

Microscopic theory of spin waves in spin glasses

S. L. Ginzburg

B. P. Konstantinov Leningrad Institute of Nuclear Physics, Academy of Sciences of the USSR, Gatchina
(Submitted 5 June 1978)
Zh. Eksp. Teor. Fiz. 75, 1497–1503 (October 1978)

Microscopic equations describing small deviations of spins from an equilibrium state in a spin glass are derived. It is shown that, in the long-wavelength limit, these equations give solutions in the form of spin waves. A formula is obtained for the velocity of the waves.

PACS numbers: 75.30.Ds

1. INTRODUCTION

A macroscopic theory of spin waves in strongly disordered magnets has been developed recently.^{1,2} This theory shows, in particular, that well-defined spin waves with a linear dispersion law should exist in spin glasses and their existence is a simple consequence of the invariance of the spin system under arbitrary rotations.

However, in many problems the macroscopic theory is insufficient and there is a need to develop a microscopic theory. We shall derive microscopic equations describing spin waves in spin glasses. We shall show that in the hydrodynamic limit they reduce to the corresponding equations obtained in the macroscopic theory.^{1,2} We shall deduce a general expression for the velocity of spin waves in terms of microscopic constants. We shall show that the derived equations are analogous to the corresponding equations for the strain tensor of amorphous substances.

2. DERIVATION OF PRINCIPAL EQUATIONS

We shall consider the following model. We shall assume that magnetic atoms (concentration ρ) are distributed at random in space and that their interaction with one another can be described by the exchange integral $J(r)$ characterized by the mean value $\langle J \rangle = 0$. This condition can be ensured in two ways. Firstly, $J(r)$ may be an oscillatory function which varies rapidly over distances much shorter than the average separation between the atoms. This case occurs in the Kittel–Ruderman interaction. Secondly, $J(r)$ can be represented in the form $J_0 f(r/r_0)$, where r_0 is the radius of the potential and J_0 is a random quantity distributed, for example, in accordance with the Gaussian law. This is the model first considered by Edwards and Anderson³ and then used in many other papers.

The Hamiltonian of the interaction is of the form

$$H = - \sum_i J_{ij} S_i S_j, \quad J_{ij} = J(\mathbf{R}_i - \mathbf{R}_j), \quad (1)$$

where S is the spin operator. Using Eq. (1) and the commutation relationships for the spin operators, we obtain the usual equations of motion for spin:

$$\partial S_i / \partial t = 2 \sum_j J_{ij} [S_i S_j], \quad (2)$$

where $[ab]$ denotes a vector product. We shall simplify the problem by assuming that the spin is large. In this case we can ignore the quantization of the spin variables and assume simply that S is a classical vector of length S , whose equations of motion are given by Eq. (2). At absolute zero the system is in equilibrium, which corresponds to zero of the right-hand side of Eq. (2). The equilibrium values of the vectors S_i will be denoted by S_{0i} . They satisfy the equations

$$\left[S_{0i} \sum_j J_{ij} S_{0j} \right] = 0, \quad (3)$$

which imply that each spin is parallel to its local molecular field.

We shall now consider small deviations from equilibrium. As shown by Andreev,¹ such deviations can be described conveniently as follows. Let a vector S_{0i} be rotated through an angle θ about a unit vector n . Then, the new vector S_i can be expressed in terms of S_{0i} as follows:

$$S_i = S_{0i} + \frac{2}{1+\varphi^2} \{ [\varphi [S_{0i}]] + [\varphi S_{0i}] \}, \quad (4)$$

where

$$\varphi = n \operatorname{tg}^{1/2} \theta, \quad 0 < \theta < \pi.$$

Let us assume that the rotation of each spin is described by its local vector φ_i . If all the vectors φ_i are equal, the energy of the system does not change and we again obtain some equilibrium position. However, if φ_i are different, a force is established which tends to return the system to its equilibrium position. The equation describing this process in the linear approximation can be derived by substituting Eq. (4) into Eq. (2) and bearing in mind that the vectors S_{0i} are independent of t :

$$\left[\frac{\partial \varphi_i}{\partial t} S_{0i} \right] = 2 \sum_j J_{ij} \{ [[\varphi_i S_{0i}]] S_{0j} + [S_{0i} [[\varphi_i S_{0i}]]] \}. \quad (5)$$

