The possibility of measurement of the intrinsic magnetic moment of a free electron

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We propose a variant of the quasiclassical equations for description of the motion of a nonrelativistic charged particle with spin in a magnetic field, from which it follows, in particular, that it should be possible to carry out a Stern–Gerlach experiment for a free electron.

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The question of the possibility of observation and measurement of the spin of a free electron was investigated in the early years of development of quantum mechanics. The point of view which developed at that time can be summed up as follows: The spin angular momentum of the electron cannot be determined by means of experiments to which the classical concept of particle trajectory is applicable.[1]

Recently Kalcar[2] has criticized this point of view. Emphasizing the need of distinguishing measurements of the mechanical and magnetic moments, he pointed out a fundamental arrangement for measurement of the spin of a free electron whose motion can be considered classical.[1)] In the present article we intend to show that the intrinsic magnetic moment of the electron also can be measured in experiments which can be described classically. As a specific example we will consider a Stern-Gerlach experiment for the free electron.

The arguments which were used to justify the impossibility of measurement of the spin and magnetic moment of the electron in the course of its classical motion are based on the uncertainty principle (see for example Refs. 5 and 6). Basically there are no objections to these arguments, but they refer to definite specific realizations of experiments and therefore do not encompass all possibilities.

We shall carry out the discussion for an arbitrary particle with spin $\frac{1}{2}$, charge e, mass m, and magnetic moment μ. The gyromagnetic ratio $\gamma=2\mu/\hbar$ we shall not necessarily consider equal to e/mc—the value which follows from the Dirac equation. The particle motion will be assumed nonrelativistic. In this case the Hamiltonian can be written as follows[7]:

$$H=\frac{1}{2m}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2-\mu\sigma\mathbf{B}. \tag{1}$$

Here σ are the Pauli matrices, $\mathbf{B}$ is the magnetic field, and $\mathbf{A}$ is its vector potential. For simplicity we shall assume the field to be static.[2)] We impose on the potential the usual condition

$$\operatorname{div}\mathbf{A}=0.$$

Since we have in mind going over to the quasiclassical approximation, we shall write the wave function in the form

$$\psi=e^{iS/\hbar}\xi, \tag{2}$$

where $S(\mathbf{x},t)$ is a scalar function and $\xi(\mathbf{x},t)$ is a spinor. Substitution of Eq. (2) into the Schrödinger equation with the Hamiltonian (1) gives

$$-\frac{\partial S}{\partial t}\xi+i\hbar\frac{\partial\xi}{\partial t}=\left[\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)^2-\mu\sigma\mathbf{B}\right]\xi$$
$$-i\hbar\left[\frac{1}{m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)\nabla\xi+\frac{\Delta S}{2m}\xi\right]-\frac{\hbar^2}{2m}\Delta\xi. \tag{3}$$

In the quasiclassical approximation we look for ξ in the form of an asymptotic expansion[8] in powers of $\hbar$. The equations for the coefficients of this expansion and for S are obtained by systematically equating terms of like powers of $\hbar$ on the two sides of Eq. (3). Here we obtain several different results, depending on whether we formally consider $\mu\sigma\mathbf{B}$ a term of zero order in $\hbar$ or, writing it in the form $\frac{1}{2}\hbar\gamma\sigma\mathbf{B}$, included in the terms containing $\hbar$ to the first power. For our purposes the first procedure, in accordance with which we have grouped the terms in Eq. (3), turns out to be more convenient. Note that this approach does not mean neglect of the quantum origin of the spin magnetic moment of the particle. Here it is necessary to have in mind, first of all, that the quasiclassical expansion in powers of $\hbar$ is essentially a formal procedure, since the small parameter (in the role of which $\hbar$ enters) must in fact be dimensionless. Therefore in actuality the expansion is carried out in the ratio of $\hbar$ to some quantities characteristic of the phenomenon considered and having the dimensions of action. In the second place, the presence in the equation of small parameters does not uniquely determine the nature of the corresponding asymptotic expansion, which can be carried out in various ways, depending on the specific features of the problem. We shall not determine the desired quasiclassical expansion completely, but shall find only the principal term which describes the classical motion of the particles and shall estimate the next term, considering it as a correction. The condition that this correction be small will permit us, on the one hand, to find limits within which the approximation adopted can be used and, on the other hand, to determine those parameters which form the actual basis of the expansion.

