to evaluate an integral of the form

$$I=\int_{-\infty}^{\infty}\frac{f(x)\,dx}{x^2-\cos^2 L(x^2-1)^{1/2}}=\int_{-\infty}^{\infty}\frac{f(x)\,dx}{\sin^2 L(x^2-1)}\frac{1}{2ix}\left(\frac{1}{m_+(x)}-\frac{1}{m_-(x)}\right), \tag{1a}$$

$$m_\pm(x)=(x^2-1)^{1/2}\operatorname{ctg} L(x^2-1)^{1/2}\mp ix.$$

We shall use the fact that the functions $m_\pm(x)$ have no zeros for $\operatorname{Im} x \gtrless 0$, respectively.[11]

We also take into account the fact that the two-valued function $g(z)=(z^2-1)^{1/2}$ has a regular branch assuming positive values on the positive axis outside the interval $(-1, 1)$. We begin by considering integrals with finite limits $-R$, R along the real axis, including the ε-intervals of the singular points

$$z_0=0,\quad z_k=\frac{k}{|k|}\left[1+\left(\frac{\pi k}{L}\right)^2\right]^{1/2},\quad (k\neq 0).$$

We have

$$I_{R\varepsilon}^{\pm}=\int_{-R}^{R}\frac{f(z)\,dz}{\sin^2 L(z^2-1)^{1/2}\cdot 2iz m_\pm(z)}. \tag{2a}$$

Substituting $R=[1+\pi^2(k+\frac{1}{2})^2/L^2]^{1/2}$, we consider the contour of integration for $I_{R\varepsilon}^{+}$ in the complex plane, which includes the semicircle of radius R for $\operatorname{Im} z\geqslant 0$, semicircles of small radius ε around the first-order poles z_0 and z_k, and the segment $[-R, R]$ along the real axis without the ε-intervals of the singular points z_0 and z_k. If

$$\left|\frac{zf(z)}{\sin^2 L(z^2-1)^{1/2}}\right|\leqslant M,\quad \operatorname{Im} z\geqslant 1, \tag{3a}$$

is satisfied on this contour, where $M=\text{const}$, the evaluation of $I_{R\varepsilon}^{+}$ reduces to the summation of the poles z_0 and z_k subject to the condition that $f(z)$ has no poles in the upper half-plane. Similarly, evaluating $I_{R\varepsilon}^{-}$ for $\operatorname{Im} z\leqslant 0$ and taking $\varepsilon\to 0$, $R\to\infty$, we obtain

$$I=-\frac{\pi f(0)}{\operatorname{sh} L\operatorname{ch} L}+\sum_{k=-\infty}^{\infty}\frac{\pi}{L[1+(\pi k/L)^2]}f\left(\frac{k}{|k|}\left(1+\left(\frac{\pi k}{L}\right)^2\right)^{1/2}\right). \tag{4a}$$

This formula was used in evaluating (21) and (22) with the function f given by (20).

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Structure of the field near a singularity arising from self-focusing in a cubically nonlinear medium

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The structure of the field is studied near a singularity that arises in the propagation of intense light beams in a nonlinear (cubic) medium. The structure near the focus is determined by a method of numerical integration of a parabolic equation with the step of the transverse coordinate changed automatically as the singularity is approached. Methods of analyzing this solution, based on extension of the scales and the use of functional relations which are invariant with respect to the exact position of the focus, make it possible to develop an idea of the formation of the field near the singularity by a bell-shaped beam with a Townes profile with adjacent weakly focusing wings, which converges to a point. On the basis of this concept the analytic form of the field near the singularity is described by the function $E\sim[\ln|(z_{sf}-z)|/(z_{sf}-z)]^{1/2}$, which is in complete agreement with the numerical results.

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In the propagation of an intense light beam in a medium with a cubic nonlinearity $[\varepsilon=\varepsilon_0(1+\varepsilon'|E|^2)]$ there can be regions where the amplitude increases without limit—foci.[1,2] The character of the singularity of the field at a focus has been discussed in the approximation of a parabolic equation

$$2i\,\partial E/\partial z=\Delta_\perp E+|E|^2E \tag{1}$$

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has been discussed in a number of papers,[3-8] but the results given are highly contradictory. There are various reasons for the discrepancies; either the calculations were not carried to very large amplitudes,[3,4] or the procedures used were not accurate enough owing to errors in determining the position of the focus z_{sf},[5-7] or, finally, the analytic approaches were based on extremely disputable *a priori* ideas.[6-8]

It seems most likely that a locally self-similar field structure with a singularity $E \sim (z_{sf} - z)^{-1/2}$ is formed in the vicinity of the focus.[9,10] In the Third School on Nonlinear Vibrations in Distributed Systems (Gorkii, March, 1975), the authors of the present paper were informed about numerical results indicating that the structure of the field near the focus was very nearly self-similar with a singularity $(z_{sf} - z)^{-1/2}$. Gorbushina, Degtyarev, and Krylov[11,12] have arrived at the same conclusion by applying the method of Lagrangian coordinates to the numerical integration of Eq. (1).

