The phase transition to the state with charge confinement in two-dimensional massless electrodynamics

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The mechanism by which charged states disappear and a hadron spectrum appears is studied in the unitarity condition of two-dimensional quantum electrodynamics of massless charges. The confinement of a massless charge is not a necessary consequence of the linear growth of the two-dimensional Coulomb potential but is due to the phase transition from the system in which the free screened charge exists. Calculations of physical quantities by perturbation theory under these conditions become erroneous as soon as it becomes necessary to take into account the contributions of the vacuum reconstruction diagrams.

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1. INTRODUCTION

As has often been noted, in two-dimensional gauge models charge confinement is an automatic consequence of the linear growth of the Coulomb potential between the charges with distance. This conclusion directly applies only to infinitely heavy classical charges and does not in any way explain what happens with light charged particles under conditions when quantum corrections are important. Furthermore, the above considerations are simply inapplicable in the case of massless charges. The vacuum current induced by these particles screens the Coulomb field at even small distances.[1] The screened field of a massless particle contracts to a point in two-dimensional space-time and there is no obvious physical obstacle to the emission of a screened charge.

Therefore the confinement of a massless charge is not the only variant of two-dimensional gauge theories. It is possible to view confinement as the result of the passage to the limit from the model in which the free charge exists. In two-dimensional quantum electrodynamics (QED_2, the Schwinger model[1]) this passage to the limit is the phase transition from the phase which is symmetric under the chiral transformation to the spontaneously broken phase (Sec. 4).

Although the importance of chiral symmetry for the characteristics of the spectrum in the Schwinger model and for the nature of the phenomena occuring in this model has been stressed repeatedly (see, for example, Refs. 2 and 3), the formal nature of the solution to the problem[2-5] has hindered its understanding. It has not made possible the study of the appearance of the asymmetric vacuum state in the symmetric phase of perturbation theory and assessment of its properties.

It is difficult to find indications of confinement within the framework of pure electrodynamics, even by exactly summing the perturbation series.[6] The charged-particle amplitudes exist on the mass shell and do not indicate the presence of any anomalies. Casher, Kogut, and Susskind[3] proposed that in studying charge confinement in two-dimensional models, the properties of the vacuum expectation value of the product of non-electromagnetic currents ("flavor") which are usually scalar, be studied:

$$S(x)=i\langle 0|T(j(x),\, j(0))|0\rangle, \tag{1}$$

which is the analog of the quantity determining e^+e^- annihilation in hadrons. In the present study we shall construct a diagram solution for $S(x)$ in that variant of QED_2 in which the vacuum of the diagram solution (the absence of electrons and massive photons[1)]) is the physical vacuum of the model and we shall study the changeover from this model to the Schwinger model.

The changes which occur in the spectrum when changing from one model to the other appear in the calculation of the imaginary part of the Fourier component of $S(x)$ in the form of a spectral sum over all the states of the model:

$$\sigma(q)=\sum_n \langle 0|j|n\rangle\langle n|j|0\rangle\delta^{(2)}(q-q_n). \tag{2}$$

In addition to the mathematical complexity of this problem, the difficulty in interpreting the spectrum of states in an invariant gauge, which we shall use, is due to the double role played by the e^+e^- pairs produced by electromagnetic (EM) means. On the mass shell, pairs e^+e^- of particles moving to one side (e_Re_R or e_Le_L, according to the terminology of Sec. 3) are equivalent to the gauge photons of quantum electrodynamics: they do not participate in the interaction and require an indefinite metric (Secs. 2 and 3). Meanwhile, their presence in the intermediate states of the unitarity condition (2), when they interfere with e_Re_L pair produced by the scalar current (Sec. 4), does not change the properties of the states themselves but does determine the transition probabilities $\langle 0|j|n\rangle$ and is an important dynamical effect. It is precisely this that ensures the transition to confinement in the invariant gauge.

As the model with charge confinement we shall consider QED_2 with N massless electrons of charges g_l, interacting via the EM field A_μ. Then the interaction of the charges is determined by the parameter

$$\alpha_l=g_l^2/2\pi M^2, \tag{3}$$

where

$$M^2=\sum_{l=1}^N g_l^2/\pi$$

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is the vector-particle mass. This model becomes the Schwinger QED_2 when α_l is changed ($\alpha_l \to \frac{1}{2}$).

The model with N electrons was proposed by Segrè and Weisberger.[4] The expression for $S(x)$ in this model (see Sec. 2) indicates the presence of massless states in its spectrum. In Ref. 4 they were arbitrarily identified with the neutral massless scalar bosons Φ^l ($\epsilon_{\mu\alpha}\partial^\alpha\Phi^l = J_\mu{}^l, l=1,2,\ldots,N-1$) and it was not noticed that the states produced by these bosons cannot appear as intermediate states in (2) because of the chiral properties of the scalar current. However, as $N\to\infty$ the model proposed in Ref. 4 permits a smooth transition to a perturbation theory in α_l, in which the presence in (2) of states of massless charged e^+e^- pairs with opposite chiralities is obvious. When α_l is changed (the part $g_l \to 0$) all the quantities which determine the features of the model change smoothly up to $\alpha_l = \frac{1}{2}$. The state produced remains the state of a pair of massless charged fermions (Sec. 4). The fact that their wave function contains $e_R e_R$ or $e_L e_L$ gauge pairs expresses in the invariant gauge, as mentioned above, the dynamical effect of the change of the probability for producing this state.

For $\alpha_l \to \frac{1}{2}$ the continuous spectrum of the system in Ref. 4 changes into the chirally noninvariant vacuum state of Schwinger QED_2. Its structure remains unknown within the scope of the investigation of the spectral formula (2).[2)] The appearance of a new physical vacuum and the phase transition indicate that for $\alpha_l = \frac{1}{2}$ the delta function $\delta^{(2)}(q)$ corresponding to the discrete state with $q_n = 0$ appears in (2). The quantity $S(x)$, obtained by summing the diagrams, ceases at $\alpha_l = \frac{1}{2}$ to describe the physical process of particle production, since it was defined in the chirally symmetric vacuum, which at $\alpha_l = \frac{1}{2}$ is not the physical vacuum of the system. The $S(x)$ of perturbation theory calculated in this manner can be compared to the physical matrix element $S(x)$ for the Schwinger model, which was formally defined in Ref. 5. The massive vector boson in the model of Ref. 4, which in the diagrammatic approach is the gluon of Schwinger QED_2 (at $\alpha_l = \frac{1}{2}$), disappears at $\alpha_l = \frac{1}{2}$ from the spectrum of states together with the charged states. The "hadron" of the Schwinger model, defined relative to the new vacuum, turns out to be pseudoscalar.[5] The corrections to the perturbation-theory calculations which are due to confinement are given by the vacuum-reconstruction diagrams. The first of these diagrams gives a contribution $\sim x^4 \ln x^2$ (the theory with a dimensional coupling constant). Therefore for large q^2 both expressions for $S(x)$ coincide up to terms $\sim 1/q^4$.

Direct calculation of the spectral formula (2) (Secs. 3 and 4) permits study of the mechanism of replacing the "gluon-quark" states in the unitarity condition by "hadron" states, which is interesting in itself. In QED_2 with massless electrons this mechanism is ensured by the presence of gauge e^+e^- pairs in the state of the produced charged particle. The interference of these pairs with the particles created by the scalar current describes the change in the probability of producing the state as α_l is changed. The destructive nature of the interference is given by the Fermi statistics for electrons.