Equation (5) is derived using the equilibrium condition (3). Substituting in Eq. (5) the expression $\varphi_j = \varphi_i + (\varphi_i - \varphi_j)$ and applying Eq. (3) again, we obtain

$$\left[\frac{\partial \varphi_i}{\partial t} S_{0i} \right] = 2 \sum_j J_{ij} \{ [(\varphi_i - \varphi_j) S_{0j}] S_{0i} \}. \quad (6)$$

The system (6) represents the equations of motion of the vectors $\varphi_i(t)$ which, according to Eq. (4), describe deviations of the initial state from equilibrium. It is clear from Eq. (6) that $\varphi_i = \text{const}$ is the solution of these

equations and this implies invariance of the system under arbitrary simultaneous rotation of all the spins through the same angle. This invariance has been discussed in detail earlier.^{1,2}

It is convenient to transform the system (6) into second-order equations. This can be done by solving Eq. (6) for φ_i . We shall show that the solutions are of the form

$$[\varphi_i S_{oi}] = \sum_j \hat{\chi}_{ij} \frac{\partial \varphi_j}{\partial t}, \quad (\hat{\chi}_{ij})^{\alpha\beta} = \frac{\partial m_i^\alpha}{\partial h_j^\beta}, \quad (7)$$

where the tensor $\hat{\chi}_{ij}$ represents a generalized susceptibility describing the appearance of a local magnetization m_i under the action of a local magnetic field h_i (for simplicity we shall assume that the gyromagnetic ratio is $\gamma=1$, i.e., that m_i is measured in units of γ and $\hat{\chi}_{ij}$ in units of γ^2). To derive Eq. (7) we shall consider the problem of the motion of spins in the presence of local magnetic fields h_i . By analogy with Eq. (2), we have

$$\begin{aligned} \partial S_i / \partial t &= [S_i, h_i] + 2 \sum_j J_{ij} [S_i, S_{oj}], \\ [S_i, h_i] + 2 \sum_j J_{ij} [S_i, S_{oj}] &= 0. \end{aligned} \quad (8)$$

The second equation in the system (8) describes new equilibrium states. These states S_{oi} can be expressed in terms of S_{ot} and local rotation angles φ_i . In the linear approximation it is clear that $\varphi_i \propto h_i$ and, therefore, we obtain from Eq. (8), by analogy of the derivation of Eq. (6) from Eq. (5),

$$[h_i S_{oi}] = 4 \sum_j J_{ij} [[(\varphi_i - \varphi_j) S_{oj}] S_{oi}]. \quad (9)$$

It is now quite clear that an applied magnetic field produces nonequilibrium moments $m_i = S_{it} - S_{ot} = 2[\varphi_i S_{oi}]$. This follows from Eq. (4) if the linear approximation is employed. By definition of the susceptibility, we have

$$m_i = \sum_j \hat{\chi}_{ij} h_i, \quad [\varphi_i S_{oi}] = \frac{1}{2} \sum_j \hat{\chi}_{ij} h_i. \quad (10)$$

The second expression in Eq. (10) follows from the definition of m_i . It gives the explicit form of the solution of Eq. (9) for φ_i .

We shall now return to Eq. (6). Comparing it with Eq. (9), we can see that the two equations are identical if h_i is replaced with $2\partial\varphi_i/\partial t$. Hence, we immediately obtain the explicit form of the solution of Eq. (6) for φ_i . This solution is Eq. (7).

We note that the above coincidence of h_i and $2\partial\varphi_i/\partial t$ has a deep physical meaning. In fact, $2\partial\varphi_i/\partial t = \omega_i$, where ω_i is the angular velocity of rotation of a spin S_{oi} and we can see that the velocity ω_i behaves exactly in the same way as the applied magnetic field h_i .

