The Clebsch-Gordan coefficients, the *d*-function of *SU*(2), and their symmetry

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An integral transformation is described, which relates the Clebsch-Gordan coefficients (CGC) and the Wigner *d*-function for the group *SU*(2). The kernel of the transformation exhibits the Regge symmetry, allowing one to establish a relation between the symmetry of the *d*-function and that of the CGC. It is pointed out that there is a relation between the form of the integral transformation and the finite difference form of the CGC.

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1. INTRODUCTION

Many reviews have been written about the theory of Clebsch-Gordan coefficients (CGC) and the Wigner *d*-functions, and tables of these have been published.[1)] There are, however, some questions which have not found a simple exposition. In the present paper we consider one of these questions: the similarities between the properties of the CGC and the *d*-functions. This relation was first pointed out in the book.[3] It was also discussed in Refs. 4 and 5.

In the present paper we obtain a transformation relating the *d*-function and the CGC. The kernel of this transformation exhibits the Regge symmetry. The meaning of this transformation is very simple: it is a transformation between a fixed and moving coordinate system, and the relation between the symmetry of these two systems and the Regge symmetry justifies the publication of this paper.

2. THE TRANSFORMATION

We start with the known formula for the *d*-functions

$$d^{J_1}_{M_1M_1'}(\beta)\,d^{J_2}_{M_2M_2'}(\beta)=\sum_{J'}\langle J',M|J_1,M_1,J_2,M_2\rangle d^{J'}_{MM'}(\beta)\langle J',M'|J_1,M_1',J_2,M_2'\rangle, \qquad (2.1)$$

where the summation extends over all values of J' appearing in the addition of the vectors J_1 and J_2. Multiplying both sides by $\langle J,M|J_1,M_1,J_2,M_2\rangle$ and summing over M_1 and M_2 for $M_1+M_2=M$ (i.e., over M_1-M_2), we obtain

$$\sum_{M_1-M_2}\langle J,M|J_1,M_1,J_2,M_2\rangle d^{J_1}_{M_1M_1'}(\beta)\,d^{J_2}_{M_2M'_2}(\beta)=d^{J}_{MM'}(\beta)\langle J,M'|J_1,M_1',J_2,M_2'\rangle. \qquad (2.2)$$

This yields a rather general formula for the *d*-functions:

$$d^{J}_{MM'}(\beta)=\sum_{M_1-M_2}\frac{\langle J,M|J_1,M_1,J_2,M_2\rangle}{\langle J,M'|J_1,M_1',J_2,M_2'\rangle}d^{J_1}_{M_1M_1'}(\beta)\,d^{J_2}_{M_2M_2'}(\beta). \qquad (2.3)$$

If one assigns all possible values to the *d*-functions in the right-hand side, one obtains all possible expressions for the *d*-function in the left-hand side.

We set

$$M_1'=J_1,\quad M_2'=-J_2. \qquad (2.4)$$

Since

$$d^{J}_{M,\pm J}(\beta)=\left(\frac{(2J)!}{(J+M)!\,(J-M)!}\right)^{1/2}\left(\cos\frac{\beta}{2}\right)^{J\pm M}\left(\pm\sin\frac{\beta}{2}\right)^{J\mp M} \qquad (2.5)$$

and

$$\langle J,J_1-J_2|J_1,J_1,J_2,-J_2\rangle=\left[\frac{(2J+1)\,(2J_1)!\,(2J_2)!}{(J_1+J_2+J+1)!\,(J_1+J_2-J)!}\right]^{1/2}, \qquad (2.6)$$

we obtain from (2.3)

$$d^{J}_{MM'}(\beta)=\left[\frac{(J_1+J_2+J+1)!\,(J_1+J_2-J)!}{2J+1}\right]^{1/2}\times\sum_{M_1-M_2}(-1)^{J_2+M_2}\langle J,M|J_1,M_1,J_2,M_2\rangle \times\frac{[\cos(\beta/2)]^{J_1+J_2+M_1-M_2}[\sin(\beta/2)]^{J_1+J_2-M_1+M_2}}{[(J_1+M_1)!\,(J_1-M_1)!\,(J_2+M_2)!\,(J_2-M_2)!]^{1/2}} \qquad (2.7)$$

This formula can be found in Ref. 1, p. 68. The above derivation seems to be the most natural one.

