$\delta N_\omega=\delta\Phi_t[(\omega-\omega_p/2)^2-2\eta_0^2]$,

and hence it follows that δN_ω is not small when $\Delta\omega < 2^{1/2}\eta_0$ and, consequently, the single-frequency solution is unstable. Moreover, we can show that even the multisatellite solutions of Eq. (4.9) obtained in Ref. 10 are also unstable. In fact, Eq. (4.9) linearized against the background of the multisatellite solution transforms into a system of linear algebraic equations and has neutrally stable solutions, corresponding to a small change in the parameter E (see Ref. 10). The determinant of the linearized system vanishes at the points ω_i where the satellites are located. Since this determinant changes sign near the first satellite for $E=E_{\min}$ (i.e., against the background of the single-frequency solution), we may expect it to change the sign also near satellites for any value of E. It thus follows that there is a range of frequencies ω in which the determinant is negative and perturbations grow.

It follows from the above analysis that the instability of single-frequency and multisatellite states is related to the nonlinear nature of the interaction of PW's: $\Phi_{\rm int}\propto N^\alpha, \alpha>1$ and, in spite of the fact that we have proved this only for specific situations, it is generally true. Therefore, the only stable (in the case of frequency broadening) state of the system is a multifrequency turbulence of PW's with a continuous frequency distribution.

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Translated by A. Tybulewicz

Fluctuations in a nonstationary nonequilibrium system near its instability threshold

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(Submitted 7 June 1978)
Zh. Eksp. Teor. Fiz. 75, 1646–1651 (November 1978)

The example of electrostatic oscillations is used in considering the growth of fluctuations in a nonequilibrium system in which transition from a stable state (characterized by a small-perturbation logarithmic decrement $\gamma_{-\infty}$) to a stable state occurs in a finite time τ. In contrast to a stationary system, fluctuations at the instability threshold are bounded even in the linear approximation. If the eigenfrequency of weakly damped fluctuations is a simple root of the permittivity (or a root of the corresponding generalized susceptibility in the case of other fluctuations), the ratio of the intensity of fluctuations at the instability threshold to the intensity in the stable region is $(\pi\gamma_{-\infty}\tau)^{1/2}$ before the onset of the transition.

PACS numbers: 41.10.Dq

1. INTRODUCTION

It is well known that when a system is not in thermodynamic equilibrium and when the small-perturbation logarithmic decrement γ tends to zero as the characteristic parameter of the system a approaches a certain value a_0, the level of fluctuations considered in the linear approximation can rise without limit for $a\to a_0$. In this case the fluctuation level is restricted only by nonlinear effects. This situation occurs, in particular, on approach to an instability threshold where γ changes its sign. However, strictly speaking, this result applies to the case when the system is stationary. In reality, the system becomes unstable only after a finite time (and it then cannot exist in a stationary state). Allowance for the nonstationary state results in limitation of fluctuations even in the linear approximation[1)] and, as shown below, there is a simple relationship between

0038-5646/78/110829-04$02.40

the fluctuation intensities at the instability threshold and in the stable region. If the transition from the stable to unstable state occurs not too slowly, the fluctuation level remains small so that the nonlinear effects may be unimportant.

We shall also assume that by a certain moment the transition to the unstable region is complete and the parameter a assumes a steady value. Then, the fluctuations which have grown by that time represent a natural initial level of perturbations in the problem of evolution of initial fluctuations in a stationary unstable system. In the case when fluctuations are small, which—as mentioned above—occurs in the case of a sufficiently fast transition, we are justified in adopting the classical linear formulation of the problem. However, if the transition to the unstable state occurs adiabatically slowly, so that the level of fluctuations at each moment depends (as in the stationary case) only on the value of a and not on the previous history of the system, this fluctuation level is governed by the nonlinear effects even near the instability threshold. Thus, the real "background" of perturbations in an unstable system is not small right from the beginning, i.e., from the instability threshold, so that the linear formulation of the problem of behavior of small initial perturbations in the system loses its physical meaning.

