Modulation instability and strong nonlinearity in media with inverse dispersion

L. I. Rudakov and V. N. Tystovich

P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow
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Linear and nonlinear theories of the modulation instability of media with dispersion of the $\omega = \omega_0(n)(1-\alpha/k^2)$ type are considered. It is shown that small-scale modulation perturbations are excited in the linear regime and the smallest scale of these perturbations is set only by the limits of validity of the theory. Sawtooth waves with sharp peaks appear in the nonlinear regime. In the final analysis, this results in the heating of thermal or almost thermal particles in a plasma.

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§ 1. INTRODUCTION

The modulation instability[1,2] has been investigated so far mainly for waves whose dispersion is

$$\omega=\omega_0(n)(1+\beta^2k^2), \quad \beta^2k^2\ll 1 \tag{1}$$

(n is the particle density). In the nonlinear stage this instability gives rise to relatively large solitons (in the one-dimensional case) or quasisolitons, called cavitons (in the three-dimensional case). Their subsequent evolution (dynamic or as a result of collisions of solitons with one another) reduces the characteristic scales of nonlinear entities. In a plasma this gives rise to diffusion of the turbulence energy to high wave numbers where the energy flux is absorbed, because of the Landau damping, by "tail" particles.[3-5] The evolutionary changes in the turbulent spectrum result in the transfer of the major part of the input energy to fast particles. This relatively common effect is important under astrophysical conditions[6] and, in particular, it explains the high efficiency of generation of fast particles and cosmic rays.

In an earlier paper[7] we also drew attention to qualitatively new features of the so-far uninvestigated modulation instability in media with the dispersion

$$\omega=\omega_0(n)-\alpha\omega_0(n)/2k^2, \tag{2}$$

which is called the inverse dispersion in Ref. 7. The present paper gives a detailed linear and nonlinear theory of this instability. A characteristic feature of the modulation instability of waves with the spectrum (2) is the excitation of a very wide spectra of k, whose maximum value has an upper bound set only by the limits of validity of the theory. It is this circumstance which is physically responsible for the heating of thermal or near-thermal particles by the modulation instability in question (in a plasma there may be direct excitation of oscillations with k close to $k_d=\omega_p/v_T$). In the nonlinear regime the modulation instability of waves (2) gives rise to entities with electric field discontinuities and these are damped by thermal particles. This distinguishes qualitatively our modulation instability from the modulation instability of the waves (1) when—in the linear state—a possible spectrum of k is limited from above the $k_{\max}$ [for the Langmuir waves we have $k_{\max}\approx k_d(W/nT)^{1/2}$, where W is the energy density in the waves (1) and nT is the thermal energy density].

Before considering the theory, we shall note the following possibilities of realization of the spectrum (2). Firstly, the heating of the fast particles by the Langmuir oscillations of a plasma with the spectrum (1) should automatically convert the spectrum (1) to the spectrum (2) if the dispersion of the Langmuir waves is governed by fast particles. This occurs when the relative number of the fast particles is small, $n'/n\ll 1$, but their average energy is higher than the thermal value:

$$n'T'\gg nT \tag{3}$$

(T' is the effective temperature of the fast particles and T is the temperature of the thermal particles). The necessary condition for the existence of the spectrum (2) is

$$1\ll\frac{T'}{T}k^2r_d^2\ll\left(\frac{n'T'}{nT}\right)^{1/2}, \quad r_d=\frac{1}{k_d}. \tag{4}$$

We then have

$$\omega_0(n)=\omega_{pe}, \quad \alpha=k_d^2Tn'/T'n. \tag{5}$$

Another example of the dispersion (2) are the so-called lower-hybrid waves currently used for plasma heating in tokamaks:

$$\omega^2=\frac{\omega_{pi}^2}{1+\omega_{pe}^2/\omega_{He}^2}\left(1+\frac{m_i}{m}\frac{k_z^2}{(k_\perp^2+k_z^2)}\right), \quad k^2\ll\frac{\omega_{He}^2}{v_{Te}^2}. \tag{6}$$

For a given value of $k_z\ll k_\perp(m/m_i)^{1/2}$, we have the dispersion (2) for $k_\perp$:

$$\omega_0(n)=\frac{\omega_{pi}}{(1+\omega_{pe}^2/\omega_{He}^2)^{1/2}}, \quad \alpha=-\frac{m_i}{m}k_z^2. \tag{7}$$

For a given $k_z\gg k_\perp$, we obtain

$$\omega_0(n)=\frac{\omega_{pe}}{(1+\omega_{pe}^2/\omega_{He}^2)^{1/2}}, \quad \alpha=k_\perp^2. \tag{8}$$

The next examples are the ion-acoustic oscillations of frequencies close to ω_{pi}:

$$\omega=\omega_{pi}\left(1-\frac{k_d^2}{2k^2}\right), \quad \omega_0(n)=\omega_{pi}, \quad \alpha=k_d^2. \tag{9}$$

The dispersion (2) is exhibited also by many branches of excitations in solids.

The starting equations in the theoretical analysis can be, firstly, the equation for the complex amplitude $\mathbf{E}$ of the field $\mathbf{E}e^{-i\omega_0 t}$:

$$\Delta\left(\frac{2i}{\omega_0}\frac{\partial\mathbf{E}}{\partial t}-2\left(\frac{\partial}{\partial n}\ln\omega_0\right)\delta n\,\mathbf{E}\right)=\alpha\mathbf{E} \tag{10}$$

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and the standard equation for the density variations ($v_s=\sqrt{T_e}/m$):

$$\frac{\partial^2}{\partial t^2}\delta n-v_s^2\Delta\delta n=\Delta\frac{|\mathbf{E}|^2}{8\pi m_i}. \tag{11}$$