One may rewrite the formula (2.7) in the form of a transformation from the CGC to the *d*-function

$$d^{J}_{MM'}(\beta)=\frac{2}{2J+1}\sum_{M_1-M_2}\langle J,M|J_1,M_1,J_2,M_2\rangle S_{M_1-M_2}(\beta), \qquad (2.8)$$

with

$$S_{M_1-M_2}(\beta)=(-1)^{J_2+M_2}[{}^1/_2(J+{}^1/_2)\,(J_1+J_2+J+1)!]^{1/2}\cdot\times\frac{[\cos(\beta/2)]^{J_1+J_2+M_1-M_2}[\sin(\beta/2)]^{J_1+J_2-M_1+M_2}}{[(J_1+M_1)!\,(J_1-M_1)!\,(J_2+M_2)!\,(J_2-M_2)!]^{1/2}} \qquad (2.9)$$

One can also write the inverse of the transformation (2.8). For this we use the "three *d*-function" theorem:

$$\int d^{J}_{-M-M'}(\beta)\,d^{J_1}_{M_1M_1'}(\beta)\,d^{J_2}_{M_2M_2'}(\beta)\,d\cos\beta=\frac{2}{2J+1}\langle J,M|J_1,M_1,J_2,M_2\rangle\langle J,M'|J_1,M_1',J_2,M_2'\rangle \qquad (2.10)$$

and substitute the particular values (2.4). As a result of this we obtain

$$\langle JM|J_1M_1J_2M_2\rangle=\frac{2J+1}{2}\left[\frac{(J_1+J_2+J+1)!\,(J_1+J_2-J)!}{2J+1}\right]^{1/2}\int d^{J}_{-M-M'}(\beta)\cdot\times\frac{[\cos(\beta/2)]^{J_1+J_2+M_1-M_2}[\sin(\beta/2)]^{J_1+J_2-M_1+M_2}}{[(J_1+M_1)!\,(J_1-M_1)!\,(J_2+M_2)!\,(J_2-M_2)!]^{1/2}}d\cos\beta. \qquad (2.11)$$

The right-hand side involves the same kernel (2.9).

All operations take an extremely simple form in symbolic writing. If one denotes the CGC by C, the "three *d*-function" theorem can be written in the form

$$d\otimes d\otimes d=C\otimes C. \qquad (2.12)$$

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Then (2.8) has the symbolic form

$$d=C\otimes C\otimes d^{-1}\otimes d^{-1}, \tag{2.13}$$

and (2.12) becomes

$$C=C^{-1}\otimes d\otimes d\otimes d. \tag{2.14}$$

The "multiplication rules" are easily established.

These equations demonstrate the relation between the d-functions and the CGC; each of them is the S-transform of the other.

3. THE REGGE SYMMETRY

The kernel $S(\beta)$ exhibits a well-known symmetry, which goes under the name Regge symmetry. We remind the reader that the symmetry of the CGC is formulated in the simplest manner in terms of the table:

$$\left\| \begin{matrix} -J+J_1+J_2 & J-J_1+J_2 & J+J_1-J_2 \\ J+M & J_1+M_1 & J_2+M_2 \\ J-M & J_1-M_1 & J_2-M_2 \end{matrix} \right\|. \tag{3.1}$$

The CGC are invariant with respect to a substitution equivalent to a permutation of two rows, two columns, or transposition of this table. The permutations of the columns of the second and third rows lead to the "trivial" symmetries, related to the permutations of the three sides of the triangle made up of the vectors $\mathbf{J}_1$, $\mathbf{J}_2$, and $\mathbf{J}$.

We are interested in the "nontrivial" operation of "transposition" (reflection in the main diagonal). It does not have a simple geometric analog, and the corresponding symmetry is proved by means of a special choice of representation of the CGC as a sum (as was done, e.g., in Ref. 5).

It was pointed out that all the symmetry properties of the CGC follow from the properties of the Whipple functions.[2] But there seems to be no simple proof in the literature.

We also note that the second "nontrivial" symmetry, corresponding to a permutation of the first and third rows, reduces to transpositions and permutations.

