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Mechanism of nonadiabatic losses in a dipole trap

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The principal mechanisms of the nonadiabatic escape of particles from a trap are considered—stochastic instability and universal Arnold instability. The boundaries of the regions of existence of these instabilities are investigated. It is shown that the results of numerical calculations of the limits of the adiabatic behavior of the particles agree with analytic estimates that follow from stochasticity theory. It is observed in experiment that introduction of azimuthal inhomogeneity of the magnetic field can lead to a decrease of the nonadiabatic losses.

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Processes that control the dynamics of particles in magnetic traps can be connected, besides the global diffusion due to the scattering by the residual gas, with the universal Arnold diffusion and with stochastic instability.[1] In particular, it was suggested[1,2] that the nonadiabatic decrease of the particle lifetime with increasing relative cyclotron radius $\chi = \rho/R_c$ (R_c is the radius of curvature of the force line), which is observed in linear traps with mirrors,[3] is due to the Arnold universal instability. To understand the reason for the nonadiabatic behavior of the particles it is necessary to determine first the limits of the regions where the proposed instability mechanisms operate. The present paper is devoted to a determination of these limits and of the form of the instability responsible for the nonadiabatic departure of a particle from a dipole trap. The questions considered here, which touch upon the general physical problem of the investigation of very subtle and universal interaction mechanisms of resonances that control stochastization processes, are of great interest, for example, in the investigation of singularities of the spatial distribution of the high-energy part of the spectrum of the charged particles in the magnetic traps of the earth and of Jupiter, etc.

1. The change of the adiabatic invariant in multiperiod systems is due to the interaction of nonlinear resonances, for example, resonances between fast Larmor rotation and higher harmonics of slow oscillations between the magnetic mirrors. In the vicinity of the separatrix of each resonance there appears the so-called stochastic layer, which constitutes a certain region of unstable motion.[1,4] In the case of axial symmetry of the field (two-dimensional motion) the thin stochastic layers of different resonances do not intersect on the phase plane, and the instability is therefore localized within the confines of a single stochastic layer, and causes only bounded oscillations of the adiabatic invariant and of the frequencies. If the parameter χ is large enough, the stochastic layers broaden to the dimensions of their resonances, neighboring resonances in phase space overlap, and strong stochastic instability results.

The boundary of the stochastic instability, determined by the criterion of overlap of the nonlinear resonances, can be represented, according to Ref. 4, in the form

$$\Delta\mu/\mu\approx\pi^{1/2}\Omega/2\omega, \tag{1}$$

where μ is the orbital magnetic moment of the particle, ω is the cyclotron frequency, and Ω is the frequency of the oscillations between the reflection points. We recognize that in a dipole magnetic field the period of the oscillation of the leading center between the reflection point is given by[5]

$$\tau_2=4R_eT(\alpha)/v, \quad 0\leqslant\alpha\leqslant\pi/2, \tag{2}$$

where v is the particle velocity, R_e is the radius of the force line in the median plane, $T(\alpha) = 1.3-0.56 \sin \alpha$, and α is the angle between the velocity vector and the

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force line in the median plane. We can then write in place of (1)

$$\frac{\Delta\mu}{\mu}\approx\frac{1.3\sin^2\alpha}{T(\alpha)}\frac{\rho_e}{R_e} \tag{3}$$

where $\rho_e = \rho(\omega = \pi/2)$ is the "total" cyclotron radius.

The formulas given in the literature for the exponentially small value of $\Delta\mu$ differ mainly in the pre-exponential factor, since the exponential itself is universal in form.[6] The exact expression for the exponential was cited by one of us,[7] and an approximate expression was obtained in Ref. 8. To determine $\Delta\mu$ one starts with the equation[8]

$$\Delta\mu_0=-\operatorname{Re}\int_{s_1}^{s_2}\frac{v_\perp}{HR_c}\left(v_\parallel^2+\frac{v_\perp^2}{2}\right)e^{i\varphi}\frac{ds}{v_\parallel},$$

$$\varphi=\varphi_0-\frac{\omega_e}{H_e}\int_0^s H\frac{ds}{v_\parallel}, \tag{4}$$

where $v_\parallel$ and $v_\perp$ are the parallel and perpendicular components of the velocity vector relative to the field H, s is the length of the force line measured from the median plane, s_1 and s_2 are the turning points, φ_0 is the phase of the particle, H_e is the magnetic field in the median plane at a the distance R_e at which the pole and the saddle point coalesce. For example, the relation

$$\bar{v}_\perp^2/2H=\text{const}=\mu_0$$

is used in Ref. 8 to reduce the integral in (4) to Γ functions. We shall integrate directly the initial equation (4) with allowance for the coalescence of the singularities, making use of Ref. 9. Expanding the integrand of (4) in powers of s, confining ourselves to second-order derivatives, and recognizing that near the zero $\tilde{s}$ of the function $H(s)$ the expansion takes the form