In accordance with the above we write

$$\xi=\xi^{(0)}+i\hbar\xi', \tag{4}$$

where $i\hbar\xi'$ is the mentioned correction. The term of zero order in $\hbar$ in Eq. (3) is the following equation:

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$$\left[\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)^2-\mu\sigma\mathbf{B}\right]\xi^{(0)}=0. \tag{5}$$

The presence of the spin matrices in this equation leads to some modification of the usual discussions for the quasiclassical approximation (cf. Ref. 8). We shall proceed as follows. In addition to the initial "laboratory" system of coordinates K we shall consider local systems $K'(\mathbf{x})$ such that the access $O'Z'$ of the system K' at each point O' with coordinates $\mathbf{x}$ is directed along the magnetic line of force. Let χ be a two-dimensional column vector of the components of the spinor with respect to the system K'. Then the relation between χ and ξ (the components of the spinor in the laboratory system) is given by

$$\xi=D(g^{-1})\chi, \tag{6}$$

where $g=g(\mathbf{x})$ is a rotation which transforms K' to a system with axes parallel to the axes of K and D is the finite-rotation matrix[7,9] for spin $\frac{1}{2}$. The rotation g can be characterized by a vector $\boldsymbol{\varphi}=\boldsymbol{\varphi}(\mathbf{x})$ directed along the rotation axis and equal in magnitude to the rotation angle. We note that g is so far determined only with accuracy to an arbitrary rotation about the $O'Z'$ axis.

In the new spinor coordinates Eq. (5) will contain the already diagonal matrix:

$$\left[\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)^2-\begin{pmatrix}\mu B & 0\\ 0 & -\mu B\end{pmatrix}\right]\chi^{(0)}=0.$$

In order that this equation have a nontrival solution ($\chi^{(0)}\neq 0$), it is necessary that

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)^2-\mu B=0 \tag{7}$$

or

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right)^2+\mu B=0. \tag{8}$$

Equations (7) and (8) have a simple meaning. They are the Hamilton-Jacobi equations for motion of a charged particle under the influence of a magnetic field B and potential $\mp\mu B$, which is the energy of the magnetic dipole μ constantly oriented parallel or antiparallel to the field. We shall return again to discussion of this circumstance but note here that for definiteness in what follows we shall assume that Eq. (7) is satisfied. Accordingly, $\chi^{(0)}$ should have the form

$$\chi^{(0)}=\begin{pmatrix}\chi_1^{(0)}\\ 0\end{pmatrix}=\begin{pmatrix}\nu\\ 0\end{pmatrix}$$

(the components of the spinors will be designated by the same letters with subscripts 1 and -1; for brevity instead of $\chi_1^{(0)}$ we shall write ν).

In what follows it is convenient to carry out the entire discussion in the local spinor coordinates. Therefore an appropriate substitution of variables must be made in the initial equation (3). Here after simple transformations we obtain

$$i\hbar\left(\frac{\partial\chi}{\partial t}+\mathbf{v}\nabla\chi\right)=\begin{pmatrix}G_1 & 0\\ 0 & G_{-1}\end{pmatrix}\chi-i\frac{\hbar}{2}[\nabla\mathbf{v}+i\sigma_i(\mathbf{v}\nabla)\varphi_i]\chi-\frac{\hbar^2}{2m}P(\chi). \tag{9}$$

In this equation G_1 and G_{-1} are respectively the left-hand parts of Eqs. (7) and (8),

$$P(\chi)=D(g)\Delta[D(g^{-1})\chi] \tag{10}$$

(Δ is the Laplace operator) and we have used the equalities

$$\mathbf{v}=\frac{1}{m}\left(\nabla S-\frac{e}{c}\mathbf{A}\right),\quad \nabla\mathbf{v}=\frac{\Delta S}{m}, \tag{11}$$

$$D(g)\nabla D(g^{-1})={}^1/_2 i\sigma_i\nabla\varphi_i, \tag{12}$$

where $\mathbf{v}$ is the particle velocity, φ_i are the components of the vector $\boldsymbol{\varphi}$ in the local coordinate system, and summation over the twice repeated vector indices is understood. The equations (11) follow from well known properties of the Hamilton-Jacobi equation, and Eq. (12) can be verified directly.