Substituting now Eq. (7) on the left-hand side of Eq. (6), we obtain

$$\sum_j \hat{\chi}_{ij} \frac{\partial^2 \varphi_j}{\partial t^2} = 2 \sum_j J_{ij} [[(\varphi_i - \varphi_j) S_{oj}] S_{oi}]. \quad (11)$$

Equation (11) is the microscopic analog of the corresponding equation obtained by Andreev.¹ We shall now find the microscopic analog of the equations deduced by

Halperin and Saslow.² It follows from Eqs. (6) and (7) that

$$\begin{aligned} \partial m_i / \partial t &= 4 \sum_j J_{ij} [[(\varphi_i - \varphi_j) S_{oj}] S_{oi}], \\ m_i &= 2 \sum_j \chi_{ij} \frac{\partial \varphi_j}{\partial t}, \quad m_i = 2[\varphi_i S_{oi}]. \end{aligned} \quad (12)$$

The second of the above equalities is the definition of a nonequilibrium magnetization m_i . The first two equations are microscopic analogs of the hydrodynamic equations obtained by Halperin and Saslow.² Summing Eq. (12) over i , we find that—on the basis of Eq. (3)—that the total moment $M = \sum_i m_i$ is an integral of motion.

3. ANALYSIS OF THE EQUATION OF MOTION

We shall now analyze the equation of motion (11). It is convenient to represent this equation in two equivalent forms

$$\begin{aligned} \sum_\beta \chi_{ij}^{\alpha\beta} \frac{\partial^2 \varphi_j^\beta}{\partial t^2} + \sum_\beta U_{ij}^{\alpha\beta} \varphi_j^\beta &= 0, \\ \sum_{j\beta} \chi_{ij}^{\alpha\beta} \frac{\partial^2 \varphi_j^\beta}{\partial t^2} + \sum_{j\beta} A_{ij}^{\alpha\beta} (\varphi_i^\beta - \varphi_j^\beta) &= 0, \end{aligned} \quad (13)$$

where

$$\begin{aligned} U_{ij}^{\alpha\beta} &= -A_{ij}^{\alpha\beta} + \delta_{ij} \sum_\ell A_{\ell\ell}^{\alpha\beta}, \\ A_{ij}^{\alpha\beta} &= 2J_{ij} \{ \delta_{\alpha\beta} (S_{oi} S_{oj}) - S_{oi}^\alpha S_{oj}^\beta \}. \end{aligned}$$

The matrices \hat{A} and \hat{U} have the following symmetry properties:

$$\begin{aligned} A_{ij}^{\alpha\beta} &= A_{ji}^{\beta\alpha}, \quad U_{ij}^{\alpha\beta} = U_{ji}^{\beta\alpha}, \quad \sum_i U_{ij}^{\alpha\beta} = \sum_j U_{ij}^{\alpha\beta} = 0, \\ \sum_i (A_{ij}^{\alpha\beta} - A_{ij}^{\beta\alpha}) &= \sum_j (A_{ij}^{\alpha\beta} - A_{ij}^{\beta\alpha}) = 0. \end{aligned} \quad (14)$$

The above relationships are derived using the equilibrium condition (3). We note that the last two properties of the matrices \hat{U} and \hat{A} are a consequence of the invariance of the system under rotations. We recall that in the problem of small vibrations of atoms in disordered systems we obtain similar equations and the main conditions imposed on the matrices \hat{U} and \hat{A} are the conditions (14) and the requirement that the matrices $\hat{\chi}$ and \hat{U} are positive and definite. This requirement is very important also in our case because, as can be seen from Eq. (13), the positive definite nature of $\hat{\chi}$ and \hat{U} ensures that the spectrum of Eq. (13) is positive, i.e., that $\omega_s^2 > 0$, where ω_s is the eigenvalue of this equation.

We shall now show that the positive definite nature of \hat{U} is a consequence of the fact that our system should be in the ground state at absolute zero and this ground state is a position of stable equilibrium.