$$H\approx H_e''\tilde{s}(s-\tilde{s}),\quad \tilde{s}=-i(2H_e/H_e''),$$

and that

$$H_e''=9H_e/R_e^2,\quad R_c''(s=0)=2/R_e,$$

we obtain

$$\Delta\mu_0=-\frac{v_\perp(14v_\parallel^2+13v_\perp^2)}{12\cdot 2^{1/2}H_e v_\parallel}\operatorname{Re}\exp\left\{i\varphi_0-\frac{i\omega_e}{H_e}\right.$$

$$\left.\times\int_0^{\tilde{s}}\frac{(1+H_e''s^2/2H_e)\,ds}{[1-\eta^{-1}(1+H_e''s^2/2H_e)]^{1/2}}\int_{-\infty}^{\infty}\frac{\exp(-3r^2/2^{1/2}R_e\rho_e}{ir}dr\right\}. \tag{5}$$

Here $v_\perp$ and $v_\parallel$ pertain to the equatorial plane, $\eta = \sin^{-2}\alpha$, $s = \tilde{s} + r$.

Taking into consideration the standard integral[9]

$$\int_{-\infty}^{\infty}\frac{\exp(-zt^2)}{t-\varepsilon}dt=i\pi\exp(-\varepsilon^2 z)\,[1-\Phi(-i\varepsilon z^{1/2})], \tag{6}$$

where $\Phi(u)$ is the Fresnel integral, $\operatorname{Re}\varepsilon = 0$, $\operatorname{Im}\varepsilon \geqslant 0$, $z>0$, we obtain the following asymptotic approximation of the integral (4)

$$\frac{\Delta\mu}{\mu}=-0.37\frac{14\eta-1}{(\eta-1)^{1/2}}\cos\varphi_0$$

$$\times\exp\left[-\psi(\alpha)\frac{R_e}{\rho_e}\right], \tag{7}$$

where

$$\psi(\alpha)=\frac{\eta^{1/2}}{3\cdot 2^{1/2}}[(\eta$$

$$+1)\operatorname{Arsh}(\eta-1)^{-1/2}-\eta^{1/2}]. \tag{8}$$

The values of the function $\Delta\mu/\mu$ obtained with the aid of (7) practically coincide with the corresponding values obtained by numerical methods.[10]

At $\eta \gg 1$ the pre-exponential factor in (7) takes the form

$$A_1\approx-5.18\sin^{-1}\alpha\cos\varphi_0. \tag{9}$$

The corresponding value obtained in Ref 8, with an arithmetic error corrected, is

$$A_2\approx-3.62(R_e/\rho_e)^{1/2}\sin^{-1}\alpha\cos\varphi_0, \tag{10}$$

which is very close to (9). Thus, allowance for the coalescence of the singularities leads to a more complicated analytic dependence of A on α and eliminates the dependence on ρ_e/R_e.

Substituting (7) in (3), we obtain the stochasticity condition in the form

$$\exp\left(-\psi\frac{R_e}{\rho_e}\right)=\frac{3.6}{T(\alpha)}\frac{\rho_e}{R_e}\frac{(\eta-1)^{1/2}}{\eta(14\eta-1)}. \tag{11}$$

The solution of (11) is shown in Fig. 1 (curve 1) and is compared with the results of numerical calculations (curves 2 and 3) of the limits of the adiabatic behavior of particles in a dipole field[11] and the experimental adiabaticity limit.[12] The boundary between the stable and unstable initial conditions was determined numerically in Ref. 11 as a function of the parameter $\xi = v_\perp^2/v^2$ and of the normalized energy. The boundary of the stable motion was approximately defined by the equation

$$\rho^2/R_e^2=0.012\xi. \tag{12}$$

It is seen from Fig. 1 that, first, the motion-stability limit that follows from stochasticity theory and from Eq. (11) agrees with the numerical results represented by Eq. (12), and, second, the experiments in the dipole trap were performed under conditions far from the region of stochastic instability. It is interesting that the instability limits come closer together with decreasing angle α. At values $\alpha \lesssim 30°$ the stochasticity condition might be attained at $\chi \lesssim 3\chi_c$ (where χ_c is the critical value of the adiabaticity parameter[12] as determined by curve 3 of Fig. 1.