The difference between our study and those of Refs. 2–5 lies in the use of the diagrammatic solution and the construction of the spectral expansion (2) using only the on-shell amplitudes. It is this approach that allows us to exhibit the physical situation which arises in the model.

2. DIAGRAMMATIC DERIVATION OF THE EXPRESSION FOR *S(x)*

The simplicity of QED_2 with a massless electron is due to the following property of the two-dimensional Dirac matrices γ_μ (γ_0 and γ_1):

$$\gamma_\mu \underbrace{\gamma_\nu\gamma_\lambda\ldots\gamma_\sigma}_{\text{odd}}\gamma_\mu \equiv 0. \tag{4}$$

This relation and gauge invariance decrease considerably the number of possible diagrams and allow us to obtain a number of the exact properties of the quantities in QED_2 (Ref. 6).

1. The exact function for the electron propagator differs from the free propagator of the massless electron only because of the longitudinal part of the photon Green's function (see also Ref. 1):

$$G(p) = \frac{1}{-\hat{p}}. \tag{5}$$

In the following we shall consider closed electron loops or amplitudes on the mass shell. In these cases it can be assumed that the exact electron propagator is equal to (5).

2. The photon polarization operator is described by the single divergent diagram of Fig. 1. Calculation of its imaginary part, which is the contribution of a massless e^+e^- pair, gives the expression

$$\mathrm{Im}\,\Pi_{\mu\nu}(k^2) = g^2 k_\mu k_\nu \delta(k^2), \tag{6}$$

so that the regularized polarization operator is

$$\Pi_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+i\varepsilon}\right)\frac{g^2}{\pi}. \tag{7}$$

The transverse part of the photon Green's function acquires a mass. In an arbitrary invariant gauge (α) the exact photon Green's function has the form

$$D_{\mu\nu}(k) = \alpha\frac{k_\mu k_\nu/k^2}{k^2+i\varepsilon} + \frac{g_{\mu\nu} - k_\mu k_\nu/(k^2+i\varepsilon)}{k^2 - g^2/\pi + i\varepsilon}. \tag{8}$$

The longitudinal parts in (8) can, of course, be omitted in calculating any gauge-invariant quantities.

3. The amplitudes for the interactions of photons with each other are identically equal to zero.[6] This result is a consequence of the transverse character of these amplitudes

$$k_{1\mu} M_{\mu\nu\lambda\ldots}(k_1, k_2, \ldots) = 0 \tag{9}$$

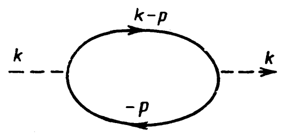

FIG. 1. The photon polarization operator.

and of the relations which follow from (4):

$$M_{\mu\mu\lambda\ldots}(k_1, k_2, \ldots)=0. \tag{10}$$

The only solution of (9) and (10) is $M_{\mu\nu\lambda}\ldots=0$.

Formula (6) and property 3 show that on-shell e^+e^- pairs created by purely EM means are gauge pairs. Therefore, closed EM loops cannot give a contribution to the imaginary parts of the amplitudes of processes; this, as usual, can be described by the introduction of an indefinite metric for e^+e^- pairs of this kind. However, as we shall see below, these pairs play a role in the unitarity condition (2), as they interfere with electrons created by nonelectromagnetic means.

All these features can be directly generalized to include QED_2 with N electrons.[4] In this model each electron interacts independently with a photon via A_μ. The only difference from QED_2 with a single electron is that the photon mass changes from g^2/π to the value

$$M^2=\sum_{i=0}^{N-1} g_i^2/\pi.$$

These features of QED_2 cause the vacuum expectation value of the product of the scalar currents j_l of one of the electrons (for example, with $l=0$)

$$j(x)=\bar\psi_0(x)\psi_0(x) \tag{11}$$

to be expressed as the sum of the contribution of the diagrams in Fig. 2. The vacuum state of this solution is obvious. It was given in the Introduction.

Let us write the quantity

$$S(x)=i\langle 0|T(j(x),\ j(0))|0\rangle=\int e^{iqx}S(q)\,d^2q/(2\pi)^2 \tag{12}$$

as an integral over the momenta p_1 and p_2 of the electrons that close the loop of Fig. 2 at the point x:

$$S(x)=i\int e^{ip_1x-ip_2x}\Phi(p_1,p_2)\frac{d^2p_1}{(2\pi)^2i}\frac{d^2p_2}{(2\pi)^2i}. \tag{13}$$

Then, after calculating the trace of the γ matrices the contribution to Φ from the sum of the graphs in Fig. 2 with n photons with all possible permutations of photon lines is equal to [3]

$$\Phi_n(p_1,p_2)=\frac{1}{2}\alpha_0^n\int\prod_{i=1}^{n}\frac{d^2k_iM^2}{2\pi i(k_i^2-M^2)}\frac{1}{n!}\Big\{\sum_{\text{perm }1}\Big[\frac{1}{p_{1+}+i\varepsilon p_{1-}^{-1}}$$
$$\times\frac{1}{(p_1-k_1)_++i\varepsilon(p_1-k_1)_-^{-1}}\cdots\frac{1}{(p_1-\sum_i k_i)_++i\varepsilon(p_1-\sum_i k_i)_-^{-1}}\Big]$$
$$\times\Big[\sum_{\text{perm }2}\frac{1}{p_{2-}+i\varepsilon(p_2-k_1)_+^{-1}}\frac{1}{(p_2-k_1)_-+i\varepsilon(p_2-k_1)_+^{-1}}$$
$$\cdots\frac{1}{(p_2-\sum_i k_i)_-+i\varepsilon(p_2-\sum_i k_i)_+^{-1}}\Big]+\text{terms }(+\rightleftarrows -)\Big\},$$
$$\alpha_0=g_0^2/2\pi M^2. \tag{14}$$

We have symmetrized the integrand with respect to the photon momenta $k_1, k_2, \ldots, k_n$ and have divided the symmetrized expression by $n!$ The sums over the permutations in (14) denote permutations of the photon lines $k_1, k_2, \ldots, k_n$ along the electron lines p_1 and p_2.

Substituting (14) into (13), we can write the expres-

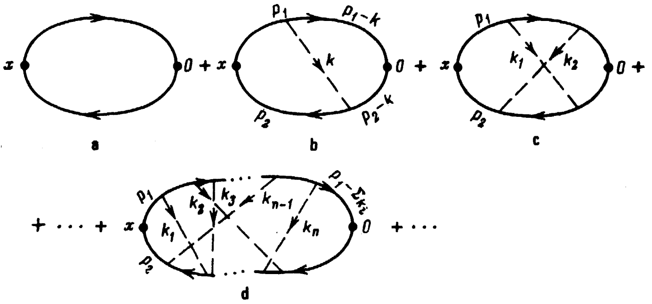

FIG. 2. Diagram for $S(x)$. The dashed lines show the contribution of the massive photon propagator (8).