In our representation the Regge symmetry follows directly from the symmetry of the kernel $S(\beta)$ with respect to transposition. Indeed, a transposition reduces to the substitutions $M_1+M_2 \rightleftharpoons J_1-J_2$, $J_1-M_1 \rightleftharpoons J_2+M_2$, leaving J_1+J_2, M_1-M_2, and J unchanged. This implies immediately that under a transposition the kernel S is only multiplied by

$$(-1)^{-J_2-M_2+J_1-M_1}=(-1)^{M'-M}. \tag{3.2}$$

But this is exactly the same factor as appears if the indices M and M' are permuted in the left-hand side of (2.8). Thus the "nontrivial" symmetry of the CGC reflects the "trivial" symmetry of the d-function with respect to the permutation of its two lower indices (permutation of the two quantization axes). In the classical limit such a relation between the formulas has been indicated in the mentioned paper of Bincer.[4] Thus, the "nontrivial" symmetry reflects the relation between the d-functions and the CGC.

This brings up the question: what symmetry of the d-functions corresponds to a trivial symmetry of the CGC, e.g., a symmetry with respect to the permutation

$$(J_1, M_1) \rightleftharpoons (J_2, M_2). \tag{3.3}$$

A simple glance at Eq. (2.9) yields the answer: the symmetry reduces to

$$d^J_{MM'}(\pi-\beta)=(-1)^{J-M}d^J_{MM'}(\beta). \tag{3.4}$$

Not so simple is the situation with the symmetry of the CGC with respect to the substitution

$$(J,M) \rightleftharpoons (J_1, M_1). \tag{3.5}$$

There is no relation for the d-functions corresponding to this operation. This symmetry gets lost for the special choice of d-functions in (2.3). Thus, in distinction from the CGC, the d-functions do not have "nontrivial" symmetries.

4. THE FINITE-DIFFERENCE FORMULA

The formula for the S-transformation allows one to demonstrate more explicitly the relation between the finite-difference representation of the CGC and the expression of the d-function in terms of derivatives.

Already Gel'fand *et al.*[3] pointed out a curious analogy between the CGC and the d-function. Let us make use of the expression for the CGC given on p. 195 of Vilenkin's book.[4] After simple manipulations one obtains from it the expression

$$\begin{aligned} &\left(\frac{J_1+J_2+J+1}{2J+1}\right)^{1/2}\langle J, M|J_1, M_1, J_2, M_2\rangle \\ &=(-1)^{J_1-J+M_2}\left[\frac{(J+M)!}{(J-M)!}\frac{1}{(J+M')!(J-M')!}\right]^{1/2}\left[\frac{1}{(J_1+J_2)^{(2J)}}\right]^{1/2} \\ &\times[(J_1-M_1)^{(M'-M)}(J_1+M_1)^{-(M'+M)}]^{1/2}\Delta^{J-M}_{M'}[(J_1+M_1)^{(J+M')}(J_1-M_1)^{(J-M')}]. \end{aligned} \tag{4.1}$$

This formula should be compared to the appropriate expression for the d-function in Ref. 1, p. 69[3]:

$$\begin{aligned} (-1)^{J-M'}d^J_{MM'}(\beta)&=\frac{(J+M)!}{(J-M)!}\left[\frac{1}{(J+M')!(J-M')!}\right]^{1/2} \\ &\times\frac{1}{2^J}(1-\cos\beta)^{(M'-M)/2}(1+\cos\beta)^{-(M'+M)/2} \\ &\times\frac{d^{J-M}}{d(\cos\beta)^{J-M}}[(1+\cos\beta)^{J+M'}(1-\cos\beta)^{J-M'}]. \end{aligned} \tag{4.2}$$

Eq. (4.1) involves the quasi-powers

$$z^{(n)}=\frac{z!}{(z-n)!} \tag{4.3}$$

and the finite differences

$$\Delta^J_M f(M)=\sum_{n=0}^{J}\frac{J!}{n!(J-n)!}f(M+n). \tag{4.4}$$

The analogy between the two formulas (4.1) and (4.2) is obtained at the value

$$\cos\beta=M_1/J_1.$$

The meaning of these formulas becomes clearer if we substitute them into the transformation equations (2.8). We see that the S-transformation maps finite differences into derivatives, a fact that can be verified by direct but tedious calculations. We shall not investigate this problem in detail.

5. THE PHYSICAL MEANING OF THE TRANSFORMATIONS

The analogy between the d-function and the CGC has a simple interpretation.