2. FORMULATION OF THE PROBLEM

By way of example, we shall consider electrostatic fluctuations. Such fluctuations are characteristic of a nonequilibrium plasma which is a medium rich in instabilities. A typical situation is one in which individual spatial modes become unstable consecutively as the parameters (plasma concentration, magnetic field, etc.) are altered. We shall be interested in fluctuations of an electric field in a single mode (one degree of freedom) near its instability threshold. For simplicity, we shall consider a homogeneous medium so that the eigenfunctions of the fluctuation field are of the form $\mathbf{E}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}$, where a discrete set of $\mathbf{k}$ is governed by the boundary (or periodicity) conditions. As shown in Sec. 4 below, the approach adopted and the results are valid in reality for normal oscillation modes in any system (including a continuous medium which may be inhomogeneous or a discrete system).

The source of fluctuations of the electrostatic field are fluctuations of the microscopic charge density

$$\sum_i q_i\delta(\mathbf{r}-\mathbf{r}_i(t)),$$

where $\mathbf{r}_i(t)$ is the trajectory of a single charge q_i and the summation is carried out over all the charges. The starting point is the Poisson equation

$$i\mathbf{k}\hat{\varepsilon}_{\parallel}\mathbf{E}_{\mathbf{k}}=4\pi\rho_{\mathbf{k}}, \tag{1}$$

where $\hat{\varepsilon}_{\parallel}$ is the longitudinal permittivity.

In the stationary (steady-state) case when $\langle E_{\mathbf{k}}(t)E^*_{\mathbf{k}}(t')\rangle$ depends only on the difference $t-t'$, we find from Eq. (1) that

$$\langle|E_{\mathbf{k}}|^2\rangle=\frac{16\pi^2}{k^2}\int\frac{\langle|\rho_{\mathbf{k}}|^2\rangle_\omega}{|\varepsilon_{\parallel}(\omega,\mathbf{k})|^2}d\omega, \tag{2}$$

where $\langle|\rho_{\mathbf{k}}|^2\rangle$ is the spectral density of the Fourier component of the charge density in a system of noninteracting particles. Calculation of this spectral density is a separate problem and we shall not consider it here.

We are interested in the contribution made to fluctuations by weakly damped normal oscillations

$$\varepsilon_{\parallel}(\omega,\mathbf{k})=0, \tag{3}$$

i.e., we shall consider the contribution made to Eq. (2) by a sharp maximum in the integrand near the eigenfrequency $\omega_{\mathbf{k}}=\Omega-i\gamma$ nearest to the real axis and satisfying Eq. (3). We shall consider only the case when $\omega_{\mathbf{k}}$ is a simple root of $\varepsilon_{\parallel}$:

$$\left.\frac{\partial\varepsilon_{\parallel}}{\partial\omega}\right|_{\omega_{\mathbf{k}}}(\omega-\Omega+i\gamma)=0. \tag{4}$$

Then,

$$\langle|E_{\mathbf{k}}|^2\rangle=|C|^2\int\frac{\langle|\rho_{\mathbf{k}}|^2\rangle_\omega}{|\omega-\Omega+i\gamma|^2}d\omega, \tag{5}$$

where

$$C=4\pi/k\left.\frac{\partial\varepsilon_{\parallel}}{\partial\omega}\right|_{\omega_{\mathbf{k}}}.$$

For small logarithmic decrements obeying

$$\gamma\frac{\partial}{\partial\omega}\ln\langle|\rho_{\mathbf{k}}|^2\rangle_\omega\ll1, \tag{6}$$

we can replace $\langle|\rho_{\mathbf{k}}|^2\rangle$ in the integration domain $|\omega-\Omega|\lesssim\gamma$ in Eq. (5) with "white noise" representing a constant quantity $\langle|\rho_{\mathbf{k}}|^2\rangle_\Omega$. In this way we find

$$\langle|E_{\mathbf{k}}|^2\rangle=\pi|C|^2\langle|\rho_{\mathbf{k}}|^2\rangle_\Omega/\gamma. \tag{7}$$