The treatment described in the present paper is based on an intuitive interpretation of the results of the numerical experiment mentioned above, by which we derive an improved formula for the field at the focus.

$$E=[|\ln(z_{sf}-z)|/(z_{sf}-z)]^{1/2},$$

which is in good agreement with the numerical data.

The numerical integration of Eq. (1) was carried out on a BESM-6 computer, with the initial condition $E = E_{in}e^{-r^2/2}$ and using for the transverse coordinate a nonuniform mesh, which changed in accordance with the structure of the function E. As the amplitude increased the size of the steps near the axis was reduced and that far from the axis was increased, keeping the total number of steps the same. The steps were varied automatically during the calculation, depending on the magnitude of the difference between the values of the amplitude on the axis and at the point of the mesh closest to it. The accuracy of the calculations of E was 10^{-4}, a satisfactory result. as determined from the number of digits that were the same in different calculations of the same case, and was obtained if the total number of points along r was at least 300. The results of the calculations are given below.

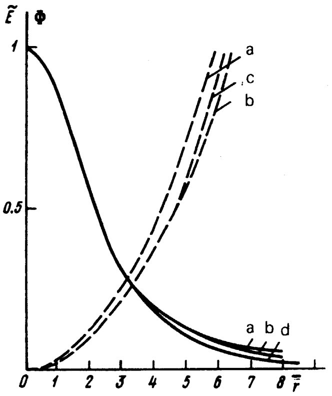

FIG. 1. Beam structure for various values of E/E in with $P=13.5\,P_{cr}$: a) $E/E_{in}=1.4\cdot10^2$; b) $E/E_{in}=(4-14)\cdot10^2$; d) Townes-profile beam; c) parabola. Solid curve is amplitude; dashed curve is phase.

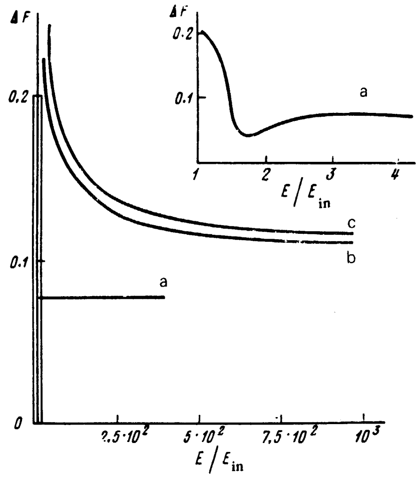

FIG. 2. Dependence of the phase difference ΔF on the ratio E/E_{in}: a) $P/P_{cr}=1.09$; b) $P/P_{cr}=13.5$; c) $P/P_{cr}=109$.

1. Figure 1 shows the structure of the function $|E|$ near the axis for a beam power $P=13.5P_{cr}$, where P_{cr} is the critical power for self-focusing. The amplitude E in Fig. 1 is normalized to the value of the function $E(r,z)$ at $r=0$, and the transverse scale is increased by a factor $E(0,z)$ $[\bar{r}=rE(0,z)]$. The structure of a uniform beam is also shown for comparison. Figure 1 shows that near the axis a self-similar structure,

$$E=E(0,z)\tilde{E}\{E(0,z)r\}e^{iE^2r^2/2}, \qquad (2)$$

is formed as the focal point is approached, which is close to that of a uniform beam with the self-similarity coefficient $E(0,z)$. However, the wings of this beam are not self-similar and fall off much more slowly than for a uniform beam.[1)] The difference in height and extent of the wings is determined by the initial structure of the beam and its power. The phase distribution in the beam is parabolic near the axis and varies slowly in coordinates E/E_{in} for large values of E/E_{in}.

The degree of approximation of the solution to the form (2) can be assessed from the phase difference ΔF in the beam between the point $r=0$ and the point $r_{0.5}$ at which the amplitude is half as large. Figure 2 shows the phase difference ΔF as a function of the field amplitude on the axis, $r=0$ for three values of the beam power. In all cases ΔF does not change much at large amplitudes.