In solids, in addition to the striction forces we have to allow also for other nonlinearities. In terms of dimensionless variables the system (10)–(11) becomes

$$\frac{\partial^2}{\partial\xi^2}\left(i\frac{\partial\varepsilon}{\partial\tau}-\nu\varepsilon\right)=\varepsilon\frac{\alpha}{|\alpha|},\qquad \frac{\partial^2\nu}{\partial\tau^2}-\frac{\partial^2\nu}{\partial\xi^2}=\frac{\partial^2}{\partial\xi^2}|\varepsilon|^2. \tag{12}$$

A dimensionless field ε and dimensionless variations of the density ν, time τ, and coordinate ξ are given by

$$\varepsilon=\left[\frac{E^2}{8\pi nT}\frac{\partial\ln\omega_0}{\partial\ln n}M^{1/3}\right]^{1/2},\quad \nu=\frac{\delta n}{2n}M^{1/3},\quad \tau=\frac{t\omega_0}{M^{1/3}},\quad \xi=r\frac{\omega_0}{v_s}\frac{1}{M^{1/3}},\quad M=\frac{2\omega_0^2}{|\alpha|v_s}. \tag{13}$$

§ 2. SOME GENERAL RELATIONSHIPS

The system (12) has a series of integrals R_i. They can be found most simply directly from Eq. (12) if the relevant laws of conservation are written in the differential form

$$\frac{\partial\rho_i}{\partial\tau}+\frac{\partial}{\partial\xi}\mathbf{w}_i=0, \tag{14}$$

where $\mathbf{w}_i$ is the flux of the conserved quantity R_i. Equating to zero the flux $\mathbf{w}_i$ across an infinitely distant surface ensures conservation of

$$R_i=\int\rho_i d\xi,\quad \frac{dR_i}{d\tau}=0, \tag{15}$$

where ρ_i is the density of the relevant quantity.

For simplicity, we shall consider the one-dimensional case when the system (12) can be written in the following form by introducing two new quantities g and v:

$$i\frac{\partial\varepsilon}{\partial\tau}-\nu\varepsilon=g,\quad \frac{\partial^2 g}{\partial\xi^2}\frac{\alpha}{|\alpha|}=\varepsilon,\qquad \frac{\partial\nu}{\partial\tau}=-\frac{\partial v}{\partial\xi},\quad \frac{\partial v}{\partial\tau}=-\frac{\partial\nu}{\partial\xi}-\frac{\partial}{\partial\xi}|\varepsilon|^2. \tag{16}$$

From Eq. (15) we can easily obtain, in the differential form of Eq. (14), the law of conservation of the number of quanta:

$$\rho_1=|\varepsilon|^2=N(\xi),\quad w_1=\frac{\alpha}{|\alpha|}i\left(g\frac{\partial g^*}{\partial\xi}-g^*\frac{\partial g}{\partial\xi}\right), \tag{17}$$

the law of conservation of momentum:

$$\rho_2=\frac{1}{2}\left(i\varepsilon\frac{\partial\varepsilon^*}{\partial\xi}-i\varepsilon^*\frac{\partial\varepsilon}{\partial\xi}+2\nu v\right)=P(\xi),\qquad w_2=\frac{\nu^2+v^2}{2}+\nu|\varepsilon|^2+\frac{\alpha}{|\alpha|}\left|\frac{\partial g}{\partial\xi}\right|^2-\frac{1}{2}(g\varepsilon^*+\varepsilon g^*), \tag{18}$$

and the law of conservation of energy:

$$\rho_3=H(\xi)=\frac{\nu^2+v^2}{2}+\nu|\varepsilon|^2-\frac{\alpha}{|\alpha|}\left|\frac{\partial g}{\partial\xi}\right|^2,\qquad w_3=\nu v+v|\varepsilon|^2+\frac{\alpha}{|\alpha|}\left(g\frac{\partial^2 g^*}{\partial\xi\partial\tau}+g^*\frac{\partial^2 g}{\partial\xi\partial\tau}\right). \tag{19}$$

In the three-dimensional case the laws of conservation (15) have the form

$$\frac{dN}{d\tau}=0,\quad N=\int N(\xi)d\xi=\int|\varepsilon|^2 d\xi,\qquad \frac{d\mathbf{P}}{d\tau}=0,\quad \mathbf{P}=\int\mathbf{P}(\xi)d\xi=\frac{1}{2}\int\left[i\varepsilon\frac{\partial\varepsilon^*}{\partial\xi}-\varepsilon^*\frac{\partial\varepsilon}{\partial\xi}i+2\nu v\right]d\xi,\qquad \frac{dH}{d\tau}=0,\quad H=\int H(\xi)d\xi=\int\left[\frac{\nu^2+v^2}{2}+\nu|\varepsilon|^2-\frac{\alpha}{|\alpha|}\left|\frac{\partial g}{\partial\xi}\right|^2\right]d\xi. \tag{15'}$$

We note also that we can introduce a dimensionless potential difference

$$\varepsilon=\frac{\partial\varphi}{\partial\xi},\quad \varphi=\frac{\alpha}{|\alpha|}\frac{\partial g}{\partial\xi}.$$

Then,

$$H=\int\left[\frac{\nu^2+v^2}{2}+\nu|\varepsilon|^2-\frac{\alpha}{|\alpha|}|\varphi|^2\right]d\xi.$$

We shall now define the field frequency Ω by

$$\varepsilon=|\varepsilon|\exp\left(-i\int^{\tau}\Omega d\tau\right).$$

Then,

$$\frac{i}{2}\left(\varepsilon^*\frac{\partial\varepsilon}{\partial\tau}-\varepsilon\frac{\partial\varepsilon^*}{\partial\tau}\right)=\Omega(\xi)|\varepsilon|^2. \tag{20}$$

We find from Eq. (16) that

$$\Omega(\xi)|\varepsilon|^2=\nu|\varepsilon|^2-\frac{\alpha}{|\alpha|}\left|\frac{\partial g}{\partial\xi}\right|^2+\frac{1}{2}\frac{\partial^2}{\partial\xi^2}|g|^2\frac{\alpha}{|\alpha|}$$

and, consequently,

$$H=\int\left[\frac{\nu^2+v^2}{2}+\Omega(\xi)|\varepsilon|^2\right]d\xi. \tag{21}$$

The meaning of the above integral is very simple. In the linear approximation, the effective frequency Ω introduced by Eq. (20) describes linear dispersion. In the nonlinear case it also includes nonlinear frequency shifts, i.e., it corresponds to the total frequency. Thus, according to Eq. (21), the total energy consists of the energy of the oscillations under consideration (equal to the density of the number of quanta $|\varepsilon|^2$ multiplied by their frequency Ω) and of the energy of deformation and motion of the medium, including the emitted acoustic oscillations.