sion for $S_n(x)$ as follows:

$$S_n(x)=i\frac{\alpha_0^n}{n!}\int\prod_{i=1}^{n}\frac{d^2k_iM^2}{2\pi i(k_i^2-M^2)}y_1(k_i,x)y_2(k_i,x), \tag{15}$$

where

$$y_1(k_i,x_+,x_-)=\int\frac{dp_-}{2\pi i}\exp(ip_-x_+)\int\frac{dp_+}{2\pi i}\exp(ip_+x_-)$$
$$\times\sum_{\substack{\text{perm}\\1,2,\ldots,n}}\frac{1}{p_++i\varepsilon p_-^{-1}}\frac{1}{(p-k_1)_++i\varepsilon(p-k_1)_-^{-1}}$$
$$\cdots\frac{1}{(p-\sum_i k_i)_++i\varepsilon(p-\sum_i k_i)_-^{-1}}, \tag{16}$$

and $y_2(k_i,x_+,x_-)=y_1(k_i,-x_-,-x_+)$. Calculation of the integral in (16) gives (see Appendix 1):

$$y_1=-\frac{1}{2\pi x_+}\prod_{i=1}^{n}\frac{\exp(ik_i\cdot x)-1}{k_i^+}. \tag{17}$$

Substituting (17) and the corresponding expression for y_2 into formula (15), after some simple transformations we find

$$S_n(x)=\frac{2i}{(2\pi)^2(-x^2)}\frac{\alpha_0^n}{n!}\Big[4M^2\int\frac{d^2k}{2\pi i}\frac{1-e^{ikx}}{k^2(k^2-M^2)}\Big]^n. \tag{18}$$

Letting $\Delta_{M^2}(x)$ denote the integral

$$\Delta_{M^2}(x)=\int\frac{d^2k}{(2\pi)^2i}\frac{e^{ikx}}{M^2-k^2}=\frac{1}{2\pi}K_0(M(-x^2+i\varepsilon)^{1/2}), \tag{19}$$

the sum of all the diagrams of Fig. 2 gives the following expression for $S(x)$:

$$S(x)=\sum_n S_n(x)=\frac{2i}{(2\pi)^2(-x^2)}$$
$$\times\exp\{8\pi\alpha_0[\Delta_{M^2}(x)-\Delta_{M^2}(0)-\Delta_0(x)+\Delta_0(0)]\}. \tag{20}$$

Using the properties of the MacDonald functions K_0, expression (20) can be rewritten as

$$S(x)=\frac{2i}{(2\pi)^2(-x^2)}\Big(\frac{M^2(-x^2)}{4}\Big)^{2\alpha_0}e^{4\alpha_0C}\exp\{8\pi\alpha_0\Delta_{M^2}(x)\}, \tag{21}$$

where C is the Euler constant.

Formulas (20) and (21) coincide, apart from a coefficient, with the solution obtained formally by Segrè and Weisberger[4] and at $\alpha_0=\frac{1}{2}$ they become the corresponding expressions for Schwinger QED_2 (Ref. 3). The singularity at $x^2=0$ in (21) indicates that the spectrum of the model in Ref. 4 contains massless excitations together with the massive vector particle M. At $\alpha_0=\frac{1}{2}$ the singularity at $x^2=0$ disappears and the spectrum of the Schwinger model consists only of massive neutral excitations, which the authors of Ref. 3 interpreted as charge confinement. However, for $\alpha_0 = \frac{1}{2}$ expression (21) has an important defect: $S(x) \neq 0$ for $x \to \infty$ (Ref. 5). This forces us to assume that the vacuum of the diagrammatic solution, which ensures that for $\alpha_0 < \frac{1}{2}$ quantities have the normal properties, cannot be the physical vacuum for Schwinger QED_2 (Ref. 2).[4)] This situation occurs if a phase transition occurs in the system. In the following sections we shall study the spectrum $\sigma(q)$ (2) for the model of Ref. 4 and its dependence on the parameter α_0 and show that this actually does occur in massless QED_2.

In order to study the spectrum it is necessary to calculate the imaginary part of the Fourier transform of (21) and interpret its spectral properties after expanding in intermediate states (2). The simple mathematical calculation then leads to a complicated and very indirect process of constructing the intermediate states in (2). Therefore, we shall arrive at the same result via a different route: we shall directly calculate the arbitrary transition amplitudes $\langle 0|j|n\rangle$ and sum them in $\sigma(q)$. The great mathematical complexity leads here directly to the construction of intermediate states. Only in this way is it possible to study the mechanism of the disappearance of the "quark" states in the unitarity condition.

3. THE TRANSITION AMPLITUDE $\langle 0|j|n\rangle$

In the invariant gauge and using the selected vacuum of the diagrammatic solution (see Sec. 1), the production of physical states (for example, vector particles like q_M in Fig. 3c) by the scalar current (11) is accompanied by an arbitrary number of EM e^+e^- pairs. In this section we shall find the amplitudes $\langle 0|j|n\rangle$ for transitions to these pairs, so that in the following section we can sum over all the states of the pairs.

The assignment of the electron to the mass shell

$$\begin{aligned} &e_R\!: \ p_-=0, \quad \gamma_-u_-=0, \quad \gamma_-v_-=0; \\ &e_L\!: \ p_+=0, \quad \gamma_+u_+=0, \quad \gamma_+v_+=0 \end{aligned} \tag{22}$$

uniquely determines the matrix structure of the amplitude and the kinematics of the electron pairs: pairs created by the EM current contain e^+ and e^- moving to one side while those created by the scalar current move in opposite directions. Different kinematical variants give identical contributions. Therefore we shall study only the variant shown in Fig. 3: particle 1 travels in the direction of e_R.

Since we are studying only on-shell amplitudes, we do not have to take into account the longitudinal terms in the propagators. Because of the nature of QED_2 (Sec. 2), for $\langle 0|j|n\rangle$ we have only diagrams of the type shown in Fig. 3. Furthermore, the interaction between different pairs (Fig. 3d) is absent because the sum of the contributions from diagrams with interchange of the photon (1 in Fig. 3d) is zero. A considerable simplification appears because the dependence on the momentum (l) of the created e^+e^- pairs or vector particles (q_M) factorizes with the dependence of the amplitude on the momenta of pair created directly by the scalar current (p_1 and p_2 in Fig. 3). This is because the propagator (5) of the massless electron is linear:

$$G_+^{-1}(p) = p_+ + i\varepsilon p_-^{-1} \ (a), \quad G_-^{-1}(p) = p_- + i\varepsilon p_+^{-1} \ (b) \tag{23}$$

for momenta $p_+(a)$ or $p_-(b)$ not equal to zero. For example, for the two diagrams of Fig. 3b the factors from the upper electron line give the expression ($p_{1-} = 0$)

$$\begin{aligned} &\frac{1}{(l+k)_- + i\varepsilon(p_1+l+k)_+^{-1}}\left[\frac{1}{k_- + i\varepsilon(p_1+k_1)_+^{-1}} + \frac{1}{l_- + i\varepsilon(p_1+l)_+^{-1}}\right] \\ &= \frac{1}{k_- + i\varepsilon(p_1+k)_+^{-1}}\,\frac{1}{l_- + i\varepsilon(p_1+l)_+^{-1}}. \end{aligned} \tag{24}$$

The factorization (24) appears if the poles in k_- and l_- do not contribute simultaneously, that is, if at least one of the momenta k_- or l_- is nonzero. However, the emitted particles are integrated in (2) over the phase space of the unitarity condition, so that their momenta can be assumed to be nonzero: the phase space of the state with $l_- = l_+ = 0$ is zero. For more complicated diagrams the factorization of the type (24) is demonstrated using the standard eikonal procedure (see, for example, Ref. 8 and Appendix 1).