We shall do this by substituting Eq. (4) into Eq. (1) and calculating the energy of a deformed state $V(\varphi_i)$ with $\varphi_i \neq 0$ in the approximation which is quadratic in φ_i :

$$\begin{aligned} V(\varphi_i) &= H(\varphi_i) - H(0) = -2 \sum_{ij} J_{ij} (\varphi_i - \varphi_j) [S_{oi} S_{oj}] \\ &+ 2 \sum_{ij} J_{ij} [(\varphi_i - \varphi_j)^2 (S_{oi} S_{oj}) - (\varphi_i - \varphi_j) [S_{oi} (\varphi_i S_{oi}) - S_{oj} (\varphi_j S_{oj})]]. \end{aligned} \quad (15)$$

The term linear in φ_i should vanish because of the necessary condition for the extremality of the ground state. Since $J_{ij} = J_{ji}$, we can see that this condition is identical with the equilibrium condition (3). Moreover, it readily follows from Eq. (13) that, if Eq. (3) is allowed for, the part of $V(\varphi_i)$ quadratic in φ_i is

$$V(\varphi_i) = 2 \sum_{ijab} U_{ij}^{ab} \varphi_i^a \varphi_j^b. \quad (16)$$

Since a sufficient condition for the ground state to be a position of stable equilibrium is the positive definite nature of the quadratic form $V(\varphi_i)$, our statement is proved. Thus, in the ground state the matrix \hat{U} must be positive definite. In fact, it is positive definite not only in the ground state but in any other state of stable equilibrium if such states do exist. The calculated energy $V(\varphi_i)$ is the potential energy. Comparing Eqs. (13) and (16), we obtain immediately the expression for the kinetic energy associated with the field φ :

$$\Gamma = 2 \sum_{ijab} \chi_{ij}^{ab} \frac{\partial \varphi_i^a}{\partial t} \frac{\partial \varphi_j^b}{\partial t} = \frac{1}{2} \sum_{ijab} \left(\frac{1}{\hat{\chi}} \right)^{ab}_{ij} m_i^a m_j^b, \quad (17)$$

where $(\hat{\chi})^{-1}$ is a matrix that is an inverse of $\hat{\chi}$. The second equality in Eq. (17) follows from Eq. (12). The formulas (16) and (17) define a second integral of motion, which is the total energy of the deformed state $E = T + U$. These formulas are the microscopic analogs of the corresponding formulas obtained by Andreev¹ and Halperin and Saslow.² (For comparison with Halperin and Saslow,² we recall that their vector θ is 2φ .) The Lagrange function for Eq. (13) is $L = T - U$.

The explicit form of the energy $E = T + U$ has an important consequence that φ_i and m_i are canonically conjugate variables and m_i are generalized momenta for generalized coordinates φ_i . We note that the definition of a nonequilibrium magnetization m_i implies that $m_i = 0$ in the ground state (otherwise we have to redefine S_{0i}). Moreover, the ground state may be stable against the appearance of spontaneous magnetic moments m_i . This means that $T(m_i \neq 0) > 0$ and $\hat{\chi}_{ij}$ must be positive definite matrix. Thus, both the kinetic and potential energies are positive definite quadratic forms. Hence, it follows that the eigenvalues of Eqs. (11) and (13) are real.

4. CALCULATION OF THE SPIN WAVE VELOCITY

We shall now consider the spectrum of low-lying excitations and derive a general formula for the spin-wave velocity. We shall do this using a method suggested for the problem of diffusion in a disordered system.⁴

The mathematical structure of the system (13) is, as mentioned above, analogous to that encountered in the problem of small vibrations in amorphous substances. Since low-lying oscillations in a macroscopically homogeneous medium are plane waves, it follows from this analogy that the corresponding eigenfunctions of our problem are modulated plane waves

$$\varphi_{ki} = u_{ki} \exp(i\mathbf{k}\mathbf{R}_i - i\omega_{ki}t), \quad (18)$$

where \mathbf{k} is the wave vector and ω_{ki} is the energy. As in Ref. 4, we shall assume that u_{ki} corresponding to small

values of \mathbf{k} can be expanded as a series in \mathbf{k} . Substituting Eq. (18) into Eq. (11), we obtain

$$\omega_{ki}^2 \sum_i \hat{\chi}_{ij} u_{kj} \exp(i\mathbf{k}(\mathbf{R}_j - \mathbf{R}_i)) = \sum_j A_{ij} [u_{kj} - u_{kj} \exp(i\mathbf{k}(\mathbf{R}_j - \mathbf{R}_i))]. \quad (19)$$

