Here the sum extends over all sites (i, j) of the lattice. The index α numbers the sets $\{k, l\}$. These sets are represented by solid lines on the graphs. The index n describes the order of the operator.

Since the contour of the simplest graph is a square, the first nonvanishing terms in the limit $T \rightarrow 0$ are of the second order in $\hat{\nu}$ and correspond to graphs 3a and 3b. Therefore in the lowest order in $\hat{\nu}$, the expansion for Z' has the form

$$Z' = 1 + \exp(-2|h|/T) (\hat{\tau}_1^{(2)} + \hat{\tau}_2^{(2)}).$$

The operators $\hat{\tau}_1^{(2)}$ and $\hat{\tau}_2^{(2)}$ are constructed in accordance with the definition (8). For each site, they are determined by the graphs of Figs. 3a and 3b. There are four different graphs per site of the first type, and

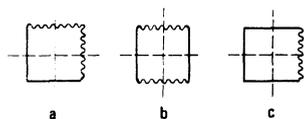


FIG. 3. Examples of graphs for the dual model.

two of the second. Therefore

$$\langle \hat{\tau}_1^{(2)} \rangle = 4Nc^2, \quad \langle \hat{\tau}_2^{(2)} \rangle = 2Nc^2.$$

The expansion of $\ln Z'$ must contain only operators proportional to N , i.e., operators $\hat{\tau}$. From products of operators $\hat{\tau}_1^{(2)}$, $\hat{\tau}_2^{(2)}$ we may not separate out new operators of the second order; therefore to the lowest order in $\hat{\nu}$, the expansion of $\ln Z'$ has the form

$$\ln[1 + e^{-2|N|/T} (\hat{\tau}_1^{(2)} + \hat{\tau}_2^{(2)})] = f_0 + \hat{\tau}_1^{(2)} f_1 + \hat{\tau}_2^{(2)} f_2.$$

First let every $\nu = 0$; this gives $f_0 = 0$. We then set equal to zero the values of almost all the ν 's excepting the two that belong to site (i, j) and correspond, for example, to the graph of Fig. 3a. Then $\nu_1^{(2)} = 1$, $\tau_2^{(2)} = 0$, whence follows

$$f_1 = \ln(1 + e^{-2|N|/T}).$$

Similarly we find $f_2 = f_1$.

The expression for the thermodynamic potential has the form

$$\begin{aligned} \frac{\Phi}{N} = -\frac{T}{N} \langle \ln Z \rangle = -2J + 4Jc - |h| - \frac{T}{N} \langle \hat{\tau}_1^{(2)} + \hat{\tau}_2^{(2)} \rangle \ln(1 + e^{-2|N|/T}) \\ = -2J + 4Jc - |h| - 6c^2 T \ln(1 + e^{-2|N|/T}). \end{aligned}$$

In the third order in $\hat{\nu}$, Z' will contain the graphs of Fig. 3c and of Figs. 4a-e. We denote the operators corresponding to them by $\hat{\tau}_0^{(3)}$, $\hat{\tau}_1^{(3)}$, $\hat{\tau}_2^{(3)}$, $\hat{\tau}_3^{(3)}$, $\hat{\tau}_4^{(3)}$, and $\hat{\tau}_5^{(3)}$. From products of operators $\hat{\tau}_1^{(2)}$ and $\hat{\tau}_2^{(2)}$ one can separate out new operators of the third order. They are represented in Figs. 4h-i and are denoted by $\hat{\tau}_6^{(3)}$, $\hat{\tau}_7^{(3)}$, $\hat{\tau}_8^{(3)}$, and $\hat{\tau}_9^{(3)}$ respectively.