It should be pointed out that the first two integrals N and $\mathbf{P}$ have the same form as for the dispersion described by Eq. (1). The negative sign of H may result not only from $\nu<0$ (lowering of the density) but also from a fairly high value of $|\varphi|^2$ in the $\alpha>0$ case.

§ 3. LINEAR THEORY OF THE MODULATION INSTABILITY

We shall use the system (12) in terms of the dimensionless variables τ and ξ. We shall consider the stability of a monochromatic wave in the case

$$\varepsilon=\varepsilon_0\exp(i\mathbf{k}_0\xi-i\omega_0\tau),\quad \omega_0=-1/k_0^2-|\varepsilon_0|^2,\quad \nu_0=-|\varepsilon_0|^2. \tag{22}$$

Here, ω_0 and $\mathbf{k}_0$ are naturally also dimensionless quantities. We shall find perturbations of the amplitude ε' and density ν' in the form

$$\varepsilon'=\varepsilon_0'\exp(i\mathbf{k}_0\xi-i\omega_0\tau+i\mathbf{k}\xi-i\omega\tau),\quad \nu'=\nu_0'\exp(i\mathbf{k}\xi-i\omega\tau),\qquad \varepsilon'^*=\varepsilon_0''^*\exp(-i\mathbf{k}_0\xi+i\omega_0\tau+i\mathbf{k}\xi-i\omega\tau),\quad \varepsilon_0''^*\neq\varepsilon_0'^*, \tag{23}$$

i.e., we shall assume that the perturbation ε' is a modulation of Eq. (22) with the wave number $\mathbf{k}$ and frequency ω.

This gives the dispersion equation ($\alpha>0$):

$$1=\frac{2k^2}{k^2-\omega^2}|\varepsilon_0|^2\frac{4\mathbf{k}_0\mathbf{k}-k_0^2k^2-k^4}{k_0^2(k^2-k_0^2)^2}\times\left(\omega-\frac{2\mathbf{k}_0\mathbf{k}-k^2}{k_0^2(\mathbf{k}-\mathbf{k}_0)^2}\right)^{-1}\left(\omega-\frac{2\mathbf{k}_0\mathbf{k}+k^2}{k_0^2(\mathbf{k}+\mathbf{k}_0)^2}\right)^{-1}. \quad (24)$$

It should be stressed that Eq. (24) is obtained from the system (12) without any additional assumptions. Therefore, it is desirable to consider the dispersion equation (24) without imposing any restrictions on $\mathbf{k}$ and $\mathbf{k}_0$. In specific cases the system (12) is valid when certain inequalities are satisfied, so that we have to use those growth rates γ (given below) which correspond to these inequalities [see, for example, Eq. (4) etc.].

In the case of subsonic growth rates subject to the condition $|\omega|\ll k$, we have

$$\omega=\frac{2(\mathbf{k}\mathbf{k}_0)^4}{(k^2-k_0^2)^2}\pm\frac{4(\mathbf{k}\mathbf{k}_0)^2-k_0^2k^2-k^4}{k_0^2(k^2-k_0^2)^2}\left(1-\frac{2k_0^2(k^2-k_0^2)^2|\varepsilon_0|^2}{k^4+k_0^2k^2-4(\mathbf{k}_0\mathbf{k})^2}\right)^{1/2}. \quad (25)$$

If $k\gg k_0$, we find that

$$\omega=\frac{2(\mathbf{k}\mathbf{k}_0)}{k^4}\pm\frac{1}{k_0^2}(1-2k_0^2|\varepsilon_0|^2)^{1/2}. \quad (26)$$

Clearly, the instability occurs if

$$|\varepsilon_0|^2>1/2k_0^2. \quad (27)$$

This criterion corresponds to the condition when the nonlinear frequency shift exceeds the linear effect.

We shall consider in greater detail two cases without assuming in advance that $k\gg k_0$:

a) the case corresponding to inequality

$$|\varepsilon_0|^2\gg 1/k_0^2, \quad (28)$$

b) the case corresponding to the opposite inequality.

We shall begin with the case when the inequality (28) is obeyed. Then, Eq. (26) gives the growth rate of the modulation instability:

$$\gamma=\gamma_{max}=(2|\varepsilon_0|^2/k_0^2)^{1/2},\quad k\gg k_0, \quad (29)$$

i.e., the growth rate is governed by the geometric mean of the linear and nonlinear frequency dispersions. The rate is constant and, consequently, any values of k may be excited, no matter how large and their maximum values are bounded from above by the limits of validity of the system (12). The growth rate (29) is obtained subject to the condition $\gamma\ll k$, i.e.,

$$k\gg k_*=(2|\varepsilon_0|^2/k_0^2)^{1/2}. \quad (30)$$

We shall also introduce

$$k_{**}=k_0^{-2}(2|\varepsilon_0|^2k_0^2)^{-1/2}. \quad (31)$$

Since $|\varepsilon_0|^2k_0^2\gg 1$, we have $k_{**}\ll k_*$. Let us assume that $k_*\gg k_{**}\gg k_0$ or

$$|\varepsilon_0|^2k_0^8\ll 1. \quad (32)$$

Then, for $k\ll k_*$, we now have $\gamma\gg k$ and the subsonic solution is inappropriate. In the range