2. At $n \geqslant 3$ degrees of freedom (say in the presence of asymmetry in the magnetic trap), then the system of the fundamental resonances is determined by three frequencies (the frequency of the azimuthal drift is added). The stochastic layers intersect in this case and form a single extended net over which the so-called Arnold diffusion[13] is possible. This diffusion makes it possible to explain the observed effect, albeit qualitatively, in accordance with the following scheme.[1]

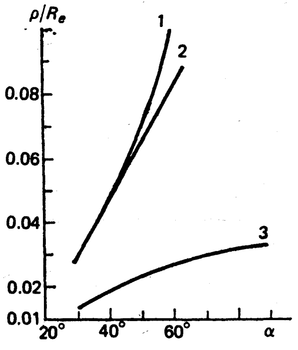

FIG. 1. Instability limits: 1—theoretical limit of stochasticity; 2—stochasticity limit obtained by numerical methods; 3—experimental nonadiabaticity region; $\rho = \rho_e \sin\alpha$.

The residual gas in the trap gives rise to global diffusion, which brings the system to the nearest stochastic layer, followed by faster Arnold diffusion along the stochastic layer, which takes the particles into the loss cone.

The existence of weak instability (Arnold diffusion) in the absence of overlap of the resonances was proved with the aid of numerical experiments using the simplest model of two linear oscillators with linear coupling.[1] Under the realistic conditions of laboratory traps, this diffusion should depend not only on the parameters of the particle and on the degree of symmetry of the field, but also on the pressure of the residual gas. A dependence on the pressure in the nonadiabatic region can actually be traced in the results of a number of studies.[3,14]

Arnold diffusion in the presence of field asymmetry (which apparently is always present in real traps) is possible in principle at all values of the parameter χ. A semi-qualitative theory of this instability yields the following estimate for the diffusion coefficient[2]:

$$D_A \sim \bar{\omega} \mu^2 \varepsilon \beta^3 \exp\left[-\frac{2\exp(1/6\varepsilon)}{(\beta\varepsilon)^{1/2}}\right], \tag{13}$$

where $\beta^2 = (\Delta H/H)_\varphi$ is the axial asymmetry of the magnetic field, and $\varepsilon \propto \chi$. If the adiabaticity parameter is small enough, the characteristic diffusion time is so long that it is impossible in practice to distinguish stable motion in a symmetrical field from unstable motion in an asymmetrical field.

With increasing χ, the situation can change radically because of the doubly exponential dependence of the coefficient D_A on the adiabaticity parameter. Because of this circumstance we can speak of a rough estimate of the instability limit. According to (13), this limit is

$$k = \rho_e/\psi R_e \sim 0.15.$$

From the experimental data (Fig. 1, curve 3) it follows that $k \sim 0.08$. From the upper bound[15] of the Arnold diffusion it follows that the system stays close to the initial position for a time

$$t \sim \exp(M^{-a}), \tag{14}$$

where

$$a = (3l+n+4)^{-1} - \sigma, \quad l \geqslant n(n-1)/2.$$

Here σ is an arbitrarily small quantity and n is the number of degrees of freedom. In our case $n = 3$; $1/a = 16$; $M = \exp(\psi R_e/p_e)$. Equation (14) yields $k \sim 0.06$, which overlaps the experimental value and is close to it.

3. To estimate the influence of the field asymmetry on the particle diffusion, we measured the electron containment time in the previously described[12] dipole trap as a function of the azimuthal inhomogeneity. The field asymmetry was produced by an iron appendage in the form of a segment with a spherically concave base. Appendages with different base diameters were constructed. The maximum value was 11 cm when the sphere (which served as the source of the magnetic field) had a 16 cm diameter. The appendage was placed on the surface of the sphere symmetrically relative to the median plane. The injector and detector were placed on the opposite side of the sphere on the leading force line with $R_e \approx 21$ cm at an angle ~20° from the median plane. The electrons were injected in the median plane with initial angle values $\alpha_0 \approx 90°$. Figure 2 shows the maximum change of the field as a function of the distance R_e in the presence of the largest appendage. It is seen from Fig. 2 that the particle drift shell is deformed in such a way that it approaches a spherical surface on the appendage side. In the distorted part of the field, on the drift trajectory with isoline $H_e \sim 25$ Oe, the value of ∇H is smaller than for the dipole field.

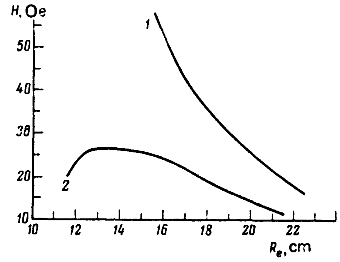

FIG. 2. Magnetic field intensity along the radius R_e: 1—initial magnetic field; 2—deformed dipole field; ($H = H_{\min}(R_e)$).

The measured lifetimes of the electrons in the trap with the magnetic field shown in Fig. 2 are given in Fig. 3. The result of the measurements is at first glance unexpected: the asymmetry increases somewhat the particle lifetime τ and the critical value χ_c of the adiabaticity parameter. This means that in this case the value of τ is determined mainly by the value of the parameter χ, and not by the asymmetry of the field: the decrease of the effective value of χ on account of the decrease of the field gradient in the distorted part leads to an increase of τ. That the roles played by the parameter χ in the field asymmetry and in the particle diffusion are unequal follows also from the theoretical estimate (13). It is seen from this estimate that the diffusion depends much more strongly on the adiabaticity parameter than on the azimuthal inhomogeneity of the magnetic field. The instability limit itself, according to (13), depends very little (logarithmically) on the asymmetry of the field, in qualitative agreement with experiment.