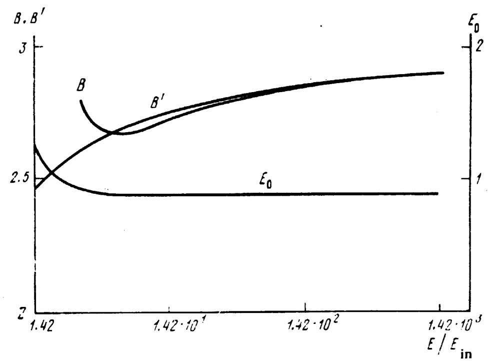

FIG. 3. Dependences of the quantities B, B', and E_0 on the ratio E/E_{in} for $P=13.5P_{cr}$.

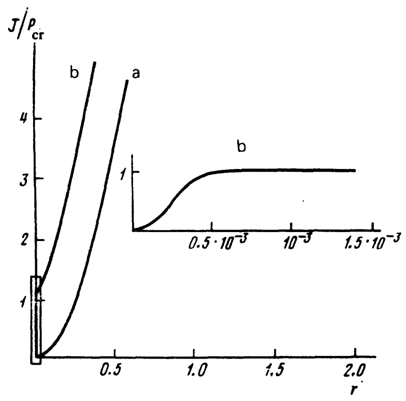

FIG. 4. Dependence of the power flux on r in a beam with $P=13.5P_{cr}$ and various values of E/E_{in}: a) $E/E_{in}=1$, b) $E/E_{in}=1.4\cdot10^2$.

It follows from Eq. (2) that the radius of curvature of the wave front varies as $1/|E|^2$. Since $\partial|E|/\partial z\sim|E|/R$, the field E increases near the focus z_{sf} approximately as $E\sim(z_{sf}-z)^{-1/2}$. The agreement with this law can be tested more precisely by analyzing the quantity $B=|E|_{zz}|E|_z^2$. This quantity is constant for any power law $E\sim(z_{cf}-z)^{-\alpha}$, and its value is uniquely related to the exponent α, $B=(\alpha+1)/\alpha$. Figure 3, in which B is plotted as a function of E/E_{in} for $P/P_0=13.5$, shows that B monotonically approaches the value 3, so that α approaches $\frac{1}{2}$. This shows that the function $E(z_{sf}-z)$ is close to a power law with exponent $\alpha=\frac{1}{2}$.

The calculations have also confirmed that a power of the order of the critical power is concentrated near the focus. This is illustrated in Fig. 4, which shows the quantity

$$J/P_{cr}\sim\int_0^r|E|^2r\,dr$$

as a function of the coordinate r for a total beam power $P=13.5P_{cr}$ and $E/E_{in}=1.4\cdot10^3$.

The solutions of Eq. (1) found numerically are not self-similar for $z\to z_{sf}$ for all values of r. The absence of self-similarity on the wings of the beam is very important, as we shall now show, in the formation of the field near the focus in bounded beams, and must necessarily be taken into account in attempts to determine the nature of the singularity analytically.

To interpret this numerical solution intuitively, it is convenient to put Eq. (1) in the form

$$-2i\partial A/\partial\tau+\Delta_\perp A+|A|^2A+\rho^2A/\Phi(\tau)=0, \tag{3}$$

by using the generalized lens transformation[13-15]

$$A(\rho,\tau)=\frac{1}{\sigma_p(\tau)}E(r,z)\exp\left(-\frac{i\rho^2}{2}\frac{\partial}{\partial\tau}\ln\sigma_p\right);$$
$$r=\rho/\sigma_p(\tau),\qquad z-z_{sf}=\sigma_s(\tau)/\sigma_p(\tau), \tag{4}$$

where $\sigma_p(\tau)$ and $\sigma_s(\tau)$ are two linearly independent solutions of the equation

$$d^2\sigma/d\tau^2-\sigma/\Phi(\tau)=0 \tag{5}$$

with the Wronskian $\sigma_s'\sigma_p-\sigma_s\sigma_p'=1$.

Using the transformation (4), we make the function $A(\rho,\tau)$ regular and nonvanishing for all τ, including

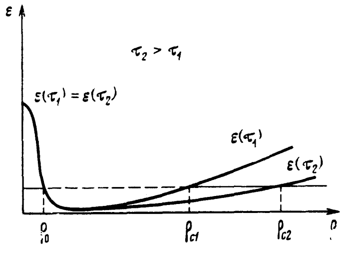

FIG. 5. Structure of dielectric waveguide for two values of τ.