$$k_0^{-2}(2|\varepsilon_0|^2k_0^2)^{-1/2}\ll k\ll k_0^{-2}(2|\varepsilon_0|^2k_0^2)^{1/2}, \quad (33)$$

we obtain

$$\gamma=\left(k^2\frac{2|\varepsilon_0|^2}{k_0^2}\right)^{1/4}=\left(\frac{k}{k_*}\right)^{1/2}\gamma_{max}, \quad (34)$$

i.e., the growth rate decreases as $k^{1/2}$ on reduction of k. In addition to the purely imaginary solution $\omega=\pm i\gamma$, there are also two completely real solutions $\omega=\pm\gamma$, which describe the second sound in the problem under discussion. The real component of the frequency is obtained also for $k\gg k_*$: $\mathrm{Re}\,\omega=2\mathbf{k}_0\mathbf{k}/k^4$; however, it is always a factor of $(k_0/k)^2(|\varepsilon_0|^2k_0^2)^{-1/2}$ smaller than the imaginary component. Finally, if $k\ll k_{**}$, the growth rate is a linear function of k:

$$\gamma=k(2|\varepsilon_0|^2k_0^2)^{1/2}\approx\gamma_{max}\left(\frac{k}{k_*}\frac{k}{k_{**}}\right)^{1/2}\gg k, \quad (35)$$

i.e., an aperiodically growing second sound is obtained and it is characterized by a linear dependence of the frequency on the wave number, as well as by a velocity $(2|\varepsilon_0|^2k_0^2)^{1/2}$ much higher than the velocity of sound. In the region adjoining $k\sim k_{**}$, all four solutions can be described by

$$\omega^2\approx\frac{2^{-1}}{k_0^4}\pm\left(\frac{1}{4k_0^8}+\frac{2k^2}{k_0^2}|\varepsilon_0|^2\right)^{1/2}. \quad (36)$$

For k of the order of $k_0\ll k_{**}$, we obtain the following angular dependence of the growth rate:

$$\omega^2=-2k^2|\varepsilon_0|^2k_0^2\frac{4\cos^2\theta-1-k^2/k_0^2}{4\cos^2\theta-k^2/k_0^2}. \quad (37)$$

For $k\ll k_0$, the angular dependence is retained and only the waves with $\cos\theta>\frac{1}{2}$ remain unstable.

We shall next consider the case $k_{**}\ll k_0\ll k_*$, when the inequality opposite to Eq. (32) is satisfied. Then, the solution of Eq. (34) is retained to a value of k of the order of k_0. If $k\ll k_0$, then

$$\gamma\approx\left(\frac{k}{k_0}|4\cos^2\theta-1|\right)^{1/2}\left(\frac{2k^2}{k_0^2}|\varepsilon_0|^2\right)^{1/4}\approx\left(\frac{k}{k_0}\frac{k}{k_*}\right)^{1/2}\gamma_{max}. \quad (38)$$

Finally, if $k_0\gg k_*$, i.e., if $|\varepsilon_0|^2\ll k_0^4$, the growth rate (29) is retained only up to $k\approx k_0$ and for $k\ll k_0$ only the perturbations with $\cos\theta>\frac{1}{2}$ are unstable.

We shall now consider the case $|\varepsilon_0|^2k_0^2\ll 1$. Then, the instability disappears for $k\gg k_0$. If k is of the order of k_0, the instability is possible in a narrow range $\Delta\theta$ of the order of $(|\varepsilon_0|^2k_0^2)^{1/2}\ll 1$ near the angle θ_0 equal to $\cos\theta_0=\frac{1}{4}(1+k^2/k_0^2)$, i.e., for $k<\sqrt{3}k_0$. If $k\ll k_0$, we obtain weakly growing (for $\theta>60°$) second sound:

$$\omega=\frac{2k\cos\theta}{k_0^3}+i\frac{k}{k_0^3}(1-4\cos^2\theta)^{1/2}(2|\varepsilon_0|^2k_0^2)^{1/2}. \quad (39)$$

However, we must bear in mind that this growth occurs only within the framework of the system (12). If we allow for the damping, which occurs in the examples given above but is ignored in the derivation of the system (12), as well as for the scatter of k_0 in respect to the magnitude and direction, we find there is no instability for $|\varepsilon_0|^2k_0^2\ll 1$. Thus, in the case of sufficiently wide spectra the instability appears in fact for $|\varepsilon_0|^2k_0^2>1$.

§4. ASYMPTOTIC (IN TIME) SOLUTIONS AND TURBULENCE SPECTRA

The problem of the turbulence spectrum is frequently solved together with a source and sink. In considering an inertial interval it is important from which end (the one corresponding to high or low wave numbers) the energy is arriving. We shall begin from some guiding ideas on the inertial-interval turbulence ignoring the effects of maintenance of oscillations from the side of high or low values of k. We shall assume that initially there is a spatially localized, in a distance L, packet of the oscillations in question with a characteristic value of $k \approx k_0$ of sufficiently high amplitude $|\varepsilon_0|^2 k_0^2 \gg 1$, so that modulation creation of short-wavelength harmonics is possible in a time much shorter than the packet spreading time [see Eqs. (22) and (29)]:

$$\frac{\gamma(k)L}{\partial\omega_0/\partial k_0} \approx \frac{Lk_0^3|\varepsilon_0|}{k_0} \approx Lk_0 \cdot k_0|\varepsilon_0| \gg 1. \quad (40)$$

The deformation of $|\varepsilon|^2$ and ν in the course of development of the instability results in the emission of acoustic oscillations in accordance with the system (12) and these can carry away energy and momentum to "infinity" at the velocity of sound assumed to be much higher than the group velocity of the packet.