Therefore, the general expression for the amplitude of producing n_1 pairs with $l_{i-} = 0$ and n_2 pairs with $l_{i+} = 0$ is written as

$$\begin{aligned} &F(p_1, p_2)\prod_{i=1}^{n_1} T(\bar{p}_{i+}', \bar{p}_{i+}')\prod_{i=1}^{n_2} T(\tilde{p}_{i-}, \tilde{p}_{i-}'); \\ &l_i = p_i + p_i', \quad q = p_1 + p_2 + \sum_i (\bar{p}_i + \bar{p}_i') + \sum_i (\tilde{p}_i + \tilde{p}_i'). \end{aligned} \tag{25}$$

The function $F(p_1, p_2)$ is a scalar form factor equal to the sum of the contribution of all the diagrams of Fig. 4a, as follows from (24),

$$T(p, p') = A(p, p')/(p+p') \tag{26}$$

and $A(p, p')$ is the exact amplitude for pair production

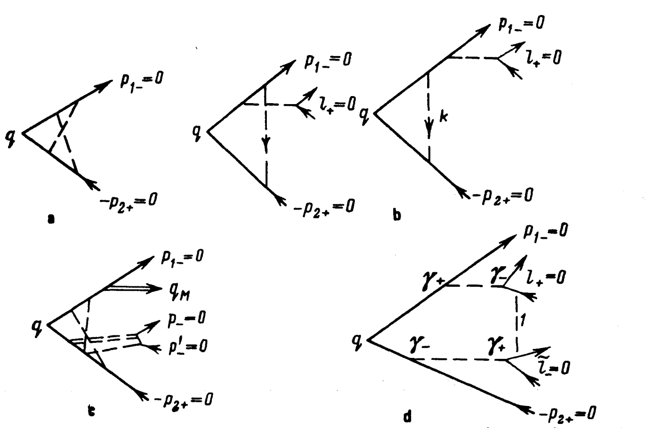

FIG. 3. Diagrams for the on-shell production amplitudes $\langle 0|j|n\rangle$.

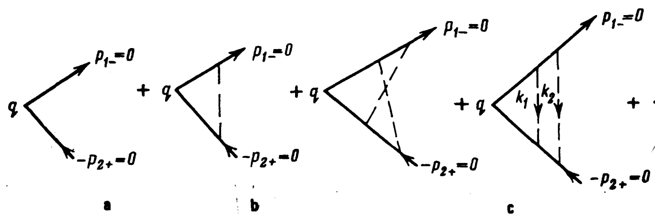

FIG. 4. Diagrams for $F(q^2)$.

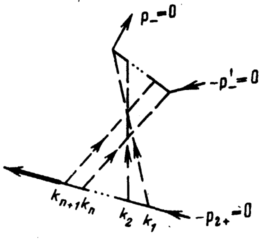

FIG. 5. The amplitude for producing a pair of virtual particles (thick arrow).

by a virtual charge (Fig. 5). The quantities $F(p_1,p_2)$ and $A(p_1,p_2)$ are difficult to calculate because for virtual photons there is no factorization like (24), since the coincidence of the poles of the electron lines plays an important role in this case.[5)]

A. Calculation of $F(p_1,p_2)$

Henceforth we shall call $F(p_1,p_2)$ the invariant coefficient of the factor $\bar{u}(p_1)v(-p_2)$. When calculated in the variables k_+ and k_-, the integral for the single-photon contribution to $F(p_1,p_2)$ (Fig. 4b) is infrared divergent. This divergence is accidental; it is due to the fact that when these variables are used, the double integral over the momentum of the photon line does not converge uniformly. The fact that this divergence is accidental is obvious because the diagram of Fig. 2b, whose imaginary part contains the form factor of Fig. 4b is infrared divergent. The correct result for the single-photon contribution,

$$F_1=-\alpha_0\ln(-q^2/M^2), \tag{27}$$

is obtained by an invariant calculation of the integral without changing to the variables k_+ and k_-.

The contribution of the two diagrams in the next highest order (Fig. 4c) is written as an integral:

$$F_2(q^2)=\frac{\alpha_0^2}{2!}\int\frac{d^2k_1}{2\pi i}\int\frac{d^2k_2}{2\pi i}[G_-(p_1+k_1)+G_-(p_1+k_2)]G_-(p_1+k_1+k_2)$$
$$\times[G_+(-p_2+k_1)+G_+(-p_2+k_2)]G_+(-p_2+k_1+k_2)\frac{M^2}{k_1^2-M^2}\frac{M^2}{k_2^2-M^2}. \tag{28}$$

Taking into account the fact that they are not factorizable, let us separate the correlators $Z_\pm$ from the G-function products:

$$[G_\pm(p+k_1)+G_\pm(p+k_2)]G_\pm(p+k_1+k_2)=Z_\pm^{(2)}(k_1,k_2)+G_\pm(p+k_1)G_\pm(p+k_2) \tag{29}$$

and substitute them in the integral (28). The integral of the product $Z^{(2)}$ is zero and the product of the independent G gives the square of F_1 (27); the combined terms are independent of the momenta and are easily calculated. The integral over one of the variables (k_-) is equal to

$$R_-(k_1,k_2)=\int\frac{dk_{1-}}{2\pi i}\int\frac{dk_{2-}}{2\pi i}Z_-(k_1,k_2)\frac{M^2}{k_1^2-M^2}\frac{M^2}{k_2^2-M^2}$$
$$=(-1)^2\theta(k_{1+}+k_{2+}-q_+)\theta(q_+-k_{1+})\theta(q_+-k_{2+})\theta(k_{1+})\theta(k_{2+}). \tag{30}$$

In calculating (30), the photon propagators under the integral can be replaced by $(-1)^2$, because the residues at the poles $k^2=M^2$ are equal to zero (see also Appendix 2).

The integral of (30) over dk_{i+} is

$$\int_0^{q_+}\frac{dk_{1+}}{k_{1+}}\int_0^{q_+}\frac{dk_{2+}}{k_{2+}}\theta(k_{1+}+k_{2+}-q_+)=\frac{\pi^2}{6}. \tag{31}$$

Therefore we find ($p_{1+}=q_+, p_{2-}=q_-, q^2=2q_+q_-$):

$$F_2(q^2)=(1/2!)[-\alpha_0\ln^2(-q^2/M^2)]^2-(-\alpha_0)^2\pi^2/6. \tag{32}$$

The summation of the contributions of all the diagrams in Fig. 4 is given in Appendix 2. It is based on factoring the correlators $Z_\pm^{(m)}$ out of arbitrary products of the G functions and leads to the following expression for $F(q^2)$:

$$F(q^2)=\frac{\exp(2\alpha_0C)}{\Gamma^2(1-\alpha_0)}\left(\frac{M^2}{-q^2}\right)^{\alpha_0}. \tag{33}$$

B. Calculation of $A(p,p')$ (Fig. 5)

Let us again factor out the trivial spinor coefficient. The principal difference between the calculation of $A(p,p')$ and of the form factor is related to the conservation law

$$k_1+k_2+\ldots+k_{n+1}=p+p'. \tag{34}$$

Because of (34) the total amplitude for the transition of the $n+1$ photons into the pair (p,p'), that is, the sum of the contributions of all diagrams within the photons permuted, is zero if at least one of the photon momenta (the component k_{i-} for the kinematics of Fig. 5) is nonzero. Then the photon propagators directly in the integral with respect to k_i can be replaced by $(-1)^{n+1}$.