$\tau\to\infty$. To do this it is sufficient to find a function $\Phi(\tau)>0$ such that the solution $\sigma_p(\tau)$ of Eq. (5) has the same type of singularity as $\tau\to\infty$ as $E(0\ z)$ has as $z\to z_{sf}$, and such that $\sigma_s(\tau)/\sigma_p(\tau)\to0$ as $\tau\to0$. On the other hand, for a solution $A(\rho,\tau)$ of Eq. (3) to exist which is finite for all τ it is necessary that the coefficient $\Phi(\tau)^{-1}$ of ρ^2 fall off sufficiently rapidly that for $\tau\to\infty$ the structure of a uniform beam can be formed.

Equation (3) describes the propagation of bounded beams of waves in a nonlinear dielectric in which the linear part of the dielectric constant has a defocusing profile. We assume that for sufficiently small $\Phi(\tau)^{-1}$ there exists near the axis $\rho=0$ a solution in the form of a quasilocalizable beam of nearly uniform structure which is propagated along τ with an almost constant phase velocity and a weak radiative damping γ. The condition for such a beam to emerge at infinity is that the integral $\int_0^\infty\gamma d\tau$ converge. The profile of the total dielectric constant $\Delta\varepsilon\sim|A|^2A+\rho^2/\Phi(\tau)$ in the field of such a beam is shown schematically in Fig. 5. In this profile there are two caustic surfaces for the beam, defined by the approximate condition $|A|^2A+\rho^2/\Phi(\tau)\approx1$, an inner surface ρ_0 and an outer surface ρ_c. Between the caustics the amplitude of the beam falls rapidly to a value proportional to the exponentially small factor $\exp(-M\rho_c)$, where the numerical coefficient M is determined by the form of the function $\Delta\varepsilon$ in the strip $\rho_0<\rho<\rho_c$.

The numerical calculations show that for $z_{sf}-z\to0$ the field changes only in a narrow region $0<r<\Delta(z-z_{sf})$, in which a power of the order of P_{cr} is localized and there are large transverse gradients of E. From the power-localization condition we have the estimate $\Delta\sim\sigma_p^{-1}$. Under the transformation (4) the range of values $r>0$ goes over into the range $\rho>\rho_c$. In this range we immediately have from Eq. (4) that $A\sim E_r/\sigma_p$, where E_r is the limiting value of the field in the region $r>\Delta$ for $z\to z_{sf}$. Since for $\rho\to\rho_c$ the nonlinear component of $\Delta\varepsilon$ is small, we have approximately $\rho_c\approx[\Phi(\tau)]^{1/2}$ and the magnitude of the field amplitude at the outer caustic is $A\sim\exp\{-M[\Phi(\tau)]^{1/2}\}/[\Phi(\tau)]^{1/4}$. Equating this value of the amplitude to E_r/σ_p in the region $\rho>\rho_c$, we get the relation

$$\exp\{-M[\Phi(\tau)]^{1/2}\}\sigma_p(\tau)/[\Phi(\tau)]^{1/4}=E_r(z_{sf})|_{\tau\to\infty}, \tag{6}$$

which together with Eq. (5) enables us to find the law of variation of $\Phi(\tau)$ as $\tau\to\infty$.

Analyzing the asymptotic behavior of the solution of (5)

$$\sigma_{p,s}=\pm[\Phi(\tau)]^{1/4}\exp\left\{\pm\int^\tau\frac{d\tau}{[\Phi(\tau)]^{1/2}}\right\}, \tag{7}$$

for $\tau \to \infty$ in the case[2] $1/\Phi > 1/\tau^2$ we get from the condition (6) the equation

$$-M\Phi^{1/2}+\int_0^\tau \frac{d\tau}{\Phi^{1/2}}=\text{const}, \tag{8}$$

which can be satisfied by the function $\Phi = 2\tau/M$.

The running value of the radiative damping coefficient of the quasilocalized beam is proportional to the square of the amplitude of the field on the outer caustic, i.e., $\exp[-2(2M\tau)^{1/2}]$. Since

$$\int_c^\infty \exp(-2(2\tau M)^{1/2})\,d\tau$$

converges, the total damping constant will be bounded, which corresponds to the formation at $\tau \to \infty$ of a uniform beam with finite power.