For an equilibrium system the quantity $\langle|\rho_{\mathbf{k}}|^2\rangle_\Omega$ is proportional, in accordance with the fluctuation-dissipation theorem,[3] to the temperature T of the system and to the imaginary part of the permittivity $\varepsilon_{\parallel}(\Omega,\mathbf{k})$, i.e., to the decrement γ, and then Eq. (7) gives the energy of the field fluctuations proportional to T, which remains finite no matter how small the decrement. In the absence of equilibrium we find that $\langle|\rho_{\mathbf{k}}|^2\rangle_\Omega$ does not generally tend to zero together with γ and the fluctuation energy then rises without limit on approach to the instability threshold.

Before considering the nonstationary case, we shall go over to the time representation. The Poisson equation (1) together with the expansion (4) can be written in the form

$$dE_{\mathbf{k}}/dt+(i\Omega+\gamma)E_{\mathbf{k}}=C\rho_{\mathbf{k}}(t), \tag{8}$$

where the constancy of the spectral density $\langle|\rho_{\mathbf{k}}|^2\rangle_\omega$ in the integration domain [condition (6)] has the consequence that $\rho_{\mathbf{k}}(t)$ is delta-correlated[4]:

$$\langle\rho_{\mathbf{k}}(t)\rho_{\mathbf{k}}^*(t')\rangle=2\pi\langle|\rho_{\mathbf{k}}|^2\rangle_\Omega\delta(t-t'). \tag{9}$$

Having solved Eq. (8) and calculated, with the aid of Eq. (9), the quantity $\langle|E_{\mathbf{k}}|^2\rangle$, we can easily show that it is identical – as expected – with Eq. (7).

We shall now consider the nonstationary case. Let us assume that at a time $t<t_0<0$ the decrement is constant and equal to $\gamma_{-\infty}$, but beginning from $t=t_0$ it decreases monotonically passing through zero at $t=0$. The following equation describes E_k:

$$dE_k/dt+(i\Omega+\gamma(t))E_k=C\rho_k(t). \tag{10}$$

Since the fluctuation level is sensitive particularly to the behavior of the decrement γ, it follows that when the condition (6) is satisfied, we can still regard the random process $\rho_k(t)$ as stationary and delta-correlated. The problem thus reduces to finding $\langle|E_k|^2\rangle$ for the function $E_k(t)$ described by Eq. (10) with the right-hand side satisfying Eq. (9).

3. CALCULATION OF FLUCTUATIONS AT THE INSTABILITY THRESHOLD

The solution of Eq. (10) has the form

$$E_k(t)=C\exp\left[-\int_{-\infty}^{t}(i\Omega+\gamma)dt'\right]\int_{-\infty}^{t}\rho_k(t')\exp\left[\int_{-\infty}^{t'}(i\Omega+\gamma)dt''\right]dt'. \tag{11}$$

The squared fluctuation

$$\langle|E_k|^2\rangle=|C|^2\exp\left[-2\int_{-\infty}^{t}\gamma dt'\right]\int_{-\infty}^{t}\int_{-\infty}^{t}\exp\left[\int_{-\infty}^{t'}(i\Omega+\gamma)d\tau+\int_{-\infty}^{t''}(-i\Omega+\gamma)d\tau\right]\langle\rho_k(t')\rho_k^*(t'')\rangle dt'dt'' \tag{12}$$

can be represented, subject to Eq. (9), in the form

$$\langle|E_k|^2\rangle=2\pi|C|^2\langle|\rho_k|^2\rangle_\Omega\int_{-\infty}^{t}\exp\left[-2\int_{t'}^{t}\gamma(t'')dt''\right]dt'. \tag{13}$$