In order to find an equation for ν, let us consider in Eq. (9) the term of first order in $\hbar$; here to begin with it is sufficient to limit ourselves just to the upper line of the corresponding spinor equation. In order to achieve simplification, we shall make use of the arbitrariness remaining in choice of the local coordinates (see above) and shall impose the condition

$$\varphi_3=0. \tag{13}$$

This condition can be satisfied by choosing the vector $\boldsymbol{\varphi}$ at each point to be orthogonal simultaneously to the OZ axis of the laboratory coordinate system and to the direction of the magnetic line of force. It is easy to see that with inclusion of (13) the equation for ν becomes the ordinary transport equation (cf. Ref. 8):

$$\frac{\partial\nu}{\partial t}+\mathbf{v}\nabla\nu+\frac{\nu}{2}\nabla\mathbf{v}=0. \tag{14}$$

Thus, we have obtained the principal equations which we need for S and ν. Note that ν can be taken as real. We shall discuss these equations later, and now we turn to investigation of the conditions of their applicability.

In correspondence to the spin-coordinate substitution made above, it is convenient to consider $\chi'=D(g)\xi'$ instead of ξ' (see Eqs. (4) and (6)). The exact equation for χ' can be obtained from (9), but solution of this equation is no easier than that of the initial equation. Therefore we shall determine χ' approximately as the term of next order in the quasiclassical expansion. Considering again in (9) the term of first order in $\hbar$, but this time the lower line of the corresponding equation, we obtain an equation for χ'_{-1}. It turns out to be not a differential equation, but simply a linear algebraic equation (cf. Ref. 8) with a solution

$$\chi_{-1}'=i\frac{(\Gamma\mathbf{v})}{4\mu B}\nu. \tag{15}$$

Here

$$\Gamma=(0\,1)\sigma_i(\nabla\varphi_i)\begin{pmatrix}1\\0\end{pmatrix}$$

and we have used the fact that in view of (7) we have $G_{-1} = 2\mu B$.

In order that the approximation discussed be applicable, it is necessary that

$$|\hbar\chi'_{-1}|\ll v. \tag{16}$$

Since we have in mind first of all the electron, we shall assume for order of magnitude estimates $\mu\sim e\hbar/mc$. Since (also in order of magnitude) $\Gamma\sim l^{-1}$, where l is the characteristic scale of inhomogeneity of the magnetic field, it is easy to see that the condition

$$mvc/eBl\ll 1 \tag{17}$$

assures satisfaction of the inequality (16). Rewriting this condition in the form

$$R\ll l, \tag{18}$$

where R is the Larmor radius, we conclude that the particle motion in a magnetic field considered here should have a drift nature[10]: The particle revolves in a circle whose center slides freely along the line of force and slowly drifts in the transverse direction.[3)]

From Eq. (17) it also follows that

$$\omega t_c\gg 1,$$

where $\omega=eB/mc$ is the cyclotron frequency and t_c is the characteristic time of variation of the magnetic field along the particle trajectory. This inequality is the adiabaticity condition for the magnetic moment,[11] which agrees completely with the results obtained previously and permits a simple interpretation for them. As we have already mentioned, Eqs. (7) and (8) describe the classical motion of a particle whose intrinsic magnetic moment adiabatically "follows" the field direction. We now see that this behavior of the magnetic moment is a natural feature of the approximation considered and follows directly from the equations of its applicability. Moreover, Eq. (14), since ν can be chosen as real, is easily seen to be equivalent to the ordinary equation of continuity for ν^2—the number density of particles moving along classical trajectories.