Integration of the conservation law (34) over k_{n+1}, that is, the substitutions

$$k_{n+1}^+=(p+p'-k_1-\ldots-k_n)_+,\quad k_{n+1}^-=-(k_1+\ldots+k_n)_- \tag{35}$$

in all the expressions, leaves the sum of the contributions of all diagrams with $n+1$ photons symmetric under permutations of $k_1,k_2,\ldots,k_n$. This allows the product of the electron propagators to be symmetrized along the p_{2+} line of Fig. 5 ($p_{2+}=0$):

$$\frac{1}{k_1^+(k_1+k_2)^+\ldots(k_1+\ldots k_n)^+}$$
$$=\frac{(-1)^n}{n!}\sum_{\substack{\text{perm}\\1,2,\ldots,n}}\frac{1}{k_1^+(k_1+k_2)^+\ldots(k_1+\ldots+k_n)^+}=\frac{(-1)^n}{n!}\frac{1}{k_1^+k_2^+\ldots k_n^+} \tag{36}$$

In (36) we have neglected terms with $i\varepsilon$ in the propagators, since only $k_i^+>0$ are significant in the integrals over dk_{i+}. As a result the sum of diagrams with $n+1$ photon lines is written as

$$A_{n+1}=-\frac{\alpha_0^{n+1}}{n!}\int\frac{dk_1^+}{k_1^+}\ldots\int\frac{dk_n^+}{k_n^+}\int\frac{dk_1^-}{2\pi i}\ldots\int\frac{dk_n^-}{2\pi i}$$
$$\times\sum_{m=0}^{n}\sum_{\substack{\text{perm}\\1,2,\ldots,n}}G_-(p-k_1)\ldots G_-(p-k_1-k_2-\ldots k_m)$$
$$\times G_-(k_{m+1}+\ldots k_n-p')\ldots G_-(k_n-p'). \tag{37}$$

(Here we are considering only electrons of charge e_0). We have factored out the correlators in the products of $G_-(p^+\cdots)$ and $G_-(-p'+\cdots)$ just as in (A2.2) (see Appendix 2). In this case A_{n+1} turns out to be the sum of the products of the integrals of the correlators y_m

(A2.4) by the integrals of the independent products of $G_-(p+k_i)$. The latter are the powers of the contribution to $A(p,p')$ from diagrams with two photons:

$$A_2(p,p') = -\alpha_0^2 \int \frac{dk_1^+}{k_1^+} \int \frac{dk_1^-}{2\pi i} [G_-(p-k_1)+G_-(k_1-p')] = -\alpha_0^2 \ln \frac{p_+'}{p_+}. \quad (38)$$

The integrals y_m do not depend on the momenta p and p'. A simple combinatorial calculation of the summation of identical contributions in (37) due to the symmetry of the correlators (as in (A2.3)) causes (37) to reduce to the double sum

$$A_{n+1}(p,p') = -\alpha_0 \sum_{l+p+s=n} \frac{(-\alpha_0)^s y_s}{s!} \frac{\alpha_0^p y_p}{p!} \frac{[\alpha_0 \ln(p_+'/p_+)]^l}{l!}. \quad (39)$$

Summation over n and calculations similar to those at the end of Appendix 2 give the following for $A(p,p')$:

$$A(p,p') = -\alpha_0 \frac{x^{\alpha_0} e^{\alpha_0 C} e^{-\alpha_0 C}}{\Gamma(1+\alpha_0)\Gamma(1-\alpha_0)} = -\frac{\sin \pi\alpha_0}{\pi} x^{\alpha_0}, \quad x = \frac{p_+'}{p_+}. \quad (40)$$

The correlator technique can also be used for calculating other on-shell amplitudes. For example, summation of all the diagrams for e^+e^+ and e^+e^- scattering gives

$$T_{e^\pm e^\pm} = 2e^{i\pi\alpha_0} \sin \pi\alpha_0, \quad T_{e^\pm e^\mp} = 2e^{-i\pi\alpha_0} \sin \pi\alpha_0, \quad (41)$$

if the particles move to different sides [the $e_R e_L$ states (22)] and $T=0$ for the $e_R e_R$ or $e_L e_L$ states. The momentum transfer is always zero in the scattering. These properties are due to pointlike nature of the interaction between two-dimensional massless particles [(41) is independent of the energy] and to the two-dimensional kinematics. Expression (41) is in complete agreement with the two-particle unitarity condition, which for the amplitudes T_{++} and T_{+-} (the invariant amplitudes without the spinor coefficients) is written as

$$\mathrm{Im}\, T_{++} = {}^1/_2 |T_{++}|^2, \quad \mathrm{Im}\, T_{+-} = -{}^1/_2 |T_{+-}|^2. \quad (42)$$

These amplitudes have the correct crossing properties and therefore the correct analytic properties.

The existence of only two-particle unitarity for the fermion amplitudes is explained by the fact that the amplitudes for the interaction of fermions on the mass shell with photons of nonzero momentum are equal to zero. Therefore, the only possible fermion amplitudes of the model are T_{++} and T_{+-} (41). The addition of an arbitrary number of $e_R e_R$ ($e_L e_L$) pairs to an electron e_R (or e_L) does not change the physical e_R (e_L) state since e^+e^- pairs moving to one side are equivalent to gauge photons. Direct calculation of the amplitudes for the scattering of these states

$$|e_R\rangle + \sum_n C_n |e_R + n e_R^+ e_R^-\rangle,$$
$$|e_L\rangle + \sum_n C_n' |e_L + n e_L^+ e_L^-\rangle \quad (43)$$

by each other leads to formula (41). An important consequence of (41) and (42) is the absence of bound states in any of the on-shell $e_R e_L$ amplitudes. The off-shell $e_R e_R$ ($e_L e_L$) amplitude has a boson pole $s = M^2$ (the $e_L e_L$ mass shell the invariant $s=0$).

In conclusion we note that formulas (33) and (40) agree in their analytic and unitary properties with expression (41) and with the two-particle unitarity condition for F and A. The amplitude A depends on the two invariants $s = p_2 p$ and $s' = p_2 p'$ and the variable is $x = p_+'/p_+ = s'/s$.

4. THE UNITARITY CONDITION (2)

The substitution of (25), (33), and (40) into (2) for the current (11) permits us to sum over the contributions of "gauge" EM e^+e^- pairs. For $q^2 < M^2$ vector particles are not created and, as we shall see, this summation leads to a contribution of charged fermion states to (2).