If $\gamma=\text{const}=\gamma_{-\infty}$, Eq. (13) gives the steady-state fluctuation level (7); this level will be denoted by $\langle|E_k|^2\rangle_{-\infty}$ and it occurs for all times satisfying $t<t_0$. The fluctuations grow for $t>t_0$. We shall find $\langle|E_k|^2\rangle$ at the instability threshold corresponding to $t=0$. We shall assume that the reduction in the decrement is not too fast[2)]:

$$d\gamma/dt\ll\gamma_{-\infty}^2. \tag{14}$$

Then, the main contribution to the integral (13) is made by the vicinity of the point $t=0$, where the argument of the exponential function is largest. If we represent the decrement for this region by the first term of the expansion

$$\gamma=\frac{d\gamma}{dt}(0)t, \tag{15}$$

we obtain

$$\langle|E_k|^2\rangle_{t=0}=2\pi|C|^2\langle|\rho_k|^2\rangle_\Omega\int_{-\infty}^{0}\exp\left(-\left|\frac{d\gamma}{dt}(0)\right|t^2\right)dt$$
$$=\pi^{3/2}|C|^2\langle|\rho_k|^2\rangle_\Omega\left|\frac{d\gamma}{dt}(0)\right|^{-1/2}. \tag{16}$$

Introducing a characteristic time for a change in the decrement

$$\tau=\gamma_{-\infty}\Big/\left|\frac{d\gamma}{dt}(0)\right|, \tag{17}$$

we finally obtain

$$\langle|E_k|^2\rangle_{t=0}=(\pi\gamma_{-\infty}\tau)^{1/2}\langle|E_k|^2\rangle_{-\infty}. \tag{18}$$

Equation (16) is dominated by the time interval

$$|t|\lesssim\tau(\gamma_{-\infty}\tau)^{-1/2}, \tag{19}$$

and since $\gamma_{-\infty}\tau\gg1$, the expansion (15) is justified.

The fluctuations grow for $t>0$. In the range of the linear time dependence of the increment given by Eq. (18) this growth occurs [sufficiently far beyond the threshold so that $t\gg(\tau/\gamma_{-\infty})^{1/2}$] proportionally to

$$\exp\left(\left|\frac{d\gamma}{dt}(0)\right|t^2\right).$$

4. DISCUSSION

We have considered fluctuations in a nonstationary system in the specific case of longitudinal oscillations of the electric field. However, the special nature of these oscillations does not affect our derivation in any way. The same derivation can be applied to oscillations of any magnitude if $\varepsilon_\parallel$ is replaced by the corresponding generalized susceptibility α (Ref. 3). The essential points of the above treatment are the following three assumptions:

1) in the time interval of interest to us only one degree of freedom (one "mode") can be unstable;

2) the intrinsic motion near the instability threshold can be described by a first-order equation;

3) in an interval $\sim\gamma$ wide near the eigenfrequency the spectral density of the "noise source" does not vary greatly and this makes it possible to use the delta-correlation approximation in the time representation.

The assumption of homogeneity of the medium made in Sec. 2 is not essential. The index **k** in our example basically shows only which mode becomes unstable. When the assumptions 1–3 above are satisfied, the main result (18) relating the intensity of fluctuations at the instability threshold to their level in the stable state before the onset of the transition to the unstable state is valid also for a single normal mode in an inhomogeneous medium and also for oscillations in a "discrete" system.

The approach can be extended in a self-evident manner to situations when the value of $\varepsilon_\parallel$ (in general α) near the instability threshold has a zero term of power $n>1$ (the assumption 2 is then disobeyed). For example, if $n=2$, we have to solve not Eq. (10) but a second-order equation with a delta-correlated right-hand side. For $n>1$, we find that in a relationship of the (18) type the term $(\gamma_{-\infty}\tau)^{1/2}$ is replaced by a term in which $\gamma_{-\infty}\tau$ has a different power exponent.