We shall consider three possible situations. In the first case the development of a modulation instability of scale L produces quasihomogeneous plane-wave turbulence with a certain spectral distribution of $|\varepsilon_k|^2$. In this situation we can always expect generation of acoustic oscillations carrying away the energy and momentum and the energy of the oscillations in question remains negative but increases in the absolute sense. However, since $|\varepsilon|^2$ cannot change greatly because of the first integral in Eq. (15′) and homogeneity of such turbulence, it follows that only $|\varphi|^2 = |\varepsilon_0|^2/k_0^2$ can increase, i.e., only the main scale of the turbulence can increase up to $_0^2 k_0^2 < 1$. Consequently, in the final analysis, the modulation instability described in Eq. (26) is suppressed and short-wavelength modes with $k > k_0 \sim 1/\varepsilon_0$ are absent from the oscillation spectrum. This apparently terminates the whole process. This is the feature that distinguishes the turbulence of the waves described by Eq. (2) from the turbulence of the waves described by Eq. (1). In the latter case we can also expect the emission of sound which can reduce the wave numbers and restore the modulation instability.

The second possible situation is represented by solutions in the form of quasisteady standing nonlinear waves. Such waves do not create a force $-\partial|\varepsilon|^2/\partial x$ which alternates with times, and, consequently, they cannot emit acoustic waves.

The third possibility is analogous to the modulation instability of the Langmuir waves and it represents self-contraction of packets with the main instability scale $1/k_0$ for $k_0^2|\varepsilon_0|^2 \gg 1$; these packets are converted into solitons in which energy is localized with a high density in narrow regions. This is permitted by the laws of conservation of N and H. In fact, in the case of packet contraction we have $\varepsilon^2 \sim 1/\xi$ and such contraction is favored because of reduction in the potential to the scale $k_0\xi \gtrsim k_0^2|\varepsilon_0|^2$. From this scale the evolution of $|\varepsilon|^2$ may proceed in the two ways described above. In the case of the first way we again return to the turbulence scale $k_0^2|\varepsilon_0|^2 < 1$.

It thus follows from the above considerations that both strong turbulence and small scale $|\varepsilon_0|^2 k_0^2 \gg 1$ may characterize quasisteady nonlinear waves. The quasisteady condition is understood to be[1)]

$$\partial|\varepsilon|^2/\partial t \ll |\varepsilon|^2/k_0^2.$$

§ 5. QUASISTEADY NONLINEAR WAVES

We shall begin by considering the problem of completely steady (standing) waves and then find more general relationships which can be used to deduce the limits of validity of the solutions when they are applied to quasisteady waves. We shall turn to the solution of the system (12). In the case of standing waves the second equation in the system (12) is integrated, $\nu = -|\varepsilon|^2$, and we then have one equation ($\alpha > 0$)

$$\frac{\partial^2}{\partial\xi^2}\left(i\frac{\partial\varepsilon}{\partial\tau} + |\varepsilon|^2\varepsilon\right) = \varepsilon. \quad (41)$$

We shall find its solution in the form

$$\varepsilon = \psi(\xi)e^{iS}, \quad \Omega = -\frac{\partial S}{\partial\tau}, \quad k(\xi) = \frac{\partial S}{\partial\xi}, \quad (42)$$

$$\frac{d^2}{d\xi^2}(\Omega + \psi^2)\psi = k^2(\Omega + \psi^2)\psi + \psi, \quad (43)$$

$$\frac{d}{d\xi}k(\Omega + \psi^2)^2\psi^2 = 0. \quad (44)$$

We shall be interested in solutions with $\Omega < 0$. It follows from Eq. (44) that

$$k = \frac{k_0|\Omega|^3}{\psi^2(-|\Omega| + \psi^2)^2}. \quad (45)$$

Equation (43) has an integral which can be represented in the form

$$\left[\left(1 - \frac{3\psi^2}{|\Omega|}\right)\frac{d\psi}{d\xi}\right]^2 = \frac{1}{6}\left(1 - \frac{3\psi^2}{|\Omega|}\right)^2 - k_0^2|\Omega|\left[-a + 1\Big/\frac{\psi^2}{|\Omega|}\left(1 - \frac{\psi}{|\Omega|}\right)^2\right], \quad (46)$$

where a and k_0^2 are the constants of integration. The expression

$$|\varphi|^2 = (|\Omega| - 3\psi^2)^2\left(\frac{d\psi}{d\xi}\right)^2 + k^2\psi^2(-|\Omega| + \psi^2)^2 = {}^1/_6(|\Omega| - 3\psi^2)^2 + k_0^2 a|\Omega|^2 \quad (47)$$

is simply the square of the modulus of the difference between the potentials, which occurs in the energy integral.

It follows from the above that the problem reduces to finding a bounded solution of Eq. (46). Consequently, we shall be interested in the solutions that minimize the energy integral

$$H = \int^l \left(-\frac{\psi^4}{2} - |\varphi|^2\right) d\xi \quad (48)$$

for a given value of the integral $\int^l \psi^2 d\xi$ in the interval $L \gg l \gg [d(\ln\psi)/d\xi]^{-1}$. The variational parameters are k_0 and a.

The value of k_0^2 characterizes the modulation amplitude $|\psi_{max} - \psi_{min}|/|\psi_{max}|$ and, as shown below, if $k_0^2 = \frac{16}{243}|\Omega|$, this amplitude vanishes, whereas for $k_0^2 \to 0$, we have $\psi_{min} \to 0$, i.e., the maximum modulation corresponds to $k_0^2|\Omega| \ll 1$. The larger the parameter a, the greater the value of $\psi_{max}/|\Omega|$. However, a has an upper bound. In fact, if ψ is to be finite throughout the interval ξ, the derivative $d\psi/d\xi$ should vanish at two values ψ_{max} and ψ_{min}. Let us assume that $\psi^2_{min}<|\Omega|/3$. The point $\psi^2=|\Omega|/3$ is singular. It cannot be crossed without violating the condition of continuity of the quantities ψ and $(|\Omega|-3\psi^2)d\psi/d\xi$, which follow from the equation for ψ. Therefore, we have to satisfy the requirement $\psi^2_{max}<|\Omega|/3$. It follows from Eq. (46) that this can be done if $a<27/4$. Consequently, (we are assuming that $a=27/4-\alpha^2$),

$$|\varphi|^2={}^1/_6(|\Omega|-3\psi^2)^2+k_0{}^2|\Omega|^3({}^{27}/_4-\alpha^2). \tag{49}$$