In relation to the discussion in Sec. 2, EM e^+e^- pairs ($e_R e_R$ or $e_L e_L$) must have an indefinite metric of states in order that intermediate states like those in Fig. 6a, which correspond to the cutting of closed EM loops, not contribute to the unitarity condition. Therefore the only effect in which EM e^+e^- pairs participate is their exchange interference with electrons created by the scalar current (Fig. 6b). In this interference, of course, only pairs of charge e_0 participate. The interference graphs are obtained from graphs like those of Fig. 6a by interchanging the electron lines. Therefore, their contribution to $\sigma(q)$ is of alternating sign and is equal to

$$\sigma_0(q) = \frac{\exp(4\alpha_0 C)}{2\Gamma^4(1-\alpha_0)} \left(\frac{M^2}{q_+ q_-}\right)^{2\alpha_0}$$
$$\times \left\{ \sum_{n=0}^{\infty} (-\lambda)^n \int_0^{\infty} \frac{dx_1 \ldots dx_{2n+1} \delta(1-x_1-x_2-\ldots-x_{2n+1})}{(x_1+x_2)(x_2+x_3)\ldots(x_{2n}+x_{2n+1})} \right.$$
$$\left. \times \left(\frac{1}{x_1}\right)^{\alpha_0} \left(\frac{x_2}{x_1}\right)^{\alpha_0} \left(\frac{x_2}{x_3}\right)^{\alpha_0} \cdots \left(\frac{x_{2n}}{x_{2n+1}}\right)^{\alpha_0} \left(\frac{1}{x_{2n+1}}\right)^{\alpha_0} \right\}^2. \quad (44)$$

Here $\lambda = (\sin^2 \pi\alpha_0)/\pi^2$ from (40), the first term in the sum ($n=0$) is equal to unity, and $\frac{1}{2}$ is the usual factor of the unitarity condition. The summation over the spins of the intermediate states (the trace of the matrices) yields a factor of $\frac{1}{2}$, which is cancelled by the doubled contribution of the diagrams of Fig. 6b to allow for the kinematical situation wherein the antiparticle moves in the direction $p_{1-} = 0(e_R)$. The square of the sum in the curly brackets is a consequence of the fact that the summations over the contributions of pairs interfering with the electron e_R^- and with the

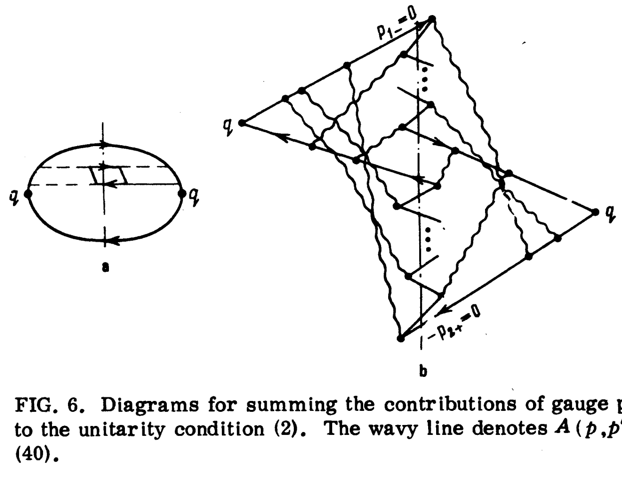

FIG. 6. Diagrams for summing the contributions of gauge pairs to the unitarity condition (2). The wavy line denotes $A(p,p')$ (40).

positron e_L^+ are independent. Therefore the integration in each sum is over only the momenta p_{i-} or only the p_{i+}. Finally, in (44) the dimensionless variables $x_i = p_{i+}/q_+$ or $x_i = p_{i-}/q_-$ are introduced in each sum.

The calculation of the sum in (44) is quite laborious (Appendix 3), but the result is simple:

$$I=\sum_{n=0}^{\infty}\ldots=\Gamma^2(1-\alpha_0)/\Gamma(1-2\alpha_0). \tag{45}$$

Consequently, for $q^2 < M^2$

$$\sigma_0(q)=\frac{\exp(4\alpha_0 C)}{2\Gamma^2(1-2\alpha_0)}\left(\frac{M^2}{q_+q_-}\right)^{2\alpha_0}, \tag{46}$$

which coincides with the answer which is easily obtained from (21).

Thus the physical contribution to the unitarity condition (46) can be written as (44), in which the factorized separation of the intermediate state into states of particles moving in the direction of e_R and particles moving in the direction of e_L is obvious. This shows that here the intermediate states are determined by the massless particles and coincide with (43). Consequently, these are states of massless charged electrons. This becomes obvious for $\alpha_0 \to 0$, when the contribution of "gauge" e^+e^- pairs is decreased.

Although the presence of e^+e^- pairs does not change the properties of the physical state created, it does determine the value of the probability for a transition into this state and has a direct dynamical meaning in (44). It is precisely the contribution of EM pairs and the destructive interference in (44) arising from the Fermi statistics that cause the contribution (46) to disappear from the spectrum (2) at $\alpha_0 = \frac{1}{2}$. In order to understand what happens at the point $q_+ = q_- = 0$ as $\alpha_0 \to \frac{1}{2}$, let us integrate (46) over some range of positive q_+ and q_-, that is, where $\sigma_0(q) = \mathrm{Im}\,S(q) \neq 0$. We then obtain the finite limit

$$\lim_{\alpha_0\to 1/2}\int\sigma_0(q)\,dq_+dq_-=\lim_{\alpha_0\to 1/2}\frac{e^{4\alpha_0 C}(M^2)^{2\alpha_0}}{2[(1-2\alpha_0)\Gamma(1-2\alpha_0)]^2}=\frac{e^{2C}m^2}{2},$$
$$m^2=g^2/\pi, \tag{47}$$

which indicates that the contribution $\delta^{(2)}(q)$ with the coefficient (47) is present in the sum (2) for the Schwinger model.

Reduction of the contribution of the continuous spectrum to the spectral sum (2) to the form $\delta^{(2)}(q)$ indicates that in the Schwinger model (at $\alpha_0 = \frac{1}{2}$) a discrete state with $q_n = 0$ appears and is different from the vacuum of the diagrammatic solution. A perturbation transfers the system to this state even from the initial vacuum, so that the new state must be a stable vacuum of Schwinger QED_2. This in turn means that as $\alpha_0 \to \frac{1}{2}$ a phase transition occurs. In Schwinger QED_2 charge confinement occurs only because of the appearance of this new vacuum and of the phase transition.

The chiral properties of the vacuum wave function in (2) coincide with the properties of the continuum states of the model at $\alpha_0 = \frac{1}{2}$, that is, with the properties of the states of the $e_R e_L$ pair of massless fermions. This state is not invariant under the transformation $\exp(i\alpha\gamma_5)$, but acquires the coefficient $\exp(2i\alpha)$. Consequently we are dealing here with spontaneously broken chiral invariance. The transition is from the chrially symmetric vacuum of the diagrammatic solution to the spontaneously broken noninvariant vacuum.

The interesting question of the structure of the spontaneously broken vacuum state, which appears in the intermediate state (44) at $\alpha_0 = \frac{1}{2}$ and $q_\pm = 0$, needs more direct study. Since the on-shell fermion amplitudes (41) do not exhibit any anomalous features, this study must apparently be concerned with quantities off the mass shell.

We can also consider the contributions of the vector states to (2). Since the sum over the contributions of gauge pairs becomes $\delta^{(2)}(k)$ as $\alpha_0 \to \frac{1}{2}$ (here k is the momentum of all the electrons of a state), the entire momentum transferred by the current is concentrated in the massive particles. The corresponding amplitude (2) for the state with a single massive boson is

$$\sigma_1(q)={}^1/_2 m^2 e^{2C}\delta(q^2-m^2). \tag{48}$$

A transformation analogous to (44)–(46) shows that massive particles are created in a new state with $q_n = 0$, as must be if this is the new stable vacuum of the system.