Let us now consider the equation for χ_1'. It is obtained from (9) in second order in $\hbar$ and by means of (15) can easily be converted to the form

$$\frac{\partial\chi_1'}{\partial t}+\mathbf{v}\nabla\chi_1'+\chi_1'\frac{\nabla\mathbf{v}}{2}=\frac{|(\Gamma\mathbf{v})|^2}{8\mu B}+\frac{1}{2m}(0\,1)P\left[\begin{pmatrix}\nu\\0\end{pmatrix}\right]. \tag{19}$$

The solution of this equation with zero initial value can be written as

$$\chi_1'=\int_0^t dt' f|_{\mathbf{x}=\mathbf{x}(t')}\exp\left\{-\int_{t'}^{t''}\frac{\nabla\mathbf{v}}{2}\bigg|_{\mathbf{x}=\mathbf{x}(t'')}dt''\right\}, \tag{20}$$

where $f(\mathbf{x},t)$ is the right-hand side of (19) and $\mathbf{x}(t)$ is the classical trajectory of the motion. Evaluating the quantity χ_1', we shall assume that the flux of particles is a narrow monochromatic beam of particles with approximately identical velocity directions. The beam diameter D will be assumed to satisfy the inequalities

$$\lambda\ll D\ll R, \tag{21}$$

where λ is the DeBroglie wavelength of the particles. The exponential factor in Eq. (20) describes the change of concentration of the particles in the beam. For simplicity we shall digress from discussion of the caustic (see Ref. 8) and shall assume that the motion of the partilces is not accompanied by a significant focusing or expansion of the bundle of trajectories. In this case the exponential factor mentioned will be close in magnitude to unity. Let us consider individually the two terms which make up f. Since D is the smallest characteristic scale of variation of the quantity occurring under the Laplace operator in f (see Eq. (10)), we must have

$$\left|(0\,1)P\left[\begin{pmatrix}\nu\\0\end{pmatrix}\right]\right|\sim\frac{\nu}{D^2}.$$

Therefore it is easy to see that $|\hbar\chi_1'|\ll\nu$, if

$$L\ll D^2/\lambda,\quad L\ll l^2/R, \tag{22}$$

where $L=vt$ is the path traversed by the particle. The first of these inequalities is the ordinary condition for geometrical optics, that the diffraction spreading of the beam be much less than its transverse dimension (we recall that the opposite case $L\gg D^2/\lambda$, corresponds to Fraunhofer diffraction, in which the geometrical representation is completely inapplicable). The second inequality is an additional specific limitation on the path length.[4)]

Let us now return to the Stern-Gerlach experiment for a free electron. At the initial point let there be a monochromatic electron beam of the type described, moving in an inhomogeneous magnetic field. The trajectory of this beam is approximately a helix of the Larmor radius wound on the magnetic line of force. As the result of action of the external field on the intrinsic magnetic moment of the electron, the initial beam gradually splits into two beams with opposite spin orientations.

Let us estimate the minimum time necessary to determine this splitting. Since motion of the particles across the magnetic lines of force is hindered by the Larmor rotation, spatial separation of the beams will occur first of all along the lines of force. The difference in the forces acting on electrons with opposite spin orientations is in order of magnitude $\mu B/l\sim e\hbar B/mcl$, so that the difference in the displacements along a line of force in a time t will amount to (in order of magnitude)

$$\Delta l\sim\frac{e\hbar B}{m^2cl}t^2=\frac{e\hbar B}{m^2cl}\frac{L^2}{v^2}$$

where L is as before the path traversed by the particles. In order that the beams with opposite spin orientations be spatially separated, Δl must be greater than D (the transverse dimension of the beam), and from this we obtain

$$L > L_{min} \cong (DlR/\lambda)^{1/3}. \quad (23)$$

This inequality gives a new limitation on the pathlength (this time a lower limit). However, it is easy to see that in principle the inequality (23) is consistent with Eqs. (17), (21), and (22), and all of these inequalities can be satisfied simultaneously, i.e., an appreciable splitting of the beam can occur before the quantum effects substantially distort the geometrical pattern of the trajectories.