We shall consider two limiting cases which can be solved completely. We shall first assume that $k_0^2|\Omega|\ll 1$, i.e., that the modulation is strong. We then have

$$|\varphi|^2\approx\frac{\Omega^2}{6}\left(1-\frac{3\psi^2}{|\Omega|}\right)^2, \tag{50}$$

$$\left(\frac{d\psi}{d\xi}\right)^2\approx\frac{1}{6}-\frac{k_0{}^2|\Omega|^2}{\psi^2}-\frac{\alpha^2k_0{}^2|\Omega|}{(1-3\psi^2/|\Omega|)^2}. \tag{51}$$

At the points $\psi\to 0$, $\psi^2\to|\Omega|/3$ we have infinite "potential barriers" corresponding to the "force"[2)]

$$\frac{d^2\psi}{d\xi^2}=2\frac{k_0{}^2|\Omega|^2}{\psi^3}-\frac{12\alpha^2k_0{}^2\psi}{|\Omega|(1-3\psi^2/|\Omega|)^3}. \tag{52}$$

The lower the values of k_0 and $k_0\alpha$, the less frequent is the reflection of a "freely moving particle" at the points $\psi=0$ and $\psi^2=|\Omega|/3$. In the vicinity of these points we have

$$\psi^3=6k_0{}^2\Omega^2+\xi^2/6, \tag{53}$$

$$[\psi-(|\Omega|/3)^{1/2}]^2=6k_0{}^2\alpha^2\Omega^2+(\xi-\xi_0)^2/6. \tag{54}$$

Between these points, we find that

$$\psi=\left(\frac{|\Omega|}{3}\right)^{1/2}\cdot\begin{cases}\xi/\xi_0\\(\xi_0-\xi)/\xi_0\end{cases};\quad \xi_0\approx|\Omega|^{1/2}. \tag{55}$$

We shall now consider the second limiting case when the modulation is weak: $|\psi_{max}-\psi_{min}|/\psi_{max}\ll 1$. The term k_0^2 in Eq. (47) for $(d\psi/d\xi)^2$ has a minimum at $\psi^2=|\Omega|/3$ and in the vicinity of this minimum we have

$$\left(\frac{d\psi}{d\xi}\right)^2=\frac{1}{6}-k_0{}^2|\Omega|\left\{\frac{\alpha^2}{(1-3\psi^2/|\Omega|)^2}+\frac{81}{32}+\frac{81}{8}\left(1-\frac{3\psi^2}{|\Omega|}\right)\right\}. \tag{56}$$

Hence, it follows that

$$k_0{}^2|\Omega|={}^{16}/_{243}(1-\beta^2)\approx{}^1/_{15}(1-\beta^2),\quad \beta^2\ll 1,$$

and we obtain

$$|\varphi|^2\leqslant{}^1/_6(|\Omega|-3\psi^2)^2+{}^4/_9\Omega^2\approx{}^4/_9\Omega^2, \tag{57}$$

$$\left(\frac{d\psi}{d\xi}\right)^2=\frac{\beta^2}{6}-\frac{1}{3}\left(1-\frac{3\psi^2}{|\Omega|}\right)-\frac{1}{15}\frac{\alpha^2}{(1-3\psi^2/|\Omega|)^2}. \tag{58}$$

In the last equation in the limit $\alpha\to 0$ the quantity ψ^2 lies between the value on the wall characterized by $\psi^2_{max}=|\Omega|/3$ and the value given by $\psi^2_{min}=\frac{1}{3}|\Omega|(1-\beta^2/3)$, where

$$\Delta\psi/\psi_{max}={}^3/_2\xi^2-{}^1/_4\beta^2, \tag{59}$$

i.e., the solution represents segments of concave parabolas which are matched at the points separated by $\xi_0=2\beta/\sqrt{6}$ and the depth of the parabolas is $\Delta\psi=\frac{3}{8}\xi_0^2$. The energy integral per unit length is then $H_0=-\frac{5}{2}\Omega^2$.

Among possible steady-state solutions of the system (12) we shall select one with discontinuities (peaks). In addition to these solutions, there are analogous solutions with amplitudes in the range $|\Omega|/3<\psi^2<|\Omega|$. For all these solutions the peaks are responsible for short-wavelength harmonics in the spectrum (we shall use the familiar expression for a sawtooth wave):

$$\varepsilon_n=\frac{2\varepsilon_0}{\pi n^2},\quad \varepsilon=\sum_{n<1/\alpha k_0\xi_0}\cos\frac{\pi n\xi}{2\xi_0}\varepsilon_n. \tag{60}$$

We must stress that a kink at $\psi^2=|\Omega|/3$ in these solutions minimizes $-|\varphi|^2$ and the energy integral [see Eq. (49)].

We shall now consider the limits of validity of our solutions within the framework of the original system (12). In the general case of a wave whose envelope moves at a constant velocity u, we have

$$\begin{gathered}\varepsilon=\psi e^{iS},\quad \psi=\psi(\zeta)=\psi(\xi-u\tau),\\ k=\frac{\partial S}{\partial\xi}=k(\zeta)=k(\xi-u\tau),\quad \frac{\partial S}{\partial\tau}=-\Omega-ku.\end{gathered} \tag{61}$$

Substituting Eq. (61) into the system (12) and separating the real and imaginary components, we obtain the integral generalizing Eq. (46):

$$\frac{1}{\psi^2}\left(\frac{d}{d\zeta}\psi^2R\right)^2+M^2=2\psi^2R+\text{const}, \tag{62}$$

$$\frac{dM}{d\zeta}=\frac{k}{\psi}\frac{d}{d\zeta}\psi^2R, \tag{63}$$

$$u\frac{d^2\psi}{d\zeta^2}=\psi kV+M, \tag{64}$$

$$V=ku+\Omega+\frac{\psi^2}{1-u^2},\quad R=ku+\frac{\Omega}{2}+\frac{3}{4}\frac{\psi^2}{(1-u^2)}. \tag{65}$$