5. CONCLUSION

Since the physical vacuum for Schwinger QED_2 differs from the vacuum of the diagrammatic solution, the physical quantities of the model must be calculated relative to the true vacuum of the system. The authors of Ref. 5 constructed such a solution for (1) on the basis of one of the operator solutions of the Schwinger model found in Ref. 2. This yields at $\alpha_0 = \frac{1}{2}$, instead of formula (21),

$$S(x)=\frac{im^2}{4\pi^2}[\cos 4\pi\Delta_{m^2}(x)-1]. \tag{49}$$

The formal nature of the solution in Refs. 2 and 5 allows us to verify only the fact that the solution (49) is defined in a chirally broken vacuum, such as we have constructed in the present study. As (47) and the study of Ref. 2 show, the hadron of the Schwinger model has pseudoscalar properties relative to this vacuum. The differences between (49) and (21) at $\alpha_0 = \frac{1}{2}$ begin with the $x^4 \ln x^2$ terms, which in fact coincides with the contribution of the first vacuum-reconstruction diagram (Fig. 2c) of our study.

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APPENDIX 1

Let $x_- > 0$. Then, integrating over dp_+ in (16) we find

$$y_1(x_+,x_-)=\int\frac{dp_-}{2\pi i}\exp(ip_-x_+)\sum_{\substack{\text{perm}\\ i_1,i_2\ldots i_n}}\left\{\frac{(-1)^n\theta(-p_-)}{k_{i_1}^+(k_{i_1}^++k_{i_2}^+)\ldots(\sum_i k_i^+)}\right.$$
$$+\frac{(-1)^{n-1}\theta(k_{i_1}^--p_-)\exp(ik_{i_1}^+x_-)}{k_{i_1}^+k_{i_2}^+(k_{i_2}^++k_{i_3}^+)\ldots(\sum_{i_2\ldots i_n}k_i^+)}$$
$$\left.+\frac{(-1)^{n-2}\theta(k_{i_1}^-+k_{i_2}^--p_-)\exp(ik_{i_1}^+x_-+ik_{i_2}^+x_-)}{(k_{i_1}^++k_{i_2}^+)k_{i_2}^+(k_{i_1}^++k_{i_2}^+)k_{i_3}^+\ldots(\sum_{i_3\ldots i_n}k_i^+)}+\ldots\right\}. \tag{A1.1}$$

Using the well-known formula of Ref. 8, we sum in each term over those indices which are not present in the numerator:

$$\sum_{\substack{\text{perm}\\ i_m, i_{m+1}\ldots i_n}} \left[k_{i_m}^+(k_{i_m}^+ + k_{i_{m+1}}^+)\ldots\left(\sum_i k_i^+\right)\right]^{-1} = k_{i_1}^+ k_{i_2}^+ \ldots k_{i_{m-1}}^+ \Big/ \prod_{i=1}^n k_i^+. \quad (A1.2)$$

We next substitute (A1.2) into (A1.1) and use the formula

$$\int \frac{dp_-}{2\pi i}\exp(ip_-x_+)\theta(a-p_-) = -\frac{1}{2\pi x_+}\exp(iax_+). \quad (A1.3)$$

The series (A1.1) now takes the form $(k\cdot x = k_+x_- + k_-x_+)$

$$y_1 = \frac{(-1)^{n+1}}{2\pi x_+ \prod_i k_i^+}\sum_{m=0}^n \sum_{\substack{\text{perm}\\ i_1\ldots i_m=1}}^n (-1)^m \exp[i(k_{i_1}+k_{i_2}+\ldots k_{i_m})\cdot x]$$

$$\times \frac{k_{i_1}^+ k_{i_2}^+ \ldots k_{i_m}^+}{(k_{i_1}^+ + k_{i_2}^+ + \ldots + k_{i_m}^+)(k_{i_2}^+ + k_{i_3}^+ + \ldots + k_{i_m}^+)\ldots k_{i_m}^+}. \quad (A1.4)$$

Summing each term over permutations of the indices as in (A1.2), we find

$$y_1 = \frac{(-1)^{n+1}}{2\pi x_+ \prod_i k_i^+}\left\{1 - \sum_{i=1}^n \exp(ik_i\cdot x) + \sum_{i_1>i_2=1} \exp[i(k_{i_1}+k_{i_2})\cdot x] - \ldots\right\}, \quad (A1.5)$$

which reduces to formula (17):

$$y_1 = \frac{(-1)^{n+1}}{2\pi x_+ \prod k_i^+}\{(1-e^{ik_1\cdot x})(1-e^{ik_2\cdot x})\ldots\}. \quad (A1.6)$$

For $x_- < 0$ the result does not change.

APPENDIX 2

The contribution to $F_n(q^2)$ of diagrams with n photon lines is given by an integral similar to (28):

$$F_n(q^2) = \frac{\alpha_0^n}{n!}\int\prod_{l=1}^n \frac{d^2k_l M^2}{(2\pi i)(k_l^2-M^2)}\left\{\sum_{\substack{\text{perm}\\ k_1\ldots k_n}} G_-(p_1+k_1)\ldots G_-(p_1+\Sigma k_i)\right\}$$

$$\times\left\{\sum_{\substack{\text{perm}\\ k_1\ldots k_n}} G_+(-p_2+k_1)\ldots G_+(-p_2+\Sigma k_i)\right\}. \quad (A2.1)$$

As in (29), we introduce the correlators

$$\sum_{\substack{\text{perm}\\ 1,2\ldots n}} G(p+k_1)G(p+k_1+k_2)\ldots G(p+k_1+k_2+\ldots+k_n)$$

$$= Z_n(k_1, k_2, \ldots k_n) + \sum_{\substack{\text{perm}\\ i_1, i_2,\ldots i_n}} Z_{n-1}(k_{i_1}, k_{i_2}, \ldots k_{i_{n-1}})G(p+k_{i_n})$$

$$+\ldots + G(p+k_1)G(p+k_2)\ldots G(p+k_n) \quad (A2.2)$$

and, taking into account the symmetry of $Z_m(k_1,\ldots,k_m)$, we find

$$F_n(q^2) = \frac{F_1^n(q^2)}{n!} - \sum_{2\le m_1+m_2\le n} (-\alpha_0)^{m_1+m_2}\frac{C_n^{m_1+m_2}[F_1(q^2)]^{n-m_1-m_2}C_{m_1+m_2}^{m_1}}{n!}y_{m_1}y_{m_2}, \quad (A2.3)$$

where F_1 denotes the contribution (25) and the numbers y_m, which are independent of q_+ and q_-, are equal to

$$y_m = \int_0^1 \frac{dk_{1+}}{k_{1+}}\ldots\frac{dk_{m+}}{k_{m+}}R_m(k_1^+, k_2^+\ldots k_m^+), \quad (A2.4)$$

$$R_m = (-1)^m \int\prod_{l=1}^m \frac{dk_{l-}}{2\pi i}\frac{M^2}{k_l^2-M^2}Z_m(k_{1-}, k_{2-},\ldots k_{m-}). \quad (A2.5)$$

We note that the integrals of the correlators over d^2k_i do not depend on p.