We shall now show that for small values of \mathbf{k} , we have

$$\omega_{ki}^2 = c^2 k^2, \quad c^2 > 0. \quad (20)$$

The quadratic dependence of ω_{ki}^2 on k^2 will be derived below; at this stage we shall assume that we know this dependence and we shall show that $c^2 > 0$. We shall multiply Eq. (19) by u_{ki}^* , sum over i , and let k on the left-hand side of Eq. (19) approach zero. Next, we shall allow for the fact that, as shown below, u_{0i} is independent of i . We then obtain

$$c^2 k^2 u_0 = \frac{1}{N} \sum_i \hat{\chi}_{ij} u_{0j} = \frac{1}{N} \sum_i A_{ij} [u_{kj} - u_{kj} \exp(i\mathbf{k}(\mathbf{R}_j - \mathbf{R}_i))]|_{k \rightarrow 0}, \quad u_{0i} = u_{ki}|_{k \rightarrow 0}, \quad (21)$$

where N is the number of particles in a crystal.

It follows from the definition of $\hat{\chi}_{ij}$ in Eq. (7) that the usual local susceptibility $\hat{\chi}_i$ and its average value $\langle \hat{\chi} \rangle$ are given by

$$\hat{\chi}_i = \sum_j \hat{\chi}_{ij}, \quad \langle \hat{\chi} \rangle = \frac{1}{N} \sum_i \hat{\chi}_i, \quad \langle \chi^{ab} \rangle = \langle \chi \rangle \delta_{ab}. \quad (22)$$

The last equality follows from the absence of any preferred direction in a spin glass. It then follows from Eqs. (18), (21), and (22) that

$$c^2 = \rho_s / \langle \chi \rangle, \quad (23)$$

where

$$\rho_s = \lim_{k \rightarrow 0} \frac{2}{N k^2 |u_0|^2} \sum_i \varphi_{ki}^* \hat{U}_{ij} \varphi_{kj}.$$

We can see that there is a positive definite form on the right-hand side of Eq. (23) and, therefore, $\rho_s > 0$. We also note that it follows from Eqs. (21) and (23) that φ_{ki} is an eigenfunction of the matrix \hat{U}_{ij} and $\rho_s k^2$ is its eigenvalue, so that once again we have $\rho_s > 0$. On the other hand, it is clear that $\langle \chi \rangle > 0$. Hence, it follows that $c^2 > 0$.

We shall now show that $\omega_{ki}^2 \propto k^2$ and that u_{ki} are independent of i . We shall do this by substituting $k = 0$ in Eq. (19):

$$\omega_{0i}^2 \sum_j \hat{\chi}_{ij} u_{0j} = \sum_j A_{ij} (u_{0j} - u_{0j}). \quad (24)$$

Summing over i and using Eq. (14), we find that $\omega_{0i}^2 = 0$. Substituting this value in Eq. (24) we can see that $u_{0i} = u_{0j}$, i.e., that u_{0i} is independent of i . Expanding Eq. (19) as a series in \mathbf{k} up to the first term and repeating exactly all the procedure given above, we find that the first term of the expansion of ω_{ki}^2 in terms of \mathbf{k} vanishes and the derivatives of u_{ki} with respect to \mathbf{k} satisfy the equations

$$\sum_i A_{ij} [\Psi_i^a - \Psi_j^a - i u_0 (R_j^a - R_i^a)] = 0, \quad (25)$$

$$\sum_i A_{ij} [\Psi_i^a - \Psi_j^a + i u_0 (R_j^a - R_i^a)] = 0,$$

where

$$\Psi_i^a = (\partial u_{ki}/\partial k_a)_{k=0}.$$

The second equation in the system (25) is associated with the first. We shall now expand Eq. (19) up to k^2 and sum again over i ; next, we shall multiply the obtained equation by u_0^* and use the second equation in the system (25). Then, simple transformations carried out on the assumption that $\rho_s = c^2 \langle \chi \rangle$ give the following expression for ρ_s :

$$\rho_s = -\frac{2}{9N} \sum_{ij} J_{ij} (S_{0i} S_{0j}) (R_i - R_j)^2 - \frac{1}{3N|u_0|^2} \sum_{ij} \Psi_i^a \cdot A_{ij} (\Psi_i^a - \Psi_j^a). \quad (26)$$

Bearing in mind that in the case of a positive definite matrix any form obeys $(\varphi^* \hat{U} \varphi) > 0$, we find from Eq. (26) that the second sum in that equation is positive and,

therefore, the expression

$$\rho_s = \frac{2}{9N} \sum_{ij} J_{ij} (S_{0i} S_{0j}) (R_i - R_j)^2 \quad (27)$$

represents an upper limit of ρ_s . This upper limit is identical with that obtained by Halperin and Saslow.²

The author is grateful to I. Ya. Korenblit for discussing this paper.