We can obtain Eq. (46) if we make the following simplifications. Firstly, in R and V of Eq. (65) we drop ku, which is small compared with the other terms. Since ψ^2 is either of the order of Ω or much smaller than Ω, the relevant criterion is

$$k\ll|\Omega|/u. \tag{66}$$

Secondly, we have to neglect the left-hand side in Eq. (64), i.e., we have to assume that $M=-\psi kV$ [the substitution of this relationship in Eq. (63) gives Eq. (45)]. This means that our solution is valid if

$$k\gg u/\xi_0{}^2|\Omega|. \tag{67}$$

Thirdly (this is not so important), we have to assume that $u\ll 1$, i.e., we have to postulate that the motion is subsonic.

The last condition is not so important because for $u\sim 1$ and even for $u\gg 1$, an integral of the (46) type is easily obtained. We shall analyze the criteria (66) and (67). Outside the peaks of a sawtooth wave we have $k\sim k_0$, $\xi_0^2\sim|\Omega|$ and, consequently, the solutions found above are valid if

$$u/\Omega^2\ll k_0\ll|\Omega|/u,\quad u\ll|\Omega|^{3/2}. \tag{68}$$

In the region of the teeth we have $k\sim k_0/k_0^2|\Omega|$, $\xi_0^2\sim k_0^2\Omega^2$, i.e.,

$$k_0\gg u/\Omega^2. \tag{69}$$

Thus, the condition (68) is sufficient to satisfy the criteria (66) and (67). Finally, the condition $k_0^2|\Omega|\ll 1$, together with Eq. (68), gives again $u\ll|\Omega|^{3/2}$. The last inequality is easily satisfied if $a_0\sim|\Omega|^{1/2}\gg 1$ not only in the subsonic but also in the supersonic range, where our solutions are no longer valid.

For supersonic waves we can also find the solution when (68) is satisfied and for $k \ll u/\Omega^2$.

§ 6. HEATING OF PARTICLES BY OSCILLATIONS IN THE CASE OF A DEVELOPED STRONG TURBULENCE OF OSCILLATIONS WITH INVERSION DISPERSION

We shall now follow our earlier treatment.[7] The appearance of short-wavelength harmonics during the growth of the modulation instability of oscillations, described by Eq. (2) and characterized by $k_0^2|\varepsilon_0|^2>1$, results, in the same way as in the asymptotic solutions, in possible transfer of energy from a source pumping the energy to the main scale k_0 of the oscillations (the pumping is by light beams and particles in the case of the modified Langmuir waves, and by the current in the case of the short-wavelength ionic Langmuir oscillations), via the Landau absorption in the short-wavelength part of the oscillation spectrum, to thermal or near-thermal particles. The last qualification is needed because Eq. (2) is usually invalid for $\omega/k\approx v_T$. Thus, for the ionic Langmuir oscillations (2) we have to include the term $k^2v_{Ti}^2/\omega_0$, so that the theory developed above is valid if

$$kv_{Ti}/\omega_0<(T_i/T_e)^{1/4}. \tag{70}$$

Substituting the spectral expansion (60) into the quasilinear equation for the particles, we obtain

$$\frac{\partial f_\alpha}{\partial t}=\frac{\pi e^2}{m_\alpha^2}\frac{\partial}{\partial v}\int dk|E|_k^2\delta(\omega_0-kv)\frac{\partial f_\alpha}{\partial v}$$
$$=\frac{\pi e^2}{m_\alpha^2}\frac{E_0^2}{\omega_0}\frac{\partial}{\partial v}\frac{k_0^2v^2}{\omega_0^3}\frac{\partial f_\alpha}{\partial v},\qquad v>v_{min}. \tag{71}$$

Hence, it follows that

$$\frac{\partial}{\partial t}\frac{3}{2}nT_\alpha\approx\omega_0\frac{E_0^2}{8\pi n}\int_{v_{min}}^{\infty}\frac{k_0^2v^2}{\omega_0^3}f_\alpha\,dv,\qquad v_{min}>\frac{\omega_0}{k_{max}}. \tag{72}$$

We shall now list some of the phenomena in which the above modulations of oscillations and of the plasma dendensity can be important:

1) the runaway electrons in tokamaks may build up the modified electron Langmuir waves and heat the majority of electrons;

2) the high-current electron beams can heat a plasma during the last stage of the quasilinear relaxation if $n_bmv_b^2>nT$;

3) the nature of turbulence may change as a result of formation of the energetic component in the excitation of the Langmuir turbulence by electron and optical beams when the condition $n'T'>nT$ is satisfied;

4) the majority of ions may be heated if the current excites a strong ion-acoustic turbulence.

We shall show how to estimate the current velocity u in the turbulent heating case. It follows from the momentum balance for ions that

$$enE_{ext}=m_i\frac{d}{dt}\int vf_i\,dv\approx\frac{k_0E_0^2k_0^2v_{Ti}^2}{8\pi\omega_{pi}^2}\approx\frac{d}{dt}nT_i\frac{1}{v_{Ti}}. \tag{73}$$

Since $dT_e/dt=euE_{ext}$, it follows that $T_i/T_e=v_{Ti}/u$ or $u/v_s=(T_e/T_i)^{1/2}$. Damping of the nonlinear oscillations by ions should be balanced out by pumping with the current. Hence,

$$\frac{u}{v_{Te}}\approx\left(\frac{T_i}{T_e}\right)^{3/2}(k_0r_d)^5.$$

Combining these relationships, we obtain

$$u/v_{Te}\approx(m/m_i)^{1/3}k_0r_d. \tag{74}$$

§ 7. CONCLUSIONS

The criteria of the modulation instability have been established for the system (10)–(11) [or, in terms of dimensionless variables, for the system (12)] and sawtooth solutions have been found: these are standing oscillations of modulated amplitude with the basic tooth scale $\xi_0\approx\varepsilon$ and a radius of rounding of the tooth peaks governed by the limit of the validity of the theory. A solution of this kind minimizes the energy integral (21) so that, in accordance with the above considerations, we may expect evolution of an extended wave packet and asymptotic establishment of such solutions.