However, if in a system obeying a second- or higher-order equation the eigenfrequencies at the instability thereshold are sufficiently far apart, the evolution of fluctuations is described by first-order equations of the (10) type, so that the results of Sec. 3 are applicable subject to a small modification. For example, let us consider an oscillator with friction $\ddot{x}+2\gamma\dot{x}+\varkappa x=0$, which becomes unstable when a potential well changes to a hump: $\varkappa=-\gamma^2$. Then, the changes in the eigenfrequencies occur as follows. As $\varkappa$ decreases, the eigenfrequencies $\omega_{1,2}$ approach one another and for $\varkappa=-0$ they merge ($\omega_{1,2}=-i\gamma$) and then they diverge along the imaginary axis. Before the onset of the instability, when ω_1 goes over to the half-plane Im $\omega>0$, the other frequency is $\omega_2=-2i\gamma$. The main contribution to fluctuations comes from the frequency ω_1 closest to the imaginary axis. In calculations of fluctuations at the instability threshold we find from Eq. (19) that the only important part of the trajectory is

$$|\mathrm{Im}\,\omega_1|\leqslant(\gamma/\tau)^{1/2}.$$

Since

$$|-\mathrm{Im}\,\omega_2|\gg(\gamma/\tau)^{1/2},$$

the contribution of the oscillations with the frequency ω_2 is exponentially small and can be ignored. Allowance for just one branch ω_1 gives the results obtained above; all that is necessary is to introduce in Eq. (18) a correction factor ~1 because the fluctuations in the "initial" state at $t=-\infty$ may include comparable contributions from both eigenfrequencies. [It is not possible to reduce the problem to one branch if the merging of the eigenfrequencies occurs at a distance $\lesssim(\gamma/\tau)^{1/2}$ from the instability threshold and, in particular, when it occurs at the threshold itself. This occurs if $\varkappa$ and γ vanish simultaneously.]

We have deliberately ignored limitation of fluctuations by the nonlinear effects because the mechanism of such limitation (and the corresponding criterion) is different for each instability.

1) This is pointed out in Ref. 1 for one specific instability. The evolution of noise with time after an abrupt application of a pump field exceeding the parametric instability threshold of a plasma is considered in Ref. 2.

2) The opposite case (instantaneous transition to an unstable state) is trivial; it follow from Eq. (13) that the intensity of the fluctuations does not change during the transition time.

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Translated by A. Tybulewicz

On actuating shock waves in a completely ionized plasma

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(Submitted 15 June 1978)
Zh. Eksp. Teor. Fiz. **75**, 1652–1668 (November 1978)

The structure of actuating shock waves in a completely ionized plasma with collisions is calculated. Two limiting cases are considered, that of a magnetized and of an unmagnetized plasma. The processes determining the structure of the front of an actuating shock wave in an unmagnetized plasma are Joule dissipation and Hall currents. The width of the shock wave front in this case is determined by Joule dissipation and is equal to the diffusion length of the magnetic field. The magnetic field vector behind the shock wave front may rotate, but (with an accuracy to $\Omega_e\tau_e<<1$) remains in the plane of the initial direction. In the case of a magnetized plasma, the end of the magnetic field vector at the shock front rotates and describes a cone-like helix which expands behind the wave front. The number of revolutions along the helix is proportional to the degree of magnetization of the plasma. For a magnetized plasma, the processes defining the structure of the shock wave front are the electronic thermal conductivity, the electron-ion temperature relaxation and the dispersion due to the Hall terms and to the thermal emf. Correspondingly, the width of the front of an actuating shock wave in a magnetized plasma is equal to the scale of the electronic thermal conductivity. The values of the critical Mach numbers for which isomagnetic discontinuities arise in the shock wave front are found. The structure of the front is investigated in these cases.

PACS numbers: 52.35.Tc

INTRODUCTION

Those shock waves in magnetohydrodynamics in which the magnetic field ahead of the wave front is directed along the normal to the plane of the front, while behind the shock-wave front there is a component of the magnetic field parallel to the plane of the front, are called actuating shock waves. The purpose of the present work is the study of the structure of actuating shock waves in a completely ionized plasma with collisions, within the framework of the hydrodynamic model with classical transport coefficients.[1] A similar problem on a shock wave in a plasma without a magnetic field and for a transverse shock wave was solved in Refs. 2 and 3. Some partial solutions for actuating shock waves were obtained in Refs. 4–8.

0038-5646/78/110832-09$02.40