¹A. F. Andreev, Zh. Eksp. Teor. Fiz. **74**, 786 (1978) [Sov. Phys. JETP **47**, 411 (1978)].

²B. I. Halperin and W. M. Saslow, Phys. Rev. B **16**, 2154 (1977).

³S. F. Edwards and P. W. Anderson, J. Phys. F **5**, 965 (1975).

⁴W. Brenig, G. Dohler, and P. Wolfle, Z. Phys. **246**, 1 (1971).

Translated by A. Tybulewicz

Properties of superfluid ${}^3\text{He-A}$ near the transition into the A_1 phase

A. D. Gongadze, G. E. Gurgenishvili, and G. A. Kharadze

Physics Institute, Georgian Academy of Sciences

(Submitted 6 June 1978)

Zh. Eksp. Teor. Fiz. **75**, 1504–1510 (October 1978)

Some characteristic singularities of the behavior of superfluid ${}^3\text{He-A}$ in the presence of a strong magnetic field are observed in the immediate vicinity of the transition into the A_1 phase. It is shown that in narrow gaps of width $L \gtrsim \xi_D$ the texture of the anisotropy vector \mathbf{l} should change abruptly when the temperature of the $A \rightarrow A_1$ transition is approached, leading to a considerable variation of the spectrum of the NMR frequencies. Coupled oscillations of the density and of the longitudinal magnetization near the $A \rightarrow A_1$ transition are investigated. These oscillations merge in the A_1 phase with the magnetosonic mode typical of triplet pairing in the state with a single spin projection.

PACS numbers: 67.50.Fi

1. INTRODUCTION

It is presently customary to identify the superfluid A phase of liquid ${}^3\text{He}$ with the Anderson-Brinkman-Morel state (see, e.g., Ref. 1). For this anisotropic state, the nine-component complex order parameter that describes the coherent phase with triplet spin pairing in the p wave has a relatively simple multiplicative structure

$$A_{\mu i} = \Delta d_{\mu} u_i, \quad (1)$$

where $\Delta = \Delta(T)$ specifies the amplitude of the order parameter, and the complex vector $\mathbf{u} = (u_1 + iu_2)/\sqrt{2}$ describes an orbital state whose projection of the relative angular momentum of the Cooper pairs along the axis $\mathbf{l} = \mathbf{u}_1 \times \mathbf{u}_2$ (\mathbf{u}_1 and \mathbf{u}_2 are real orthogonal unit vectors) is equal to +1. As for the spin part of the order parameter, in the absence of a magnetic field it is

described by a real (unit) vector \mathbf{d} , along which the projection of the summary spin of the pair is equal to zero.

It must be recognized, however, that in the presence of a sufficiently strong magnetic field, and close enough to the temperature of the transition to the normal phase, the pairing amplitudes of the quasi-particles in states with summary-spin projection $S_z = \pm 1$ are not equal ($\Delta_+ \neq \Delta_-$) (Ref. 2). It is then no longer possible to describe the spin part of the order parameter by a single real vector, and the vector \mathbf{d} in (1) must be represented by a linear superposition

$$\mathbf{d} = c_+ \mathbf{d}_+ + c_- \mathbf{d}_-, \quad (2)$$

where the complex vectors $\mathbf{d}_+ = (\mathbf{d}_1 + i\mathbf{d}_2)/\sqrt{2}$ and $\mathbf{d}_- = \mathbf{d}_+^*$ describe states of pairs whose summary-spin projection on the direction $\mathbf{s} = \mathbf{d}_1 \times \mathbf{d}_2$ is equal to +1 or -1,