Thus, a Stern-Gerlach experiment for a free electron turns out to be achievable in principle, but it is necessary to have in mind that from Eqs. (22), (23), and (18) it follows that $L_{min} > l \gg R$ and therefore for spatial separation of the beams the electrons must execute a rather large number of turns in the magnetic field. This circumstance explains, in particular, why the arguments mentioned above, which were given by Pauli[5] and by Mott and Massey,[6] do not exclude the possibility of carrying out a Stern-Gerlach experiment for a free electron: In Refs. 5 and 6 an experimental arrangement is discussed in which the electron crosses the region with inhomogeneous magnetic field once.

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1) See also the discussion which followed this work.[3,4] In regard to the possibility of measuring the intrinsic magnetic moment of a free electron, Kalcar holds to the previous negative point of view.

2) Also for the sake of simplicity we shall not assume an electric field in the calculation. It would not be difficult to take it into account, and no important changes would be introduced in the subsequent discussions.

3) More precisely: R is the maximum possible value of the Larmor radius for a given velocity, determined by the expression $mv_\perp c/eB$, where $v_\perp$ is the projection of the velocity perpendicular to the line of force. We note further that Eq. (17) can be written in terms of action: $\mu Bl/v$ coincides in order of magnitude with that part of the change of action of the particle ΔS_μ on its passage through a region of characteristic size l which is due to interaction of the intrinsic magnetic moment with the external field. Therefore Eq. (17) can be represented in the form $\Delta S_\mu \gg \hbar$. However, it must be kept in mind that there is some arbitrariness in this interpretation of Eq. (17). While it is suitable for a trajectory of "general type" in which, in particular, $v_\perp$ and $v_\parallel = (v^2 - v_\perp^2)^{1/2}$ are comparable in magnitude (and both of the order of v), this may become inadequate in some special cases, for example, for $v_\parallel \ll v$, when the time of flight through the characteristic region of inhomogeneity of the magnetic field turns out to be much greater than l/v.

4) The limitations obtained above, strictly speaking, do not exhaust all conditions of applicability of the quasiclassical approximation considered. It is necessary also to be convinced that the terms discarded in derivation of Eqs. (15) and (19) are actually small. Verification of this will, generally speaking, lead to the necessity of imposing an additional limitation, which can be written as $l \ll D^2/\lambda$. However, for successful accomplishment of a Stern-Gerlach experiment (see below) it is necessary that $l < L$ and therefore we obtain nothing new in comparison with Eq. (22).

1W. Pauli, General Principles of Wave Mechanics, Russ. transl., Gostekhizdat, 1947.

2J. Kalcar, Nuovo Cimento 8A, 759 (1972).

3P. S. Farago, Lett. Nuovo Cimento 4, 991 (1972).

4J. Kalcar, Lett. Nuovo Cimento 5, 645 (1972).

5W. Pauli, Works on Quantum Theory, Russ. transl., Nauka, 1977, p. 131.

6N. F. Mott and H. S. W. Massey, The Theory of Atomic Collisions, Oxford, 1949. Russ. transl., Mir, 1969.

7L. D. Landau and E. M. Lifshitz, Kvantovaya mekhanika (Quantum Mechanics), Nauka, 1975.

8V. P. Maslov and M. V. Fedoryuk, Kvaziklassicheskoe priblizhenie dlya uravneniĭ kvantovoĭ mekhaniki (The Quasiclassical Approximation for the Equations of Quantum Mechanics), Nauka, 1976.

9G. Ya. Lyubarskiĭ, Teoriya grupp i ee primenenie v fizike (Group Theory and Its Applications in Physics), Fizmatgiz, 1958.

10A. I. Morozov and L. S. Solov'ev, in: Voprosy teorii plazmy (Problems of Plasma Theory), Gosatomizdat, 1963, p. 177.

11A. Abragam, Principles of Nuclear Magnetism, Oxford, 1961. Russ. transl., IIL, 1963.

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