It is natural to consider the question of validity of the above conclusions in the case of a real three-dimensional system. In the case of the Langmuir waves with the usual dispersion (1) it is found that the one-dimensional solutions become unstable in respect of packet contraction when applied in two and three dimensions. The physical reason for this is that the modulations size l and the amplitude of a Langmuir wave E are related by $l\propto E^{-1}$, and an increase in the amplitude increases the depth of the well in accordance with Eq. (11) as well as reduces the frequency and energy integral (21), i.e., the process of "collapse" is favored by the energy considerations. In the case of waves with negative dispersion there is no reason for the collapse because the relationship between the modulation size and amplitude $l\propto E$ is opposite to that required in the dynamic collapse process. Therefore, the solutions found above should apply to real three-dimensional systems.

The limits of validity of the representation (2) in the cases known to us are limited on the short-wavelength side of the spectrum by the effects of thermal motion of particles, which can be allowed for by additional terms in Eq. (2) of the type

$$\left[1+i\frac{\pi\omega}{2kv_{Ti}}f\left(\frac{\omega}{kv_{Ti}}\right)\right]\frac{k^2v_{Ti}^2}{\omega_0},\qquad\frac{\omega}{kv_{Ti}}\gg 1.$$

The imaginary part of the above expression corresponds to weak damping of oscillations by particles allowed for in § 6. The real part limits the radius of rounding of the teeth in our solutions. For example, in the case of the ionic Langmuir oscillations it cannot be less than $r_d(T_i/T_e)^{1/4}$.

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[1]) The necessary criteria are derived more rigorously below.
[2]) In this case we are discussing the analogy with a nonlinear

oscillator in which ψ corresponds to the coordinate and ξ corresponds to time.

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Collective oscillations and instability of a single-frequency state of parametrically excited waves

V. S. L'vov and V. B. Cherepanov

Institute of Automation and Electrometry, Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk

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Collective oscillations of parametrically excited waves are investigated with allowance for their scattering on one another and on random inhomogeneities of the medium. It is shown that in addition to oscillations investigated earlier in the *S* theory (self-consistent field) approximation, there are also relatively low-frequency oscillations (according to the *S* theory their frequency is zero). It is shown that distributions of parametrically excited waves with singular frequencies are unstable when their spectrum is broadened.

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Considerable attention is currently given to the phenomena which appear on parametric excitation of waves in ferromagnets,[1-3] antiferromagnets,[4,5] plasma,[6,7] ferroelectrics,[8] and other nonlinear media. In some important cases the wave dispersion law $\omega_{\mathbf{k}}$ is of the nondecaying type and an external field (pump wave) can be regarded as spatially homogeneous and monochromatic:

$$h(\mathbf{r},t)=h(t)=h\exp(-i\omega_p t). \tag{1}$$

A relatively simple theory, based on the self-consistent field approximation (*S* theory), is developed for this case in Refs. 1 and 9. This theory is in good qualitative and quantitative agreement with many experimental observations on ferromagnets and antiferromagnets (for details see the review in Ref. 1 and also Refs. 4 and 5). However, recent experiments require interpretation which goes beyond this theory. For example, measurements have been made of the spectral density N_ω of parametrically excited waves,[10]

$$N_\omega=\int n_{\mathbf{k}\omega}\,d\mathbf{k}, \tag{2}$$

where $n_{\mathbf{k}\omega}$ is the Fourier component of the correlation function $n_{\mathbf{k}}(\tau)$ of the complex amplitudes $a_{\mathbf{k}}(t)$ of traveling waves:

$$n_{\mathbf{k}}(\tau)\delta(\mathbf{k}-\mathbf{k}')=\langle a_{\mathbf{k}}(t)a_{\mathbf{k}'}^*(t+\tau)\rangle. \tag{3}$$

The experimental results show that the N_ω line has a finite width $\Delta\omega$ which depends in a certain way on the pump amplitude and other experimental conditions. However, the *S* theory, which describes correctly the integral characteristics of parametrically excited waves (PW's), predicts a singular distribution of PW's in the $\mathbf{k}$-ω space:

$$n_{\mathbf{k}\omega}\sim\delta(\tilde\omega_{\mathbf{k}}-\omega_p/2)\delta(\omega-\omega_p/2), \tag{4}$$

where $\tilde\omega_{\mathbf{k}}$ is the frequency $\omega_{\mathbf{k}}$ renormalized to the interaction [see Eq. (1.13) below].

L'vov[11] used the diagram technique to formulate integral equations for $n_{\mathbf{k}\omega}$ generalizing the *S*-theory equations by a systematic allowance for the Hamiltonian of the interaction of PW's given by Eq. (1.5). These equations give rise to a finite width of the distribution $n_{\mathbf{k}\omega}$ in respect of the modulus k:

$$\Delta\omega_{\mathbf{k}}\propto\nu,\quad \nu^3\propto\gamma^2(TN)^2/kv. \tag{5}$$

Here, γ is the logarithmic decrement of PW's; $N=\int n_{\mathbf{k}\omega}d\mathbf{k}\,d\omega$ is the total number of PW's; k is the characteristic wave vector of PW's ($2\tilde\omega_{\mathbf{k}}=\omega_p$); v is the group velocity.

The distribution of N_ω in respect of ω is more complex. The generalized equations have, like the *S*-theory equations, a "single-frequency" solution singular in ω:

$$N_\omega=N\delta(\omega-\omega_p/2), \tag{6}$$

which is investigated in Ref. 11. However, as shown earlier,[10] this solution is not the only one. In addition to the central line of Eq. (6), the solution of N_ω may