Returning by means of Eq. (4) from the variables ρ, τ to r, z and using the asymptotic form of $\sigma_p(\tau)$, we find that

$$E(z)\,|_{z\to z_{\mathrm{sf}}} \sim \sigma_p \sim E_0 [\,|\ln(z_{\mathrm{sf}}-z)\,|/(z_{\mathrm{sf}}-z)\,]^{1/2} \tag{9}$$

and the structure of the field in the neighborhood of the focus is of the form

$$E \sim \left[\frac{|\ln(z_{\mathrm{sf}}-z)\,|}{z_{\mathrm{sf}}-z}\right]^{1/2} E_\tau\left\{r\left[\frac{|\ln(z_{\mathrm{sf}}-z)\,|}{z_{\mathrm{sf}}-z}\right]^{1/2}\right\} \exp\left[-\frac{ir^2|\ln(z_{\mathrm{sf}}-z)\,|}{2(z_{\mathrm{sf}}-z)}\right], \tag{10}$$

where E_T is the field of the uniform beam.

3. Let us compare the data from the numerical calculations with the asymptotic formula (9). In Fig. 3 we have constructured the quantity $B' = |E|_{zz}|E|/|E|_z^2$ as determined from Eq. (9). For large E/E_{in} it agrees well with the function $B(E/E_{\mathrm{in}})$ found from the numerical calculations. In the same diagram we show the value of E_0 calculated by Eq. (8) from the numerical data.[3] For large amplitudes this quantity is constant to a high degree of accuracy and depends weakly on the power of the Gaussian beam, decreasing from $E_0 \sim 0.88$ for $P = 13.5\,P_{\mathrm{cr}}$ to $E_0 \sim 0.78$ for $P = 109\,P_{\mathrm{cr}}$. The speed of approach to the asymptotic form (9) is found from the calculations to decrease with increasing beam power.

We shall give an estimate of the limits of applicability of this asymptotic form. The dimensionless equation (1) is derived from the dimensional equation by introducing the coordinates x/a, y/a, z/ka^2 and the field $E_{\mathrm{new}} = ka(\varepsilon')^{1/2}E_{\mathrm{old}}$ where a is a characteristic dimension of the beam at $z=0$. Neglecting the vector character of the field, but including longitudinal diffusion of the amplitude, near the focus we would have to examine the equation

$$2iE_z' = \Delta E + |E|^2E + E_{zz}''/k^2a^2. \tag{11}$$

The condition for the last term in Eq. (11) to be negligible in comparison with the nonlinear term will be the inequality

$$k^2a^2E_0^2(z_{\mathrm{sf}}-z)\,|\ln(z_{\mathrm{sf}}-z)\,| \gg 1$$

in the case of the dependence (8), or in dimensional coordinates (assuming $E_0 \sim 1$)

$$k(z_{\mathrm{sf}}-z)\left|\ln\frac{z_{\mathrm{sf}}-z}{ka^2}\right| \gg 1.$$

For $ka \gg 1$ this inequality is satisfied for practically all values $k(z_{\mathrm{sf}}-z) > 1$.

These asymptotic formulas can be used to estimate the field strengths reached at the focus in the case when these are limited by multiphoton absorption and induced scattering. The estimates can be made approximately by assuming that the rates of increase of the field owing to self-focusing and of its decrease owing to the limiting effect are equal. We find the position and value of the maximum by equating these rates. In the case of K-photon absorption, for which

$$\varepsilon = \varepsilon_0(1+\varepsilon'|E|^2+\varepsilon''_{2K-2}|E|^{2K-2}),$$

the field at the focus is unbounded for $K<2$ and finite for $K>2$:

$$E_f^2 \approx (E_0)^{-2/(K-2)}\left\{\varepsilon'/\varepsilon''_{2K-2}\ln\frac{\varepsilon''_{2K-2}}{(\varepsilon')^{K-1}}E_0^{2K-2}\right\}^{1/(K-2)}. \tag{12}$$

The field E_f^2 found in this way with three-photon absorption ($K=3$) is in quantitative agreement with the numerical results of Ref. 1. For two-photon absorption and induced scattering this procedure cannot be used to estimate the field in the focal region, since the region where the asymptotic form (10) holds can be reached only when the cubic nonlinearity predominates strongly over the nonlinearity from two-photon absorption or induced scattering, and this gives field values $E_f^2 \gg 1/\varepsilon$, at the focus, which are beyond the range in which the original equation (1) can be applied.

[1]The analysis shows that the wings fall off approximately as $r^{-1/2}$.