Study of the integrals (A2.4) and (A2.5) shows that the integration in (A2.4) is over the region in which the integral remains finite. This region is determined by the correlator R_m, which is equal to

$$R_m = \theta(k_1+k_2+\ldots+k_m-1) - \sum_{i_m}^m \theta(k_{i_1}+k_{i_2}+\ldots+k_{i_{m-1}}-1)$$

$$+\sum_{i_{m-1}\neq i_m}^m \theta(k_{i_1}+k_{i_2}+\ldots+k_{i_{m-2}}-1)+\ldots(-1)^{m-2}\sum_{i_1\neq i_2=1}^m \theta(k_{i_1}+k_{i_2}-1), \quad (A2.6)$$

$$1\geq k_i \geq 0.$$

The correlators $Z_m = 0$ if at least one of the momenta $k_{i-} \neq 0$. Then in calculating (A2.5) (or (30)) we can directly replace the product of the photon propagators by $(-1)^m$ under the integral in (A2.5). Summation over n in (A2.3) gives the expression

$$F(q^2) = \exp[F_1(q^2)]\left\{1 - \sum_m \frac{y_m(-\alpha_0)^m}{m!}\right\}^2 = \left(\frac{M^2}{-q^2}\right)^{\alpha_0} r^2. \quad (A2.7)$$

The coefficient r can be calculated if it is noticed that the sum of finite integrals

$$P(\gamma) = \sum \frac{(-\alpha_0)^n}{n!}\int_0^1 \theta(1-x_1-x_2-\ldots-x_n)\frac{dx_1}{x_1^{1-\gamma}}\ldots\frac{dx_n}{x_n^{1-\gamma}} \quad (A2.8)$$

can be reduced to a sum analogous to (A2.7) by separating the factorizable contribution in terms of the correlators R_m:

$$P(\gamma) = e^{-\alpha_0/\gamma}\left\{1 - \sum_{m=2}^\infty \frac{(-\alpha_0)^m}{m!}y_m(\gamma)\right\} = r(\gamma)e^{-\alpha_0/\gamma}. \quad (A2.9)$$

Here $y_m(\gamma)$ is the integral of (A2.4) in which the denominators k_i are replaced by $k_i^{1-\gamma}$.

Using the formula[9]

$$\int_0^1 \theta(1-x_1-x_2-\ldots x_n)\frac{dx_1}{x_1^{1-\gamma}}\frac{dx_2}{x_2^{1-\gamma}}\ldots\frac{dx_n}{x_n^{1-\gamma}} = \frac{\Gamma^n(\gamma)}{\Gamma(1+n\gamma)} \quad (A2.10)$$

and the representation for the inverse Γ function

$$\Gamma^{-1}(1+n\gamma) = \frac{i}{2\pi}\int_C (-t)^{-1+n\gamma}e^{-t}dt, \quad (A2.11)$$

we sum over n in (A2.8). Then

$$r = \lim_{\gamma\to 0} r(\gamma) = \lim_{\gamma\to 0} P(\gamma)e^{\alpha_0/\gamma} = e^{\alpha_0 C}/\Gamma(1-\alpha_0). \quad (A2.12)$$

Expressions (A2.7) and (A2.12) give (33).

APPENDIX 3

Let us transform the n-th term of the series in (44), by substituting in it the formula for the δ function

$$\delta(x) = \frac{1}{2\pi}\int e^{ikx}dk \quad (A3.1)$$

and the representation

$$(x_i + x_{i+1})^{-1} = \int_0^\infty \exp[-(x_i+x_{i+1})t_i]dt_i. \quad (A3.2)$$

After integrating over x_i, rotating the contours of integration with respect to t_i $(t_i \to iky_i)$, and integrating over over dk, the n-th term becomes

$$\left(-\lambda\frac{2\pi\alpha_0}{\sin 2\pi\alpha_0}\right)^n\int_0^\infty\frac{dy_1}{(1+y_1)^{1-2\alpha_0}}\int_0^\infty\frac{dy_2}{(1+y_1+y_2)^{1+2\alpha_0}}$$

$$\cdot\cdot\int_0^\infty\frac{dy_{2n-1}}{(1+y_{2n-2}+y_{2n-1})^{1-2\alpha_0}}\int_0^\infty\frac{dy_{2n}}{(1+y_{2n-1}+y_{2n})^{1+2\alpha_0}(1+y_{2n})^{1-2\alpha_0}}. \quad (A3.3)$$

We now use the formula[9]

$$\int_0^\infty\frac{dy_1}{(1+y+y_1)^{1-2\alpha_0}(1+y_1+y')^{1+2\alpha_0}}=-\frac{1}{2\alpha_0}\frac{1}{y-y'}\left[1-\left(\frac{1+y}{1+y'}\right)^{2\alpha_0}\right] \quad (A3.4)$$

and integrate over odd y_i in (A3.3). The problem of finding the sum (44) immediately reduces to solving the integral equation

$$I(x)=\frac{1}{(1+x)^{1-2\alpha_0}}+\frac{\pi\lambda}{\sin 2\pi\alpha_0}\int_0^\infty\frac{dx'}{x-x'}I(x')\left[1-\left(\frac{1+x}{1+x'}\right)^{2\alpha_0}\right], \quad (A3.5)$$

and the sum (44) is equal to

$$1+\frac{2\pi\alpha_0\lambda}{\sin 2\pi\alpha_0}\int_0^\infty\frac{dy_1}{(1+y_1)^{1-2\alpha_0}}\int_0^\infty\frac{dy_2}{(1+y_1+y_2)^{1+2\alpha_0}}I(y_2). \quad (A3.6)$$

Solution of (A3.5) by the standard Wiener-Hopf method gives the answer

$$I(x)=\frac{\Gamma^2(1-\alpha_0)}{\Gamma(1-2\alpha_0)}\int\frac{\Gamma^2(1-\alpha_0-p)}{\Gamma(2-2\alpha_0-p)\Gamma(1-p)}(1+x)^{-p}\frac{dp}{2\pi i}. \quad (A3.7)$$

The contour in (A3.7) is drawn to the left of all the poles of the integrand:

$$I(x)=0, \quad -1\leqslant x\leqslant 0. \quad (A3.8)$$

Substituting (A3.7) into (A3.6) and integrating over y_2 after using (A3.8) (the lower limit of the integration over y_2 can be replaced by -1), we find that the contour integral is the well known[9] representation of the hypergeometric function. This reduces the latter integration to the tabulated integral[9]:

$$\Sigma\ldots=(1-2\alpha_0)\int_0^\infty F(\alpha_0,\alpha_0,1,-y_1)\,dy_1/(1+y_1)^{2-2\alpha_0}. \quad (A3.9)$$

1) In two-dimensional models with massless electrons this vacuum state is completely analogous to the vacuum of perturbation theory. The "diagrammatic" solution differs from perturbation theory in that it takes the "photon" mass into account exactly.

2) The sum (2) acquires a component of the physical vacuum of the model with chirality ±2 (Refs. 2 and 5).

3) In this article we are using the light cone variables

$$p_\pm=(p_0\pm p_1)/\sqrt{2},\quad \gamma_\pm=(\gamma_0\pm\gamma_1)/\sqrt{2},\quad \gamma_\pm^2=0$$
$$\gamma_+\gamma_-+\gamma_-\gamma_+=2,\quad \mathrm{Tr}\,\gamma_+\gamma_-=2,$$
$$p_a\tilde{p}_a=p_+\tilde{p}_-+p_-\tilde{p}_+.$$

The calculations of two-dimensional models are simplified considerably when these variables are used.

4) The physical vacuum of the model proves to be a superposition of states with different chirality (0, ±2, ±4, ... (Ref. 2); see also Ref. 7).

5) The neglect of these facts led to a number of errors in Ref. 6.

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