We write the expansion of $\ln Z'$, to terms of the third order in $\hat{\nu}$, in the form

$$\begin{aligned} \ln \left[1 + e^{-2|N|/T} (\hat{\tau}_1^{(2)} + \hat{\tau}_2^{(2)}) + e^{-(4J-2|N|)/T} \hat{\tau}_0^{(3)} \right. \\ \left. + e^{-4|N|/T} \sum_{i=1}^5 \hat{\tau}_i^{(3)} \right] \\ = (\hat{\tau}_1^{(2)} + \hat{\tau}_2^{(2)}) \ln(1 + e^{-2|N|/T}) + \sum_{i=0}^5 f_i \hat{\tau}_i^{(3)}. \end{aligned}$$

As in the preceding example, we set equal to zero the values of almost all the $\hat{\nu}$'s, excepting those that correspond, for example, to the graph of Fig. 3c for $\hat{\tau}_0^{(3)}$ at some site. We get directly

$$\tau_1^{(2)} + \tau_2^{(2)} = 3, \quad \tau_0^{(3)} = 1, \quad \tau_i^{(3)} = 0, \quad 0 < i \leq 9.$$

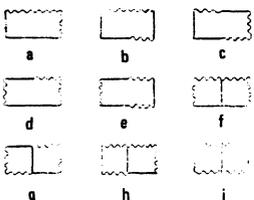


FIG. 4. Third-order graphs for the dual model.

This gives

$$f_0 = \ln(1 + 3e^{-2|N|/T} + e^{-(4J-2|N|)/T}) - 3 \ln(1 + e^{-2|N|/T}).$$

On performing analogous operations for the other $\hat{\tau}$'s, we obtain the expansion of $\langle \ln Z' \rangle$, and accordingly of Φ , in powers of the concentration:

$$\begin{aligned} \Phi/N = -2J + 4Jc - |h| - 6c^2 T \ln(1 + e^{-2|N|/T}) \\ - 4c^3 T \ln(1 + 3e^{-2|N|/T} + e^{-(4J-2|N|)/T}) \\ - c^3 T \{ 18 \ln(1 + 2e^{-2|N|/T}) - 84 \ln(1 + e^{-2|N|/T}) \\ + 36 \ln(1 + e^{-2|N|/T} + e^{-4|N|/T}) \}. \end{aligned} \quad (9)$$

One can obtain similarly a series expansion to higher powers of the concentration. We note that generalization of the procedure described to a cubic lattice is trivial. We remark also that in the calculations, the smallness of $e^{-2J/T}$ was used repeatedly. If this quantity is of order unity, then one must take practically all the graphs into account even in the second order in $\hat{\nu}$.

4. CALCULATION OF PHYSICAL CHARACTERISTICS

The expression (9) enables us to calculate the susceptibility, the magnetic moment, the entropy, and the energy of the ground state as power series in the concentration. We expand (9) as a power series in $|h|/T \ll 1$, and, in each coefficient, separate out the principal term with respect to $e^{2J/T}$:

$$\begin{aligned} E/N|_{h=0} = J(-2 + 4c - 16c^3), \\ S/N|_{h=0} = 6c^2 \ln 2 + c^3(54 \ln 3 - 84 \ln 2), \\ M|_{h=0} = 1 - 6c^2 - 20c^3, \\ \chi|_{h=0} = 2T^{-1}(3c^2 + 14c^3). \end{aligned} \quad (10)$$

As is seen from (10), the susceptibility diverges at $T \rightarrow 0$, as for a paramagnet. This fact is easily understood in a simple example. We consider the case in which two antiferromagnetic bonds are located one after another in a ferromagnetic matrix, for example as in Figs. 3a-b. Then at the end spins three bonds are ferromagnetic, while at the middle one two bonds are ferromagnetic and two antiferromagnetic. Therefore for the end spins it is advantageous to align along the total moment, while for the middle one the energy is the same for orientations up and down. Therefore this will behave like a paramagnet, making a contribution $6c^2 \ln 2$ to the entropy (since there are six different bond configurations per site) and insuring divergence of the susceptibility at $T \rightarrow 0$. If $T = 0$, an arbitrarily weak field orients all paramagnetic spin configurations and thereby removes the degeneracy. For this case, we have at $h \rightarrow 0$

$$\begin{aligned} E/N|_{h \rightarrow 0} = -J(2 - 4c + 16c^3), \\ M|_{h \rightarrow 0} = 1 - 8c^3. \end{aligned}$$

The energy has not changed, since the paramagnetic configurations make no contribution to it. The moment has increased, since the paramagnetic configurations

have become ordered; the residual entropy has disappeared. The susceptibility is infinite, as is insured by paramagnetic spins.