[2]For $\Phi(\tau) > \tau^2$ Eq. (6) can be satisfied only for $E_r = 0$. This means that the entire power of the beam is contained in the localized part, i.e., $P = P_{\mathrm{cr}}$. A localized beam with $P = P_{\mathrm{cr}}$ can exist only when its structure is that of a uniform beam for all τ. In this special case the variation of the beam near the focus follows the law $E \sim (z_{\mathrm{sf}}-z)^{-1}$ (Ref. 10).

[3]The quantities E_0 and z_{sf} were calculated from values of $|E|$ and $|E|_z$ taken from the numerical results.

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Electron-electron collisions in a weakly ionized plasma

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Electron-electron collisions in a weakly ionized plasma are studied in the case when relaxation in momentum occurs on neutral atoms. It is shown that if the electron mean free path becomes less than the Debye length, the form of the diffusion coefficient in energy space changes: the Coulomb logarithm is cut off at the mean free path and, in addition, a nonlogarithmic contribution from large impact parameters appears. The results of the work are applicable also to semiconductors when momentum diffusion occurs, for example, in acoustic phonons and the energy relaxation is the result of electron-electron collisions.

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1. INTRODUCTION

It is well known that in a rather highly ionized plasma the relaxation of the electrons in energy is determined by electron-electron collisions, while the relaxation of nonequilibrium electrons in momentum occurs both on electrons and on ions. Since these collisions occur at large impact parameters, i.e., in each event there is a small transfer of energy and momentum, the operator of these collisions can be represented in differential form as was done by Landau.[1]

In a weakly ionized plasma the relaxation of the electrons in momentum occurs in collisions with neutral atoms, while electron-electron collisions continue to remain important for energy relaxation even in this case. The question arises of how the finite electron mean free path affects the form of the Landau operator. It is clear that as long as the electron mean free path l is much greater than the Debye screening radius $\varkappa^{-1}$, collisions with neutral atoms do not affect the nature of electron-electron collisions. In the present work we investigate the opposite limiting case: $\varkappa l \ll 1$, and derive the Landau operator for energy relaxation.

The principal result is that the electron-electron collision operator has the usual form

$$\left(\frac{\partial n_\varepsilon}{\partial t}\right)_{\text{coll}} = \frac{1}{\nu(\varepsilon)}\frac{\partial}{\partial\varepsilon}\int_0^\infty d\varepsilon'\nu(\varepsilon')\nu(\varepsilon)D(\varepsilon,\varepsilon')\left[n_{\varepsilon'}\frac{\partial n_\varepsilon}{\partial\varepsilon} - n_\varepsilon\frac{\partial n_{\varepsilon'}}{\partial\varepsilon'}\right], \quad (1)$$

where $\nu(\varepsilon) = (2m^3\varepsilon)^{1/2}/\pi^2\hbar^3$ is the density of electron states and n_ε is the electron distribution function. However, $D(\varepsilon,\varepsilon')$ differs from the standard expression.[1] It turns out that

$$D(\varepsilon,\varepsilon') = {}^2/_3 e^4 (vv')^{1/2} f(v'/v), \quad (2)$$

where $v = (2\varepsilon/m)^{1/2}$ is the electron velocity. The function $f(v'/v)$ has the obvious property $f(v'/v) = f(v/v')$ and for $v' \leqslant v$ we have the form

$$f\left(\frac{v'}{v}\right) = \left(\frac{v'}{v}\right)^{3/2}\ln\left(0.751\frac{l}{\rho_0}\right) + \varphi\left(\frac{v'}{v}\right). \quad (3)$$

Here ρ_0 is of the order of the electron wavelength for $e^2/\hbar v \ll 1$, and in the reverse case it is equal to the value of the impact parameter of the Coulomb scattering problem, at which the scattering angle becomes of the order of π.[2]

A plot of the function $\varphi(v'/v)$ is shown in Fig. 1. It is important to note that in the case considered φ does not depend on the parameter l/ρ_0 (with accuracy to terms of order ρ_0/l). For small v'/v we have

$$\varphi(v'/v) = \sqrt{3}v'/v. \quad (4)$$

It is evident from Eqs. (2) and (3) that, in contrast to the usual case, the Coulomb integral is cut off at the mean free path l and not at the Debye length. In addition, $D(\varepsilon,\varepsilon')$ contains a nonlogarithmic contribution which for a sufficiently high velocity ratio becomes greater than the logarithmic contribution. Physically the presence of the two terms in Eq. (3) is due to the fact that as long as the impact distances are small in comparison with the mean free path l, Coulomb colli-