The experimental and calculated data show thus that the most probable mechanism responsible for the observed nonadiabatic effects in the dipole trap[14] is the mechanism of universal instability of the multidimensional ($n > 2$) Hamiltonian system (the Arnold diffusion). The particle containment time depends relatively little

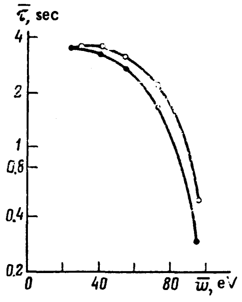

FIG. 3. Plots of $\bar{\tau}(\bar{w})$: ●—symmetrical field, ○—asymmetrical field, residual gas pressure $p \sim 2 \cdot 10^{-10}$ Torr.

on the asymmetry of the magnetic field and is determined mainly by the effective value of the parameter χ on the drift shell.

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Effect of cutoff radius of Coulomb scattering cross section on the distribution function and on the conductivity of nonequilibrium electrons

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The distribution function and the mean values are obtained for a system of noninteracting electrons situated in a strong electric field, whose momentum is scattered by ionized impurities (in semiconductors) or ions (in a plasma) and whose energy is dissipated as a result of extreme inelastic scattering at $\varepsilon > \varepsilon_0 \gg T$, ε_c (where ε_c is the characteristic energy of the electron-ion interaction and depends on the model and on the cutoff radius r_c of the Coulomb scattering cross section). It is shown that the dependence of the averaged quantities (the conductivity σ, the average energy $\bar{\varepsilon}$, and others) on E and on the parameters of the material is determined by the value of ε_c and by the form of the "Coulomb logarithm" $\Lambda(\varepsilon/\varepsilon_c)$. For a quite realistic form of $\Lambda(\varepsilon/\varepsilon_c)$ and not too small ε_c, the conductivity and the average energy have power-law dependences on $\varepsilon_c, \delta \propto \varepsilon_c^{1/2}$ and $\bar{\varepsilon} \propto \varepsilon_c^{1/2}$, with both $\sigma(E)$ and $\bar{\varepsilon}(E)$ constant. For some cutoff models it is shown that the dependence of the conductivity of the nonequilibrium electrons on the ion concentration N and on the longitudinal magnetic field intensity H can differ noticeably from the standard relations. The relations obtained for $\sigma(E, N, H)$ are in satisfactory agreement with the available experimental data.

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1. INTRODUCTION. FORMULATION OF PROBLEM

It is known that the calculation of the transport scattering cross section $\sigma_{tr}(\varepsilon)$ and of the pair-collision frequency $\tau_i^{-1}(\varepsilon)$ in the case of Coulomb scattering of electrons by ions (in a plasma) or by randomly distributed ionized centers (in semiconductors) encounters a characteristic difficulty, namely the logarithmic divergence of $\sigma_{tr}(\varepsilon)$ and $\tau_i^{-1}(\varepsilon)$ at small scattering angles. This difficulty is avoided by assuming that the Coulomb potential acts only up to distances $r < r_c$, so that it is possible to introduce a minimal scattering angle $\theta_{min}(\varepsilon, r_c)$ that depends on r_c and on the energy ε. This yields

$$\sigma_{tr}(\varepsilon) = \frac{\pi}{2}\left(\frac{e^2}{\varkappa\varepsilon}\right)^2 \ln\left(1+\mathrm{ctg}^2\frac{\theta_{min}}{2}\right)$$

($\varkappa$ is the permittivity). This relation is valid in both classical and quantum theory, except for the different connection between θ_{min} and r_c (Ref. 1). Different values are chosen for the Coulomb-potential cutoff radius r_c frequently on the basis of intuitive physical considerations, e.g., the Debye radius λ_D, the Larmor radius r_L (in the presence of an external magnetic field), or half the average distance between the ions $\frac{1}{2}N^{-1/3}$ (N is the ion concentration).[1-5] In all the foregoing cases the logarithmic factor (hereafter designated $\Lambda(\varepsilon)$) which enters in $\sigma_{tr}(\varepsilon)$ and $\tau_i^{-1}(\varepsilon)$ can be represented at $\varepsilon \gtrsim \varepsilon_c$ in the form

$$\Lambda(\varepsilon) = \ln\left[1+(\varepsilon/\varepsilon_c)^\nu\right] + \mathrm{const},$$

where $\nu \gg 1$ and ε_c is a quantity on the order of the en-

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