In Eq. (2.3) the functions $d^{J_1}_{M_1 J_1}(\beta)$ and $d^{J_2}_{M_2 -J_2}(\beta)$ should be considered as two quantum unit vectors. Indeed, in a quantum system (intrinsically) one can define a direction in space only by means of eigenstates of angular momentum vectors J_1 or J_2, corresponding to the maximally possible value of the projection on the moving axis. (This signifies that the corresponding vector has been chosen as a moving axis.) In the second d-function this projection has to be taken with a minus sign, since $J_1 - J_2 < J$, and $J_1 + J_2 > J$, which is inadmissible.

Thus, the product of two d-functions determines a moving coordinate system which maximally approaches in its physical meaning such a system in the theory of the spinning top.

The function $d^{J}_{M J_1 - J}(\beta)$ which does not depend on the projections M_1 and M_2 describes the states in a fixed coordinate system (the projection M). If one adopts such an interpretation, the CGC takes on the meaning of a transformation amplitude of the wave function between the two coordinate systems: a moving one and a fixed one. The Regge symmetry reflects the symmetry between these two coordinate systems. Such a symmetry is explicitly visible for the d-function and is hidden for the CGC. One may add to this that the d-function itself can be considered as the amplitude describing the transformation between two fixed coordinate systems. In the classical limit, when the difference between the quantum and classical vectors disappears, the distinction between the d-functions and the CGC also disappears (cf. Ref. 5).

There also appears the question of finding the transformation amplitude between two moving coordinate systems. Such a coefficient must be completely independent of the projections of J. It is nothing but the Racah coefficient or the $6j$-symbol

$$\begin{Bmatrix} J_1 & J_2 & J_{12} \\ J_3 & J & J_{23} \end{Bmatrix} \tag{5.1}$$

The $6j$-symbol describes the transition from the plane spanned by the quantum vectors $\mathbf{J}_1$, $\mathbf{J}_2$, $\mathbf{J}_{12}$ to the plane spanned by $\mathbf{J}_2$, $\mathbf{J}_3$, $\mathbf{J}_{23}$ for given total angular momentum J. In the classical limit it turns into $(24\pi V)^{-1}$, where V is the volume of the tetrahedron with the edges the vectors entering into the $6j$-symbol (or it is equal to zero for $V<0$, cf. Ref. 6, p. 423, or Ref. 1, p. 259). One can also show that the symmetry of the tetrahedron yields all 72 Regge symmetries (including the nontrivial ones).

The higher $3nj$-symbols are related to spaces of higher dimension. Thus, the $9j$-symbol is a CGC of the group $O(4)$ and can be given a physical interpretation. But this goes beyond the scope of this paper. We only note that if one considers the group $O(n)$ for $n>3$ there appears a hierarchy of functions of the same type: the d-function, the CGC, the $6j$-symbol, just as for $SU(2)$, but now for "unphysical" values of the angular momenta.[8]

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[1)] Cf. the tables[1] and the review[2] which contains a bibliography.
[2)] The recent paper of Raynal[9] is dedicated to the use of Whipple functions in connection with the CGC.
[3)] The equations (4.1) and (4.2) go over into each other for J_1, $J_2 \gg J$, cf. Ref. 4.

[1] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskiĭ, Kvantovaya teoriya uglovogo momenta (Quantum theory of angular momentum), Nauka, Leningrad, 1975.
[2] Ya. A. Smorodinskiĭ and L. A. Shelepin, Usp. Fiz. Nauk 106, 3 (1972) [Sov. Phys. Uspekhi 15, 1 (1972)].
[3] I. M. Gel'fand, R. A. Minlos, and Z. Ya. Shapiro, Predstavleniya gruppy vrashcheniya i gruppy Lorentsa (Representations of the rotation group and of the Lorentz group), Fizmatgiz, Moscow, 1958 [Engl. Transl. available].
[4] N. Ya. Vilenkin, Spetsial'nye funktsii i teoriya predstavleniĭ grupp (Special functions and group representations), "Nauka", Moscow 1965 [Engl. Transl. by AMS ca. 1970].
[5] A. M. Bincer, J. Math. Phys. 11, 1835 (1970).
[6] E. P. Wigner, Group Theory, Academic Press, N. Y. 1959 (Russian Transl., 1961 quoted in text).
[7] L. S. Kil'dyushov, Yad. Fiz. 15, 197 (1972) [Sov. J. Nucl. Phys. 15, 113 (1972)].
[8] G. I. Kuznetsov and Ya. A. Smorodinskiĭ, ZhETF Pis. Red. 25, 500 (1977).
[9] J. Raynal, J. Math. Phys. 19, 467 (1978).

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