The case of an antiferromagnet with a certain concentration of ferromagnetic bonds is solved in similar fashion. By the substitution $\hat{\nu} \rightarrow 1 - \hat{\nu}$ and simple transformations, the partition function (7) can be reduced to a partition function of the form

$$Z = 2^{-2N} \exp\{2JN(1-2c)/T\} \sum_{\epsilon = \pm 1} \prod_{2i, 2j} Z_{2i, 2j}^{(+)} Z_{2i, 2j+1}^{(-)} Z_{2i+1, 2j+1}^{(+)} Z_{2i+1, 2j}^{(-)}$$

$$Z_{k,l}^{(\pm)} = [1 + \exp(\pm 2|h|/T) s_{k-\frac{1}{2}, l} s_{k+\frac{1}{2}, l} s_{k+\frac{1}{2}, l} s_{k, l-\frac{1}{2}}]$$

$$\times [1 + \{e^{-2J/T} + \hat{\nu}_{k-\frac{1}{2}, l} (e^{2J/T} - e^{-2J/T})\} s_{k-\frac{1}{2}, l}]$$

$$\times [1 + \{e^{-2J/T} + \hat{\nu}_{k, l-\frac{1}{2}} (e^{2J/T} - e^{-2J/T})\} s_{k, l-\frac{1}{2}}].$$

Actually this is the same partition function (7), only there are two identical sublattices; to a site of one of them corresponds a factor $\exp(-2|h|/T)$, of the other a factor $\exp(2|h|/T)$.

The calculations give the following results for the case $|h|/T \ll 1$:

$$E/N|_{h=0} = -J(2-4c+16c^3)$$

$$S/N|_{h=0} = 6c^2 \ln 2 + c^3(54 \ln 3 - 84 \ln 2)$$

$$\chi|_{h=0} = 2T^{-1}(3c^2 - 2c^3)$$

As in the case of a ferromagnet, there is a paramagnetic divergence of the susceptibility. Physically, this can be explained with exactly the same example as in the case of a ferromagnet; it is necessary only to make the ferro- and antiferromagnetic bond change places.

We shall estimate the effect of graphs of higher orders. For example, let there be a graph in the form of a square with all solid lines. It will encircle r^2 lattice sites. The factor occurring in the argument of the logarithm will be of the order of $\exp\{8Jr - 2|h|r^2/T\}$. In a field larger than $|h| > 4J/r$, this term will disappear at $T \rightarrow 0$. When $|h| < 4J/r$, it will make a contribution to the energy and to the moment. When $|h| = 4J/r$, it will correspond to a paramagnetic configuration. These deductions are easily verified by taking the corresponding configuration of

bonds in the lattice and considering the possible configurations of spins. Thus on increase of the field, there will occur at $|h| = 4J/r$ a discontinuity in the moment and in the energy, and also the appearance and disappearance of an entropy term, as well as a term in the susceptibility. These will all be of order c^{4r} in the concentration. It is clear that there may be an arbitrary number of such graphs in an infinite lattice. Therefore whenever $|h|/J$ becomes equal to any rational number, the corresponding discontinuities will occur. On increase of $|h|$, when $|h| > 4J$ all dependence of the physical properties on the field will disappear, and all the spins will align along the field direction.

All the behavior characteristics set forth for a square lattice will occur also for a cubic lattice. The corresponding calculations lead to the following expressions for the case of antiferromagnetic bonds in a ferromagnetic matrix, to terms of order c^4 :

$$E/N|_{h=0} = -3J(1-2c) - 60c^4$$

$$S/N|_{h=0} = 20c^3 \ln 2$$

$$M|_{h=0} = 4 - 20c^3 - 30c^4$$

$$\chi|_{h=0} = 20c^3/T$$

In conclusion, we note that an increase of the susceptibility at $T \rightarrow 0$ has been observed in calculations by the Monte Carlo method.³ The absence of a significant difference between the two and three-dimensional cases has been noticed by Kinzel and Fischer.⁴

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⁴W. Kinzel and K. H. Fischer, *J. Phys. C* **11**, 2115 (1978).

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